# INVERSION AND REPRESENTATION THEOREMS FOR A GENERALIZED LAPLACE INTEGRAL

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**1. Varma [8] introduced a generalization** of the Laplace integral

$$\mathscr{F}(x) = \int_0^\infty e^{-xt}\phi(t)dt \tag{1}$$

in the form

$$F(x) = \int_0^\infty (xt)^{m-1/2}e^{-xt/2}W_{k,m}(xt)\phi(t)\,dt \tag{2}$$

where $\phi(t) \in L(0, \infty)$, $m > -1/2$ and $x > 0$. This generalization is a slight variant of an equivalent integral introduced earlier by Meijer [7] and reduces to (1) when $k + m = 1/2$. In a recent paper [1] Erdélyi has pointed out that the nucleus of (2) can be expressed as a fractional integral of $e^{-xt}$ in terms of the operators of fractional integration introduced by Kober [6]. In this note two theorems have been given-one giving an inversion formula for the transform (2) and another giving necessary and sufficient conditions for the representation of a function as an intgral of the form (2) by considering its nucleus as a fractional integral of $e^{-xt}$.

**2. The operators are defined as follows.**

$$I^+_{\eta,\alpha}\mathscr{F}(x) = \frac{1}{\Gamma(\alpha)}x^{-\eta-\alpha}\int_0^x (x-u)^{\alpha-1}u^{\eta}\mathscr{F}(u)du$$

$$K^-_{\zeta,\alpha}\mathscr{F}(x) = \frac{1}{\Gamma(\alpha)}x^{\zeta}\int_x^\infty (u-x)^{\alpha-1}u^{-\zeta-\alpha}\mathscr{F}(u)du$$

where $\mathscr{F}(x) \in L_p(0, \infty)$, $1/p + 1/q = 1$ if $1 < p < \infty$, $1/q = 0$ if $p = 1$, $\alpha > 0$, $\eta > -1/q$, $\zeta > -1/p$.

The Mellin transform $\overline{M}_t\mathscr{F}(x)$ of a function $\mathscr{F}(x) \in L_p(0, \infty)$ is defined as

$$\overline{M}_t\mathscr{F}(x) = \int_0^\infty \mathscr{F}(x)x^{it}dx \qquad (p = 1)$$

and

$$= \underset{X\to\infty}{\overset{\text{index } q}{\text{l.i.m}}} \int_{1/X}^X \mathscr{F}(x)x^{it-1/q}dx \qquad (p > 1)$$

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The inverse Mellin transform $\bar{M}^{-1}\phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

$$\bar{M}^{-1}\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t) x^{-it} dt \qquad (q = 1) \tag{3}$$

and

$$= \frac{1}{2\pi} \underset{T\to\infty}{\overset{\text{index } p}{\text{l.i.m}}} \int_{-T}^{T} \phi(t) x^{-it-1/p} dt \qquad (q > 1).$$

If the Mellin transform is applied to Kober's operators and the orders of integration are interchanged we obtain, under certain conditions,

$$\bar{M}_t\{I_{\eta,\alpha}^{+}\mathscr{F}(x)\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} \bar{M}_t \mathscr{F}(x)$$

and

$$\bar{M}_t\{K_{\zeta,\alpha}^{-}\mathscr{F}(x)\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} \bar{M}_t \mathscr{F}(x) .$$

But

$$\bar{M}_t(e^{-x}) = \int_0^\infty e^{-x} x^{it-1/q} dx = \Gamma\left(\frac{1}{p} + it\right) \text{ if } \frac{1}{p} > 0 .$$

Therefore

$$\bar{M}_t\{I_{\eta,\alpha}^{+}(e^{-x})\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]}$$

and

$$\bar{M}_t\{K_{\zeta,\alpha}^{-}(e^{-x})\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} .$$

By (3)

$$I^{+}_{\eta,\alpha}(e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} x^{-it-1/p}dt$$

and

$$(4)\qquad K^{-}_{\zeta,\alpha}(e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} x^{-it-1/p}dt$$

provided that $1/p > 0$, $\eta + 1/q > 0$ and $\zeta + 1/p > 0$.

It has also been shown by Erdélyi [2] that if the integral in (4) is evaluated by the calculus of residues then it can be expressed in terms of a confluent hypergeometric function. In particular,

$$K^{-}_{2m,(1/2)-m-k}(e^{-x}) = x^{m-1/2}e^{-x/2}W_{k,m}(x)$$

where $x > 0$, $(1/2) - m - k > 0$.

3. THEOREM 1. *Assume $\phi(t) \in L_p(0, \infty)$, $1 \leqq p < \infty$, $x > 0$. If $2m > -1/q$ when $(1/2) - m - k > 0$ and $(1/2) + m - k > -1/q$ when $(1/2) - m - k > 0$, then $K^{-}_{2m,(1/2)-m-k}[\mathscr{F}(x)]$ exists and is equal to*

$$\int_0^{\infty} K^{-}_{2m,(1/2)-m-k}(e^{-xt})\phi(t)dt = F(x)$$

*where $\mathscr{F}(x)$ and $F(x)$ are given by (1) and (2) respectively.*

*Proof.* *Case* I $(1/2) - m - k > 0$, $1 < p < \infty$.

If $\phi(t) \in L_p(0, \infty)$, $1 \leqq p < \infty$ and $x > 0$ it is easy to see that $\mathscr{F}(x)$ exists. Therefore

$$K^{-}_{2m,(1/2)-m-k}[\mathscr{F}(x)] = \frac{x^{2m}}{\Gamma((1/2) - m - k)} \times \int_x^{\infty} (u - x)^{-(1/2)-m-k} u^{-(1/2)-m+k} \left\{\int_0^{\infty} e^{-ut}\phi(t)dt\right\} du\,.$$

But from a theorem of Hardy [5] we know that if $\phi(t) \in L_p(0, \infty)$, $1 < p < \infty$ then $u^{1-2/p}\mathscr{F}(u) \in L_p(0, \infty)$ and therefore $(u - x)^{\alpha}u^{\beta}\mathscr{F}(u) \in L_p(x, \infty)$ provided that $\alpha + \beta = 1 - 2/p$ and $\alpha p > -1$. Therefore the integral

$$\int_x^{\infty} (u - x)^{-(1/2)-m-K} u^{-(1/2)-m+K} \mathscr{F}(u)du$$
$$= \int_x^{\infty} \{(u - x)^{-(1/2)-m-k-\alpha} u^{-(1/2)-m+k-\beta}\}\,\{(u - x)^{\alpha}u^{\beta}\mathscr{F}(u)\}du$$

will exist if the expressions within the brackets in the integrand belong to $L_p(x, \infty)$ and $L_q(x, \infty)$ respectively. The conditions for these are $(-(1/2) - m - k - \alpha)q > -1$, $(-1 - 2m - \alpha - \beta)q < -1$ and $\alpha + \beta = 1 - 2/p$, $\alpha p > -1$ which reduce to $2m > -1/q$ and $(1/2) - m - k > 0$. Hence under these conditions the integral converges absolutely and we can change the order of integration. Therefore

$$K^{-}_{2m,(1/2)-m-K}[\mathscr{F}(x)] = \frac{x^{2m}}{\Gamma((1/2) - m - k)} \int_0^\infty v^{-(1/2)-m-k}(x + v)^{-(1/2)-m+k}e^{-vt}$$
$$\times \left\{\int_0^\infty e^{-xt}\phi(t)dt\right\} dv = \frac{x^{2m}}{\Gamma((1/2) - m - k)} \int_0^\infty e^{-xt}\phi(t)$$
$$\times \left\{\int_0^\infty v^{-(1/2)-m-k}(x + v)^{-(1/2)-m+k}e^{-vt}dv\right\}dt$$
$$= \int_0^\infty (xt)^{m-(1/2)}e^{-(1/2)xt}W_{k,-m}(xt)\phi(t)dt = F(x)$$

as $W_{k,-m}(x) = W_{k,m}(x)$.

If $p = 1$, it is similarly seen that the change in the order of integration is justified if $2m > 0$ and $(1/2) - m - K > 0$.

*Case* II. $(1/2) - m - k < 0$, $1 < p < \infty$.

If $\alpha < 0$ then the operator $K^{-}_{\eta,\alpha}\{\mathscr{F}(x)\}$ is defined as the solution, if any, of the integral equation $\mathscr{F}(x) = K^{-}_{\eta+\alpha,-\alpha}\{g(x)\}$. Now

$$K^{-}_{(1/2)+m-k,-(1/2)+m+k}[F(x)]$$
$$= \frac{x^{(1/2)+m-k}}{\Gamma(-(1/2) + m + k)} \int_0^\infty (u - x)^{-(3/2)+m+k}u^{-2m}$$
$$\times \left\{\int_0^\infty (ut)^{m-(1/2)}e^{-(1/2)ut}W_{K,m}(ut)\phi(t)dt\right\}du .$$

Again from a result of Hardy [5] we know that if

$$F(x) = \int_0^\infty K(xy)\phi(y)dy$$

then

$$\int_0^\infty x^{p-2}\{F(x)\}^p dx < \left\{\psi\left(\frac{1}{q}\right)\right\}^p \int_0^\infty \{\phi(y)\}^p dy$$

where

$$\psi(s) = \int_0^\infty x^{s-1}K(x)dx .$$

If

$$K(x) = |x^{m-(1/2)}e^{-(1/2)x}W_{k,m}(x)|$$

then

$$\psi(s) = \frac{\Gamma(2m+s)\Gamma(s)}{\Gamma\left(m-k+\frac{1}{2}+s\right)}$$

by Goldstein's formula [4]. Therefore

$$\int_0^\infty x^{p-2}\{F(x)\}^p dx < \left\{\frac{\Gamma\left(2m+\frac{1}{q}\right)\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(m-k+\frac{1}{2}+\frac{1}{q}\right)}\right\}^p \int_0^\infty \{\phi(y)\}^p dy$$

provided that $2m > -1/q$, or $x^{1-(2/p)}F(x) \in L_p(0, \infty)$ if $\phi(y) \in L_p(0, \infty)$ $(p > 1)$. Hence $(u-x)^\alpha u^\beta F(u) \in L_p(x, \infty)$ if $\alpha + \beta = 1 - (2/p)$ and $\alpha > -1/p$. Also $(u-x)^{-(3/2)+m+k-\alpha}u^{-2m-\beta} \in L_q(x, \infty)$ if $(-(3/2)+m+k-\alpha)q + 1 > 0$ and $(-(3/2)-m+k-\alpha-\beta)q + 1 < 0$. These four conditions reduce to $m + k - (1/2) > 0$ and $m - k + (1/2) > -1/q$. So the integral $\int_x^\infty (u-x)^{-(3/2)+m+k}u^{-2m}F(u)du$ exists under these conditions and

$$\begin{aligned} &K^{-}_{(1/2)+m-k,-(1/2)+m+k}[F(x)] \\ &= \frac{x^{(1/2)+m-K}}{\Gamma(-(1/2)+m+k)}\int_0^\infty t^{m-(1/2)}\phi(t)dt \\ &\times \int_x^\infty (u-x)^{m+k-(3/2)}u^{-m-(1/2)}e^{-(1/2)ut}W_{k,m}(ut)du \end{aligned}$$

on changing the order of integration which is permissible since the integral is absolutely convergent. But [4]

$$\int_x^\infty u^{\lambda-1}(u-x)^{k-\lambda-1}e^{-u/2}W_{k,m}(u)du = \Gamma(k-\lambda)x^{k-1}e^{-x/2}W_{\lambda,m}(x)$$

where $k > \lambda$ and $x$ is positive. Therefore

$$\begin{aligned} K^{-}_{(1/2)+m-k,-(1/2)+m+k}[F(x)] &= \int_0^\infty (xt)^{m-(1/2)}e^{-(xt/2)}W_{-m+(1/2),m}(xt)\phi(t)dt \\ &= \int_0^\infty e^{-xt}\phi(t)dt \end{aligned}$$

under the conditions $m + k - (1/2) > 0$, $m - k + (1/2) > -1/q$, $x > 0$.

If $p = 1$, the change in the order of integration is justified if $m + K - (1/2) > 0$ and $(1/2) + m - k > 0$.

Hence $K^{-}_{(1/2)+m-k,-(1/2)+m+k}[F(x)] = \mathscr{F}(x)$ and the theorem is proved.

THEOREM 2. *Under the conditions of Theorem* 1 *we have*

$$(5)\qquad \int_0^\infty e^{-xt} I^{+}_{2m,(1/2)-m-K}\{\phi(t)\}dt = \int_0^\infty K^{-}_{2m,(1/2)-m-K}(e^{-xt})\phi(t)dt \ .$$

This is a consequence of Theorem 2 of Erdélyi [3] and is proved similarly.

**4. We are now in a position to** give inversion and representation theorems for the transform.

We have seen that, under certain conditions,

$$K^{-}_{(1/2)+m-k,\,-(1/2)+m+k}[F(x)] = \mathscr{F}(x) \ .$$

Also $\mathscr{F}(x)$ has derivatives of all orders for $x$ sufficiently large and vanishes at infinity. So we can apply the Post-Widder operator $L_{\lambda,u}$ defined by the relation

$$L_{\lambda,u}[\mathscr{F}(x)] = \frac{(-1)^\lambda}{\lambda\,!}\mathscr{F}^{(\lambda)}\left(\frac{\lambda}{u}\right)\left(\frac{\lambda}{u}\right)^{\lambda+1}$$

(where $\lambda$ is a positive integer and $u$ a real positive number) to $\mathscr{F}(x)$ and obtain an inversion theorem.

LEMMA. *If* $\phi(t) \in L_p$ *in* $(0 \leqq t < \infty)$ *and*

$$\psi(u) = \int_0^\infty |\phi(ut) - \phi(t)|^p dt$$

*then*

$$\text{(i)}\qquad \left|\frac{u\psi(u)}{1+u}\right| \leqq \|\phi\|_p^p \ \textit{for}\ u \geqq 0$$

*and*

$$\text{(ii)}\qquad \psi(u) \to 0 \ \textit{as}\ u \to 1$$

*where* $\|\mathscr{F}\|_p$ *denotes the norm of the function* $\mathscr{F}(t) \in L_p(0,\infty)$, *that is*

$$\|\mathscr{F}\|_p = \left\{\int_0^\infty |\mathscr{F}(t)|^p dt\right\}^{(1/p)} .$$

*Proof*. We have

$$|\psi(u)| \leqq \int_0^\infty |\phi(ut)|^p dt + \int_0^\infty |\phi(t)|^p dt = \left(1 + \frac{1}{u}\right)\int_0^\infty |\phi(t)|^p dt$$

which proves (i).

Also, by a change of variable,

$$\psi(e^y) = \int_{-\infty}^{\infty} |\phi(e^{x+y}) - \phi(e^x)|^p e^x dx \ .$$

If $\alpha(x) = e^{(x/p)}\phi(e^x)$ then

$$\int_{-\infty}^{\infty} |\alpha(x)|^p dx = \int_{-\infty}^{\infty} |\phi(e^x)|^p e^x dx = ||\phi||_p^p$$

and so $\alpha(x) \in L_p(-\infty, \infty)$. Again

$$\begin{aligned}\{\psi(e^y)\}^{1/p} &= \Big[\int_{-\infty}^{\infty} |\{\alpha(x+y)e^{-(y/p)} - \alpha(x)e^{-(y/p)}\} \\ &\quad + \{\alpha(x)e^{-(y/p)} - \alpha(x)\}|^p dx\Big]^{1/p} \\ &\leqq e^{-(y/p)}\Big[\int_{-\infty}^{\infty} |\alpha(x+y) - \alpha(x)|^p dx\Big]^{1/p} \\ &\quad + |e^{-(y/p)} - 1|\Big[\int_{-\infty}^{\infty} |\alpha(x)|^p dx\Big]^{1/p}\end{aligned}$$

by Minkowski's inequality. And $\int_{-\infty}^{\infty} |\alpha(x+y) - \alpha(x)|^p dx \to 0$ as $y \to 0$ if $\alpha(x) \in L_p(-\infty, \infty)$ and so does $|e^{-y/p} - 1|$. Therefore $\psi(e^y) = o\ (1)$ as $y \to 0$ or $\psi(u) \to 0$ as $u \to 1$.

THEOREM 3. *Assume $\phi(t) \in L_p$ $(1 \leqq p < \infty)$ in $0 \leqq t \leqq R$ for every positive $R$. If the integral $\mathscr{F}(x)$ converges for $x > 0$ and $2m > -1/q$ when $(1/2) - m - k > 0$; $(1/2) + m - k > -1/q$ when $(1/2) - m - k < 0$, then, for almost all positive $t$,*

$$\operatorname*{l.i.m}_{\lambda\to\infty}^{\text{index } p}\ L_{\lambda,t}[K^{-}_{(1/2)+m-k,-(1/2)+m+k}\{F(x)\}] = \phi(t)\ .$$

*Proof.* We have seen in the proof of Theorem 1 that, under the conditions of the theorem,

$$K^{-}_{(1/2)+m-k,-(1/2)+m+k}\{F(x)\} = \mathscr{F}(x)\ .$$

Therefore

$$\begin{aligned}L_{\lambda,t} &\equiv L_{\lambda,t}[K^{-}_{(1/2)+m-k,-(1/2)+m+k}\{F(x)\}] \\ &= \frac{1}{\lambda\,!}\left(\frac{\lambda}{t}\right)^{\lambda+1}\int_0^{\infty} e^{-(\lambda u/t)} u^{\lambda}\phi(u)du\end{aligned}$$

by simple computation and

$$\begin{aligned}|L_{\lambda,t} - \phi(t)| &\leqq \frac{1}{\lambda\,!}\left(\frac{\lambda}{t}\right)^{\lambda+1}\int_0^{\infty} e^{-(\lambda u/t)} u^{\lambda}|\phi(u) - \phi(t)|du \\ &= \frac{1}{\lambda\,!}\lambda^{\lambda+1}\int_0^{\infty} e^{-\lambda v}v^{\lambda}|\phi(vt) - \phi(t)|dv\ .\end{aligned}$$

Therefore

$$|L_{\lambda,t} - \phi(t)|^p \leqq \left|\frac{\lambda^{\lambda+1}}{\lambda\,!}\int_0^\infty e^{-\lambda v}v^\lambda|\phi(vt) - \phi(t)|dv\right|^p$$
$$\leqq \left[\frac{\lambda^{\lambda+1}}{\lambda\,!}\int_0^\infty e^{-\lambda v}v^\lambda|\phi(vt) - \phi(t)|^p dv\right]\left[\frac{\lambda^{\lambda+1}}{\lambda\,!}\int_0^\infty e^{-\lambda v}v^\lambda dv\right]^{p/q}$$
$$\frac{\lambda^{\lambda+1}}{\lambda\,!}\int_0^\infty e^{-\lambda v}v^\lambda|\phi(vt) - \phi(t)|^p dv\ .$$

Hence

$$\int_0^\infty |L_{\lambda,t} - \phi(t)|^p dt \leqq \frac{\lambda^{\lambda+1}}{\lambda\,!}\int_0^\infty dt\int_0^\infty e^{-\lambda v}v^\lambda|\phi(vt) - \phi(t)|^p dv$$
$$= \frac{\lambda^{\lambda+1}}{\lambda\,!}\int_0^\infty e^{-\lambda v}v^\lambda dv\left\{\int_0^\infty |\phi(vt) - \phi(t)|^p dt\right\}\ .$$

In changing the order of integration, this becomes

$$\frac{\lambda^{\lambda+1}}{\lambda\,!}\int_0^\infty e^{-\lambda v}v^\lambda \psi(v)dv \tag{6}$$

where $\psi(v)$ is defined as in the lemma. From the lemma it is easily seen that

$$\psi(u) = 0(1) \qquad (u \to \infty)$$
$$= 0(u^{-1}) \quad (u \to 0+)\ .$$

Therefore $\int_0^\infty e^{-\lambda v}v^\lambda\psi(v)dv$ converges for $\lambda \geqq 1$ and the inversion of the order of integration is justified by Fubini's theorem. By a familiar result [9, Theorem 3c, p. 283] the integral (6) approaches $\psi(1)$ as $\lambda \to \infty$. But, by the lemma, $\psi(u) = o(1)$ as $u \to 1$. Therefore $L_{\lambda,t}$ converges in mean to $\phi(t)$ with index $p$ on $0 \leqq t < \infty$ and the result is proved.

THEOREM 4. *The necessary and sufficient conditions for a function* $F(x)$ *to have the representation* (2) *with* $\phi(t) \in L_p(0, \infty)$, $p \geqq 1$, $x > 1$, *and with* $2m > -1/q$ *when* $1/2 - m - K > 0$ *and* $m - k + 1/2 > -1/q$ *when* $1/2 - m - k < 0$ *are*

(i) $K^{-}_{1/2+m-K,-1/2+m+K}\{F(x)\} \equiv G(x)$ *exists, has derivatives of all orders in* $0 < x < \infty$ *and vanishes at infinity and*

(ii) *there exist constants* $M$ *and* $p$ $(p \geqq 1)$ *such that*

$$\int_0^\infty |L_{\lambda,t}[G(x)]|^p dt < M \qquad (\lambda = 1, 2, \cdots)\ .$$

*Proof.* First let $F(x)$ have the representation (2). Then, from Theorem 1,

$$G(x) \equiv K^{-}_{1/2+m-k,\,-1/2+m+k}\{F(x)\} = \mathscr{F}(x)$$

and as in the proof of Widder [9, Theorem 15a, pp. 313–14] we see that the conditions are satisfied.

Conversely, let the conditions be satisfied. Then again, as in the proof of Widder's theorem referred to before, we see that

$$G(x) = \int_0^\infty e^{-xt}\phi(t)dt = \mathscr{F}(x) .$$

Therefore [3, p. 300]

$$F(x) = (K^{-}_{(1/2)+m-k,\,-(1/2)+m+k})^{-1}\mathscr{F}(x) = K^{-}_{2m,1/2-m-k}\{\mathscr{F}(x)\}$$
$$= \int_0^\infty (xt)^{m-1/2}e^{-xt/2}W_{K,m}(xt)\phi(t)dt$$

by Theorem 1; and the theorem is proved.

COROLLARY. *If the fractional derivatives or integrals*

$$K^{-}_{(1/2)+m-k+r,\,-(1/2)+m+k-r}\{F(x)\}$$

*exist for* $r = 0$ *and every positive integer, then the integral in the condition* (ii) *of Theorem* 4 *can be replaced by*

$$\int_0^\infty \left| \frac{(-1)^\lambda}{\lambda !}\left(\frac{\lambda}{t}\right)\sum_{r=0}^{\lambda}(-1)^r A_r K^{-}_{(1/2)+m-k+r,\,(1/2)+m+k-r}\left\{F\left(\frac{\lambda}{t}\right)\right\}\right|^p dt$$

*where*

$$A_r = {}^\lambda C_r(m-k+(1/2))(m-k-(1/2))\cdots(m-k-\lambda+(3/2)+r)$$
$$(r = 0, 1, \cdots, \lambda-1), \quad A_\lambda = 1 .$$

For [6]

$$t^a K^{-}_{\zeta,\alpha}\{\mathscr{F}(t)\} = K^{-}_{\zeta+a\,\alpha}\{t^a\mathscr{F}(t) .$$

Therefore

$$K^{-}_{\zeta,\alpha}\{F(x)\} = x^\zeta K^{-}_{0,\alpha}\{x^{-\zeta}F(x)\}$$

and

$$\frac{d^\lambda}{dx^\lambda}\left[K^{-}_{\zeta,\alpha}\{F(x)\}\right] = \frac{d^\lambda}{dx^\lambda}(x^\zeta)\left[K^{-}_{0,\alpha}\{x^{-\zeta}F(x)\}\right]$$
$$+ {}^\lambda C_1\frac{d^{\lambda-1}}{dx^{\lambda-1}}(x^\zeta)\frac{d}{dx}\left[K^{-}_{0,\alpha}\{x^{-\zeta}F(x)\}\right] + \cdots$$
$$+ {}^\lambda C_{\lambda-1}\frac{d}{dx}(x^\zeta)\frac{d^{\lambda-1}}{dx^{\lambda-1}}\left[K^{-}_{0,\alpha}\{x^{-\zeta}F(x)\}\right]$$
$$+ x^\zeta\frac{d^\lambda}{dx^\lambda}\left[K^{-}_{0,\alpha}\{x^{-\zeta}F(x)\}\right] .$$

By Leibnitz's theorem this becomes

$$
\begin{aligned}
&= \zeta(\zeta-1)\cdots(\zeta-\lambda+1)x^{\zeta-\lambda}[K^{-}_{0,\alpha}\{x^{-\zeta}F(x)\}] \\
&- {}^{\lambda}C_1\zeta(\zeta-1)\cdots(\zeta-\lambda+2)x^{\zeta-\lambda+1}[K^{-}_{0,\alpha-1}\{x^{-\zeta-1}F(x)\}] \\
&+ \cdots + (-1)^{\lambda}x^{\zeta}[K^{-}_{0,\alpha-\lambda}\{x^{-\zeta-\lambda}F(x)] .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{(-1)^{\lambda}}{\lambda^1} x^{\lambda+1}\frac{d^{\lambda}}{dx^{\lambda}}\Big[K^{-}_{\zeta,\alpha}\{F(x)\}\Big] \\
&= \frac{(-1)^{\lambda}}{\lambda\,!}\sum_{r=0}^{\lambda}(-1)^r A_r x^{\zeta+r+1}\Big[K^{-}_{0,\alpha-r}\{x^{-\zeta-r}F(x)\}\Big]
\end{aligned}
$$

where

$$
\begin{aligned}
&A_r = {}^{\lambda}C_r\zeta(\zeta-1)\cdots(\zeta-\lambda+r+1) \\
&A_{\lambda} = 1, \qquad\qquad (r = 0, 1, \cdots, \lambda-1) ,
\end{aligned}
$$

and

$$
\begin{aligned}
L_{\lambda,t}\Big[K^{-}_{\zeta,\alpha}\{F(x)\}\Big] &= \frac{(-1)^{\lambda}}{\lambda\,!}\sum_{r=0}^{\lambda}(-1)^r A_r\Big(\frac{\lambda}{t}\Big)^{\zeta+r+1}\Big[K^{-}_{0,\alpha-r}\Big\{\Big(\frac{\lambda}{t}\Big)^{-\zeta-r}F\Big(\frac{\lambda}{t}\Big)\Big\}\Big] \\
&= \frac{(-1)^{\lambda}}{\lambda\,!}\Big(\frac{\lambda}{t}\Big)\sum_{r=0}^{\lambda}(-1)^r A_r\Big[K^{-}_{\zeta+r,\alpha-r}\Big\{F\Big(\frac{\lambda}{t}\Big)\Big\}\Big] .
\end{aligned}
$$

Putting $\zeta = m - k + 1/2$ and $\alpha = m + k - 1/2$ we have the required result.

THEOREM 5a. *If $F(x)$ has representation (2) with the conditions of Theorem 4 on $\phi(t)$, $x$, $k$ and $m$ satisfied and if the fractional derivatives or integrals $K^{-}_{(1/2)+m-k+r,-(1/2)+m+k-r}\{F(x)\}$ exist for $r = 0$ and every positive integer, than*

$$
\lim_{\lambda\to\infty}\int_0^{\infty}\left|\frac{(-1)^{\lambda}}{\lambda\,!}\Big(\frac{\lambda}{t}\Big)\sum_{r=0}^{\lambda}(-1)^r A_r\Big[K^{-}_{(1/2)+m-k+r,(-1/2)+m+k-r}\Big\{F\Big(\frac{\lambda}{t}\Big)\Big\}\Big]\right|^p dt = \Big\|\phi\Big\|_p^p .
$$

*where the $A_r$ 's have values as in the Corollary to Theorem 4.*

*Proof.* The proof is similar to that of Widder [9, Theorem 15b, p. 314]

THEOREM 5b. *If the function $F(x)$ has representation (2) with the conditions of Theorem 4 on $\phi(t)$, $x$, $k$ and $m$ satisfied, then*

$$
\lim_{\lambda\to\infty}\int_0^{\infty}|L_{\lambda,t}\{F(x)\}^p dt = \int_0^{\infty}|I^{+}_{2m,(1/2)-m-k}\{\phi(t)\}|^p dt .
$$

*Proof.* If $F(x)$ has the representation (2), then, by Theorem 2 we have

$$F(x) = \int_0^\infty e^{-xt} I^+_{2m,(1/2)-m-k}\{\phi(t)\} dt \; .$$

Also if $\phi(t) \varepsilon L_p(0, \infty)$ so does $I^+_{2m,(1/2)-m-k}\{\phi(t)\}$ provided that $2m > -1/q$. Therefore, as in Widder [9, Theorem 15b, p. 314], we can prove again that

$$\lim_{\lambda\to\infty} \int_0^\infty | L_{\lambda,t}\{F(x)\}|^p dt = \int^\infty | I^+_{2m,(1/2)-m-k}\{\phi(t)\}^p dt \; .$$

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