EXTREME POINTS AND EXTREMUM PROBLEMS IN H_1

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The class H_1 consists of all functions f which are analytic in the open unit disc, and for which

$$||f|| = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta$$

is finite. With this norm, H_1 is a Banach space, whose unit sphere will be denoted by S; that is, S is the set of all $f \in H_1$ with $||f|| \leqq 1$.

We are concerned in this paper with (a) the identification of the extreme points of S and some geometric properties of the set of these extreme points, (b) the closure of Pf (the set of all functions of the form $p \cdot f$, where p ranges over the polynomials and f is a fixed function in H_1) in various topologies, and (c) the structure of the set of those $f \in S$ which maximize a given bounded linear functional on H_1.

We find that the factorization $f = M_f Q_f$ (see Lemma 1.3), which was apparently first used by Beurling [1], is of basic importance in these problems.

Our results are summarized at the beginning of Sections II, III, and IV.

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I. PRELIMINARIES

1.1 Let C be the boundary of the open unit disc U in the complex plane. If $f \in H_1$, then $f(e^{i\theta})$, which we define to be $\lim_{r \to 1} f(re^{i\theta})$, exists almost everywhere on C and differs from 0 for almost all $e^{i\theta}$, unless f is identically 0. Moreover, the one-to-one correspondence between an $f \in H_1$ and its boundary function is an isometric embedding of H_1 in L_1, the Banach space of all Lebesgue integrable functions on C, normed by

$$||f|| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta \, . \tag{1.1.1}$$

Thus (1.1.1) may be taken as the norm in H_1. We also have

$$\lim_{r \to 1} \int_{-\pi}^{\pi} |f(re^{i\theta}) - f(e^{i\theta})| d\theta = 0 \tag{1.1.2}$$

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for every $f \in H_1$, and

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(e^{i\varphi})}{1-e^{-i\varphi}z} d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})P_r(\theta - \varphi)d\varphi \ , \tag{1.1.3}$$

where $z = re^{i\theta}$ and the Poisson kernel is defined by

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} = \Re\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right] . \tag{1.1.4}$$

For proofs of these facts we refer to [6] and [10; Section 7.5].

1.2. Inner functions and outer functions. A *Blaschke product* is a function of the form

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{a_n - z}{1-\bar{a}_n z} \cdot \frac{|a_n|}{a_n} \qquad (z \in U) \tag{1.2.1}$$

where m is a non-negative integer, $0<|a_n|<1$, and $\sum(1-|a_n|)<\infty$. The set $\{a_n\}$ may be finite, or even empty. If $\{a_n\}$ is finite, we call B a *finite Blaschke product*.

A function of the form

$$M(z) = B(z) \exp\left\{- \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right\} \qquad (z \in U) \ , \tag{1.2.2}$$

where B is a Blaschke product and μ is a non-negative singular (with respect to Lebesge measuree) measure on C, is called an *inner function* [1]. A function f, analytic in U, is an inner function if and only if f is bounded in U, f has radial limits of modulus 1 almost everywhere on C, and the first non-zero Taylor coefficient of f is positive [9].

An *outer function* [1] is a function of the form

$$Q(z) = c \cdot \exp\left\{\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} h(\theta)\, d\theta\right\} \qquad (z \in U) \ , \tag{1.2.3}$$

where $c \neq 0$ is a constant and $h \in L_1$.

The following factorization is crucial for what follows (see [1] and [9]).

1.3. LEMMA. *Each $f \in H_1$ (except $f = 0$) has a unique factorization of the form $f = M_f Q_f$, where M_f is an inner function and Q_f is an outer function; there is a real α such that*

$$Q_f(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})|d\theta + i\alpha\right\} \qquad (z \in U); \tag{1.3.1}$$

also, $Q_f \in H_1$, and $\|Q_f\| = \|f\|$.

It is known [1] that $f=Q_f$ if and only if

$$\log|f(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{i\theta})|d\theta \ . \tag{1.3.2}$$

Indeed, if $f=Q_f$ then (1.3.1) leads immediately to (1.3.2); on the other hand, if $M_f \neq 1$, then the left member of (1.3.2) is less than the right member.

1.4. LEMMA. *If $f \in H_1$, either of the following two conditions implies that $f=Q_f$:*

(i) $1/f \in H_1$

(ii) $\Re[f(z)]>0$ *for all* $z \in U$.

Proof. If $g=1/f$ and $g \in H_1$, then $1=fg=M_fM_gQ_fQ_g$. By Lemma 1.3, the factorization of 1 is unique, so that $Q_fQ_g=M_fM_g=1$. This implies $M_f=1$, so $f=Q_f$.

If $\Re[f(z)>0]$ define $f_\varepsilon(z)=f(z)+\varepsilon$ for $z \in U$. Then $1/f_\varepsilon$ is bounded and by (i) we have

$$f_\varepsilon(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f_\varepsilon(e^{i\theta})|d\theta + i\arg f_\varepsilon(0)\right\} \qquad (z \in U) \ . \tag{1.4.1}$$

As $\varepsilon \to 0$, the integrable functions $\log|f_\varepsilon|$ decrease monotonically to $\log|f|$; by the Lebesgue convergence theorem, (1.4.1) thus remains valid if f_ε is replaced by f, so that $f=Q_f$.

1.5. Some topologies on H_1. Besides the norm topology, we shall be most concerned with the weak* topology, which may be described as follows: The space L_1 can be isometrically embedded in M, the space of bounded Borel measures on C; M is the dual space of the space of all continuous functions on C; and H_1 is a closed subspace of M, in the weak* topology of M. The restriction of this topology to H_1 will be called the weak* topology of H_1. Since M is a dual space, its unit sphere is weak*-compact. Hence S, the unit sphere of H_1, is weak*-compact. The fact that the space of all continuous functions on C is separable implies that S is metrizable in the weak* topology. Thus, when discussing weak* convergence in S, it suffices to consider simple countable sequences.

There is also the weak topology of H_1, i.e., the weakest topology in which all bounded linear functionals on H_1 are continuous. The weak topology is actually stronger than the weak* topology: S is weak*-compact, but S is not compact in the weak topology [8; p. 54].

The following lemma describes the weak* topology on S in a manner which will be useful to us.

1.6. LEMMA. *Suppose $f_n \in S(n=1, 2, 3, \cdots)$. Each of the following* *four properties implies the other three:*

(i) $f_n \to f$ *in the weak* topology of* H_1.

(ii) $f_n(z) \to f(z)$ *or every* $z \in U$.

(iii) $f_n(z) \to f(z)$ *uniformly on all compact subsets of* U.

(iv) $\lim_{n\to\infty} a_{n,k} = a_k$ *for* $k = 0, 1, 2, \cdots$, *where*

$$f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k,\ f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (z \in U)\,.$$

Proof. (i) means, by definition, that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f_n(e^{i\theta})\phi(e^{i\theta})\,d\theta = \int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta})\,d\theta \tag{1.6.1}$$

for every continuous function ϕ on C. Since, for every $z \in U$, the function $\phi(e^{i\theta}) = (1 - e^{-i\theta}z)^{-1}$ is of this type, (1.1.3) shows that (i) implies (ii).

Since the functions $\{f_n\}$ are bounded in norm, they are uniformly bounded on every compact subset of U [8 ; p. 51] and hence form a normal family, so that (ii) implies (iii).

That (iii) implies (iv) follows immediately from the Cauchy integral formulae for the derivatives of f_n at the origin.

Finally, if (iv) holds, then (1.6.1) holds whenever ϕ is a trigonometric polynomial. Since every continuous function can be uniformly approximated on C by trigonometric polynomials, the boundedness of $\{\|f_n\|\}$ implies that (1.6.1) holds for every continuous ϕ. Thus (iv) implies (i).

II. THE EXTREME POINTS OF S.

2.1. An element f **of** S **is called** an *extreme point* of S if f is not an interior point of any line segment that lies in S. Since S is weak*-compact and convex, the Krein-Milman theorem [2 ; p. 84] guarantees the existence of extreme points. However, the following more detailed information will be established, without use of the Krein-Milman theorem.

THEOREM 1. *A function* $f \in H_1$ *is an extreme point of* S *if and only if* $\|f\| = 1$ *and* $f = Q_f$.

THEOREM 2. (a) *If* $\|f\| = 1$ *and* f *is not an extreme point of* S, *then there exist extreme points* f_1 *and* f_2 *such that* $f_1 + f_2 = 2f$. (b) *If* $\|f\| < 1$, *then* f *is a convex combination of some two extreme points of* S.

THEOREM 3. *A function* $f \in H_1$ *lies in the norm closure of the set of all extreme points of* S *if and only if* $\|f\| = 1$ *and* $f(z) \neq 0$ *for all* $z \in U$.

THEOREM 4. *A function* $f \in H_1$ *lies in the weak*-closure of the set* *of all extreme points of S if and only if $f \in S$ and $f(z) \neq 0$ for all $z \in U$, or if f is identically* 0.

These results should be contrasted with the easily established fact that the unit sphere of L_1 has no extreme points at all, and that in the unit spheres of the H_p-spaces (for $1<p<\infty$) every boundary point is an extreme point. The extreme points of the unit spheres of the space H_∞ of all bounded analytic functions in U and of the subspace of all uniformly continuous functions have recently been determined (see § V).

2.2. **For convenience, we shall now display** some relations which furnish the key to several parts of our paper.

Suppose $f \in H_1$, $f = M_f Q_f$, and $M_f \neq 1$. Choose a real α such that

$$\int_{-\pi}^{\pi} |f(e^{i\theta})| \Re[e^{i\alpha} M_f(e^{i\theta})] d\theta = 0 \; ; \tag{2.2.1}$$

this can be done, since the left member of (2.2.1) is a real continuous function of α which changes sign on the interval $[0, \pi]$. Put

$$u(z) = e^{i\alpha} M_f(z) \qquad (z \in U) \tag{2.2.2}$$

and

$$g(z) = \frac{1}{2} e^{-i\alpha} Q_f(z)(1 + u^2(z)) \qquad (z \in U) \; . \tag{2.2.3}$$

Then $g \in H_1$ and $g \neq 0$. Note that $e^{-i\alpha} Q_f = f/u$, that $|u(e^{i\theta})|=1$ a.e. on C, and that

$$2\Re[u] = u + \bar{u} = u + \frac{1}{u} = \frac{1+u^2}{u} \tag{2.2.4}$$

whenever $|u| = 1$. These facts imply

$$g(e^{i\theta}) = f(e^{i\theta}) \Re[u(e^{i\theta})] \qquad (\text{a.e. on } C) \; , \tag{2.2.5}$$

so that

$$|f(e^{i\theta}) \pm g(e^{i\theta})| = |f(e^{i\theta})|(1 + \Re[u(e^{i\theta})]) \qquad (\text{a.e. on } C) \; . \tag{2.2.6}$$

By (2.2.1) we have, therefore,

$$||f + g|| = ||f - g|| = ||f|| \; . \tag{2.2.7}$$

Suppose next that λ is a real number, satisfying $\lambda \geqq 1$. Then there exists a real β such that

$$f \pm \lambda g = \frac{1}{2} e^{-i\alpha} Q_f \cdot (\pm \lambda u^2 + 2u \pm \lambda) = \pm \frac{\lambda}{2} e^{-i\alpha} Q_f \cdot (1 \pm e^{i\beta} u)(1 \pm e^{-i\beta} u), \tag{2.2.8}$$

Lemma 1.4 (ii) shows that each of the last two factors is an outer function.

We conclude that $f+\lambda g$ and $f-\lambda g$ are outer functions, if $\lambda \geqq 1$.

2.3. *Proof of Theorem* 1. Suppose $\|f\|=1$ and $f=Q_f$. To prove that f is an extreme point, it evidently suffices to show that the conditions

$$\|f+h\| = \|f-h\| = 1 \,, \tag{2.3.1}$$

where $h \in H_1$, imply $h=0$.

Let us assume that h is not identically 0 and that (2.3.1) holds. Define

$$k(z) = h(z)/f(z) \qquad (z \in U) \tag{2.3.2}$$

and let $k(e^{i\theta})$ denote the boundary values of k, which exist a.e. on C. By (2.3.1) we have

$$\int_{-\pi}^{\pi} \{|1+k(e^{i\theta})| + |(1-k(e^{i\theta})| - 2\}|f(e^{i\theta})|d\theta = 0 \,. \tag{2.3.3}$$

Since $f(e^{i\theta}) \neq 0$ a.e. on C, (2.3.3) implies that $k(e^{i\theta})$ is real a.e. on C, and in fact that

$$-1 \leqq k(e^{i\theta}) \leqq 1 \qquad \text{(a.e. on } C) \,. \tag{2.3.4}$$

Thus $\log|h(e^{i\theta})| \leqq \log|f(e^{i\theta})|$ a.e. on C, and the factorization $k = M_h Q_h/Q_f$, combined with (1.3.1), shows that k is bounded in U. Having real boundary values a.e. on C, k must therefore be constant. But (2.3.1) then implies that $(1+k)\|f\| = (1-k)\|f\|$, so that $k=0$, and therefore $h=0$. Consequently, f is an extreme point of S.

It is clear that $\|f\|=1$ if f is an extreme point of S. To prove the converse, let us therefore assume that $\|f\|=1$ and $f \neq Q_f$. If g is then defined as in § 2.2, we see from (2.2.7) that $f+g \in S$ and $f-g \in S$, so that f is not an extreme point of S. Theorem 1 is thus proved.

2.4. *Proof of Theorem* 2. Suppose $\|f\|=1$ and f is not an extreme point of S. By Theorem 1, f is not an outer function; define g as in § 2.2, and put $f_1 = f+g$, $f_2 = f-g$. By (2.2.8), f_1 and f_2 are outer functions, (2.2.7) shows that $\|f_1\| = \|f_2\| = 1$, so that f_1 and f_2 are extreme points of S and $2f = f_1 = f_2$.

Suppose next that $\|f\| = t$, with $0 < t < 1$ (the case $t=0$ is trivial). If f is an outer function, so are the functions $f_1 = f/t$ and $f_2 = -f_1$, and f is clearly on the segment bounded by f_1 and f_2, that is, by extreme points of S. If f is not an outer function, define g as in § 2.2, and choose $\lambda_1 > 1$ and $\lambda_2 > 1$ such that

$$\|f+\lambda_1 g\| = \|f-\lambda_2 g\| = 1 \ . \tag{2.4.1}$$

By (2.2.8), $f+\lambda_1 g$ and $f-\lambda_2 g$ are outer functions, hence extreme points of S, and f lies on the segment bounded by them.

This completes the proof of Theorem 2.

2.5. *Proof of Theorem* 3. Suppose

$$\lim_{n\to\infty} \|f_n - f\| = 0 \ , \tag{2.5.1}$$

where $\{f_n\}$ is a sequence of extreme points of S. Then $\|f\| = 1$, and since $f_n \to f$ uniformly on compact subsets of U and no f_n has a zero in U, we conclude that either f has no zero in U or f is identically 0. The last alternative contradicts the fact that $\|f\| = 1$.

To establish the converse, suppose $\|f\| = 1$ and f has no zero in U. For each $r < 1$, define $f_r(z) = f(rz)$ for $z \in U$, and put $g_r = f_r/\|f_r\|$. Then g_r is bounded away from 0, and is therefore an outer function, by Lemma 1.4, and an extreme point of S, by Theorem 1. Since

$$f - g_r = f_r\{1 - 1/\|f_r\|\} + (f - f_r) \ , \tag{2.5.2}$$

and since $\|f_r\| \to 1$ as $r \to 1$ (see (1.1.2)), we have

$$\lim_{r\to 1} \|f - g_r\| = 0 \ . \tag{2.5.3}$$

This proves Theorem 3.

2.6. *Proof of Theorem* 4. Suppose $\{f_n\}$ is a sequence of extreme points of S and $f_n \to f$ in the weak* topology. Since S is weak*-compact, $f \in S$. By Lemma 1.6, $f_n \to f$ uniformly on compact subsets of U, so that f either has no zero in U or f is identically 0.

To establish the converse, suppose $\|f\| = 1$, f has no zero in U, and λ is a real number satisfying $0 \leqq \lambda < 1$. Choose θ such that

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) \tag{2.6.1}$$

exists and is not 0 and define

$$h(z) = \frac{f(z)}{z - e^{i\theta}} \qquad (z \in U) \ . \tag{2.6.2}$$

If h were in H_1, then the Poisson integral representation (1.1.3) of h would lead to the relation

$$\lim_{r\to 1} (1-r)h(re^{i\theta}) = 0 \ , \tag{2.6.3}$$

so that $f(e^{i\theta}) = 0$, a contradiction. Thus $h \notin H_1$.

For each $a>1$, define

$$f_a(z) = f(z)\left\{\lambda - \frac{\varepsilon_a e^{i\theta}}{z-ae^{i\theta}}\right\} \qquad (z \in U), \tag{2.6.4}$$

where ε_a is chosen positive and so that $\|f_a\| = 1$. Note that the second factor in (2.6.4) is 0 only when $z=(a+\varepsilon_a/\lambda)e^{i\theta}$, and this number is not in U. Thus (by Theorem 3) each f_a is in the norm closure, and hence in the weak*-closure, of the extreme points of S.

Since $h \notin H_1$, $\varepsilon_a \to 0$ as $a \to 1$, so that $f_a(z) \to \lambda f(z)$ as $a \to 1$, for all $z \in U$. By Lemma 1.6, this implies that λf is in the weak*-closure of the extreme points of S. Since this is true for all $f \in H_1$ with $\|f\|=1$ and without zeroes in U, and for all λ with $0 \leqq \lambda < 1$, it is also true for $\lambda = 1$, and the theorem is proved.

III. THE SETS *Pf*.

3.1. **For any $f \in H_1$ the collection** of all functions g of the form

$$g(z) = p(z)f(z) \qquad (z \in U), \tag{3.1.1}$$

where p is a polynomial, will be denoted by Pf. In other words, Pf is the linear subspace of H_1 which is generated by the functions $z^n f(z)(n = 0, 1, 2, \cdots)$. We are concerned with finding conditions on f under which Pf is dense in H_1.

THEOREM 5. *If $f \in H_1$ and $f \neq 0$ the following three statements are equivalent:*

(i) $f = Q_f$.

(ii) *Pf is dense in H_1 in the norm topology.*

(iii) *Pf is dense in H_1 in the weak* topology.*

The corresponding problem for H_2 in the norm topology was solved by Beurling [1]; here too a necessary and sufficient condition is that $f = Q_f$.

If $\|f\| = 1$, Theorems 1 and 5 imply that Pf is dense in H_1 in either of these topologies (and hence in any intermediate one) if and only if f is an extreme point of S. One would like to have a direct proof of this equivalence (i.e., a proof not involving the Poisson integral and the factorization $f=M_fQ_f$), but we have been unable to find such a proof. Indeed, there may not exist one, since the analogous statement is false in H_2, where every f with $\|f\| = 1$ is an extreme point of the unit sphere.

3.2. *Proof of Theorem* 5. Suppose $f=Q_f$, and suppose ϕ is a bounded measurable function on C such that

(3.2.1)
$$\int_{-\pi}^{\pi} p(e^{i\theta})f(e^{i\theta})\phi(e^{i\theta})d\theta = 0$$

for all polynomials p. By the Theorem of F. and M. Riesz [10 ; p. 158] there is a function $h \in H_1$ such that $h(0) = 0$ and

(3.2.2.)
$$f(e^{i\theta})\phi(e^{i\theta}) = h(e^{i\theta}) \qquad \text{(a.e. on } C) .$$

The function g defined by

(3.2.3.)
$$g = \frac{h}{f} = M_h \frac{Q_h}{Q_f}$$

is analytic in U and has radial boundary values equal to $\phi(e^{i\theta})$ a.e. on C. By (3.2.2), $\log |h(e^{i\theta})| - \log |f(e^{i\theta})$ is bounded above on C, so that (3.2.3), combined with (1.3.1), implies that g is bounded in U. Also, $g(0) = 0$ since $h(0) = 0$.

Consequently, for any $k \in H_1$ we have

(3.2.4)
$$\int_{-\pi}^{\pi} k(e^{i\theta})\phi(e^{i\theta})\,d\theta = \int_{-\pi}^{\pi} k(e^{i\theta})g(e^{i\theta})d\theta = 0 .$$

In other words, every bounded linear functional on H_1 which annihilates Pf also annihilates H_1, so that Pf is dense in H_1, in the norm topology. Thus (i) implies (ii).

It is trivial that (ii) implies (iii).

Suppose next that Pf is weak*-dense in H_1 but that $f \neq Q_f$. By Lemma 1.6, f cannot have a zero in U. Hence

(3.2.5)
$$M_f(z) = \exp\left\{ - \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right\} \qquad (z \in U)$$

for some positive singular measure μ. There is a closed subset E of C with $\mu(E)>0$, whose Lebesgue measure is 0. Define

(3.2.6)
$$M_1(z) = \exp\left\{ - \int_E \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right\}$$

and $M_2(z) = M_f(z)/M_1(z)$. Then $f = M_1 M_2 Q_f$.

There exists an outer function Q_1 which is uniformly continuous in U, such that $Q_1(e^{i\theta}) = 0$ for every $e^{i\theta} \in E$ (compare [9 ; p. 433]). Define

(3.2.7)
$$\phi(e^{i\theta}) = \overline{M_1(e^{i\theta})} \cdot Q_1(e^{i\theta})e^{i\theta} .$$

Then ϕ is continuous on C ; since $f = M_1 M_2 Q_f$, we have $(pf\overline{M}_1 Q_1)(e^{i\theta}) = (pM_2 Q_f Q_1)(e^{i\theta})$ a.e. on C, and since $pM_2 Q_f Q_1 \in H_1$,

$$\int_{-\pi}^{\pi} p(e^{i\theta})f(e^{i\theta})\phi(e^{i\theta})d\theta = 0 \tag{3.2.8}$$

for every polynomial p. Since Pf is weak*-dense in H_1 and ϕ is continuous, (3.2.8) implies

$$\int_{-\pi}^{\pi} h(e^{i\theta})\phi(e^{i\theta})d\theta = 0 \tag{3.2.9}$$

for every $h \in H_1$. By (3.2.7), it follows that there is a bounded analytic function g in U such that

$$g(e^{i\theta}) = \overline{M_1(e^{i\theta})}Q_1(e^{i\theta}) \qquad \text{(a.e. on } C). \tag{3.2.10}$$

Thus $Q_1 = M_1 g$. Since Q_1 is an outer function and M_1 is a non-constant inner function, we have arrived at a contradiction.

Thus (iii) implies (i), and Theorem 5 is proved.

3.3. **Additional remarks.** We wish to point out that the full analogue of Theorem 1 of [1] is valid in our situation. Since it can be established by the same methods which were used to prove our Theorem 5, we content ourselves with the statement of the result:

THEOREM 6. *For any $f \in H_1$, the closures of Pf in the norm topology and in the weak* topology are identical. Moreover, the closure of Pf contains the closure of Pg if and only if M_g/M_f is bounded in U* (i.e., *if M_f divides M_g*).

Finally, the analogue of Theorem 4 of [1] is also valid in this context. Again we simply state the result, this time since the proof is almost identical with that on p. 432 of [9]:

THEOREM 7. *Each closed linear subspace X of H_1 which is invariant under multiplication by z is the closure of some Pf, where f is an inner function which is uniquely determined by X.*

IV. EXTREMUM PROBLEMS IN H_1

4.1. **We shall now apply some of the material** of Section II to extremum problems in H_1 and will obtain some results which go beyond those of [5] and [7].

If ϕ is a bounded measurable function on C, we shall denote by T_ϕ the functional defined on H_1 by

$$T_\phi(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta})d\theta\ . \tag{4.1.1}$$

T_ϕ is a bounded linear functional and, conversely, every bounded linear functional on H_1 is of the form (4.1.1). The norm of T_ϕ is

$$\|T_\phi\| = \sup_{s\in S} |T_\phi(f)| , \tag{4.1.2}$$

and we let S_ϕ denote the set of all $f \in S$ for which

$$T_\phi(f) = \|T_\phi\| . \tag{4.1.3}$$

The set S_ϕ is simply the set of solutions of the extremum problem: "Maximize $T_\phi(f)$ for f in S," where we restrict ourselves to those f for which $T_\phi(f) \geqq 0$.

If $f \in H_1$ and $\|f\| = 1$, S^f will denote that S_ϕ which contains f. (We will see in 4.2 that every f with $\|f\| = 1$ belongs to one and only one S_ϕ.)

All results of this section deal with the structure of the sets S_ϕ.

Let us note right away that S_ϕ may be empty; the function

$$\phi(e^{i\theta}) = \begin{cases} e^{i\theta} & (0 \leqq \theta < \pi) , \\ 0 & (\pi \leqq \theta < 2\pi) , \end{cases} \tag{4.1.4}$$

leads to an example of this sort (see [7; p. 308]).

We need one more definition before we state our results. An outer function $f \in H_1$ is a *strong outer function* if for every a with $|a| = 1$ the function g_a defined by

$$g_a(z) = (z - a)^{-2} f(z) \qquad (z \in U) \tag{4.1.5}$$

fails to be in H_1.

Our first theorem concerns the question of uniqueness of the solution of an extremum problem of the above type:

THEOREM 8. (a) *If $\|f\| = 1$ and $|f(z)| > \delta$ for all $z \in U$ and some $\delta > 0$, then S^f consists of f alone.*

(b) *If S^f consists of f alone, then f is a strong outer function.*

Unfortunately, the gap between these two conditions seems quite large.

THEOREM 9. *If S_ϕ contains more than one function, then S_ϕ contains infinitely many outer functions, and for every $a \in U$ there is an $f \in S_\phi$ with $f(a) = 0$.*

Lemma 4.6 contains some more information along these lines.

THEOREM 10. *If ϕ is continuous on C, then the following assertions are true:*

(a) *S_ϕ is weak*-compact and not empty.*

(b) *There is a non-negative integer n such that no $f \in S_\phi$ has more* *than n zeros in U; for every $f \in S_\phi$ the function M_f is a finite Blaschke product.*

(c) *There exists a unique strong outer function g with $\|g\| = 1$ and with the following property: for any choice of points $a_1, \cdots, a_n$ in U (where n is the smallest integer for which* (b) *holds) there is a uniqne $f \in S_\phi$ such that $f(a_1) = \cdots = f(a_n) = 0$, and this f is of the form*

$$f(z) = \lambda g(z) \prod_{i=1}^{n} (a - z)(1 - \overline{a}_i z) \qquad (z \in U)\,, \tag{4.1.6}$$

where λ is a positive constant.

If some a appears more than once in the sequence $a_1, \cdots, a_n$, it is of course understood that f is to have a zero of the appropriate multiplicity at a.

Finally, if we strengthen the conditions on ϕ even more, we obtain a complete description of S_ϕ; we do not know whether the conclusion of Theorem 11 holds even if ϕ is merely continuous but not analytic.

THEOREM 11. *If ϕ is analytic in $|z| > R$, for some $R < 1$, there is a non-negative integer n and a unique strong outer function g, such that every $f \in S_\phi$ is of the form*

$$f(z) = \lambda g(z) \prod_{i=1}^{n} (a_i - z)(1 - \overline{a}_i z) \qquad (z \in U)\,, \tag{4.1.7}$$

where $|a_i| \leqq 1$ $(1 \leqq i \leqq n)$ and λ is a positive constant. Conversely, every f of the form (4.1.7) *is in S_ϕ, provided $\|f\| = 1$.*

4.2. Before proceeding to the proofs, we briefly present some of the background material.

Suppose S_ϕ is not empty. The functional T_ϕ can be extended to L_1 in a norm-preserving manner; hence there is a function ψ with $|\psi(e^{i\theta})| \leqq \|T_\phi\|$ a.e. on C, such that for every $f \in S_\phi$

$$\|T_\phi\| = T_\phi(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\psi(e^{i\theta})d\theta \leqq \|f\|\cdot\|T_\phi\| = \|T_\phi\|\,. \tag{4.2.1}$$

Thus the equality sign holds in the above inequality, which implies that $|\psi(e^{i\theta})| = \|T_\phi\|$ a.e. on C and that

$$f(e^{i\theta})\psi(e^{i\theta}) \geqq 0 \qquad \text{(a.e. on } C) \tag{4.2.2}$$

for every $f \in S_\phi$ (this type of argument is the basis of the work in [7]).

Consequently, f and g belong to the same S_ϕ if and only if

$$\arg f(e^{i\theta}) = \arg g(e^{i\theta}) \qquad \text{(a.e. on } C)\,; \tag{4.2.3}$$

of course, we must also have $\|f\| = \|g\| = 1$.

Furthermore, if $\|f\| = 1$ and $\phi(e^{i\theta}) = |f(e^{i\theta})|/f(e^{i\theta})$, then $f \in S_\phi$. It follows that every f with $\|f\| = 1$ belongs to some S_ϕ.

It is obvious that every S_ϕ is convex.

It is well known (and easy to prove) that every function of the form

$$w(z) = \prod_{i=1}^{n} \frac{(z-b_i)(1-\bar{b}_i z)}{(z-a_i)(1-\bar{a}_i z)} \tag{4.2.4}$$

where $|a_i| \leqq 1$ and $|b_i| \leqq 1$, is real and non-negative on C; conversely, if f is meromorphic in $|z| < R$ for some $R > 1$, and if $f(z) \geqq 0$ for all $z \in C$ (except for poles), then f is a positive multiple of some function of the form (4.2.4).

In this connection we state the following lemma which will be used frequently.

4.3. LEMMA. *Suppose* $f \in H_1$, $\|f\| = 1$ *and*

$$f(a_1) = f(a_2) = \cdots = f(a_n) = 0 ,$$

where $a_i \in U$ $(1 \leqq i \leqq n)$. *If* $|b_i| \leqq 1$ $(1 \leqq i \leqq n)$ *then a positive multiple of the function* g *defined by*

$$g(z) = f(z) \prod_{i=1}^{n} \frac{(z-b_i)(1-\bar{b}_i z)}{(z-a_i)(1-\bar{a}_i z)} \qquad (z \in U)$$

is in S^f.

The proof follows immediately from (4.2.3) and the positivity of (4.2.4) on C.

4.4. *Proof of Theorem* 8. If $g \in S^f$, then $g(e^{i\theta})/f(e^{i\theta}) \geqq 0$ a.e. on C, by (4.2.4). If $|f(z)| > \delta$ in U, then $1/f$ is bounded, so that $g/f \in H_1$. Having real boundary values a.e. on C, g/f if therefore constant, and since $\|f\| = \|g\|$, this constant is 1. Thus $g = f$, and part (a) is proved.

Suppose next that S^f consists of f alone. If $f \neq Q_f$, we can choose $g \neq 0$ as in § 2.2, and (2.2.5) implies that

$$\arg\,(f(e^{i\theta}) \pm g(e^{i\theta})) = \arg f(e^{i\theta}) \qquad \text{(a.e. on } C) . \tag{4.4.1}$$

Since $\|f \pm g\| = 1$, by (2.2.7), we see that $f \pm g \in S^f$, so that S^f contains more than one function.

This contradiction shows that f is an outer function. Suppose f is not a strong outer function. Then some g_a defined by (4.1.5) is in H_1, and so is the function h defined by

$$h(z) = \frac{-az}{(z-a)^2} f(z) \qquad (z \in U) . \tag{4.4.2}$$

Since

$$\arg h(e^{i\theta}) = \arg f(e^{i\theta}) \qquad \text{(a.e. on } C) \, , \tag{4.4.3}$$

some constant multiple λh of h is in S^f. But $\lambda h \neq f$, which is again a contradiction.

Thus part (b) is proved.

4.5. *Proof of Theorem* 9. If S_ϕ contains more than one function, then since S_ϕ is convex it contains a line segment and hence infinitely many functions f which are not outer functions (by Theorem 1). To each such f we assign a g as in § 2.2; arguing as in the proof of Theorem 8, we see that $f+g$ and $f-g$ are outer functions which belong to S_ϕ. Thus S_ϕ contains infinitely many outer functions.

If $f(b) = 0$ for some $f \in S_\phi$ and some $b \in U$, then Lemma 4.3 shows that S_ϕ contains, for every $a \in U$, a function f_1 such that $f_1(a) = 0$.

The proof of the theorem will therefore be complete if we can show that S_ϕ contains a function which has a zero in U. Since S_ϕ contains functions which are not outer functions, the required conclusion is a consequence of the following lemma, which actually proves a little more.

4.6. LEMMA. *Suppose* $\|f\| = 1$ *and* M_f *is not a finite Blaschke product* (*in particular*, $M_f \neq 1$). *Then* S^f *contains a function* h *with infinitely many zeros in* U. *Also, for any* n *prescribed points of* U, S^f *contains a function* k *which vanishes at those points.*

Proof. If g is associated with f as in § 2.2, then, as we saw in the proof of Theorem 8, $f+g$ and $f-g$ are in S^f, and so is every h of the form

$$h = \lambda(f+g) + (1-\lambda)(f-g) \qquad (0 \leqq \lambda \leqq 1) \tag{4.6.1}$$

since S^f is convex. We shall show that h has infinitely many zeros in U if λ is chosen appropriately.

If $\lambda \neq 1/2$, (2.2.3) leads to the following representation of h:

$$h = \left(\lambda - \frac{1}{2}\right) e^{-i\alpha} Q_f \left\{ u^2 + \frac{2}{2\lambda - 1} u + 1 \right\} ; \tag{4.6.2}$$

we recall that $u = e^{i\alpha} M_f$, so that $|u(z)| < 1$ in U and $|u(e^{i\theta})| = 1$ a.e. on C.

Let $x(\lambda)$ denote that solution of the equation

$$x^2 + \frac{2}{2\lambda - 1} x + 1 = 0 \qquad \left(0 < \lambda < 1, \ \lambda \neq \frac{1}{2}\right) \tag{4.6.3}$$

which lies in U; the set of these points $x(\lambda)$ covers an arc $\Gamma \subset U$. Since Γ has positive logarithmic capacity, a theorem of Frostman [4;

p. 111] implies that for some λ_0 the equation

$$u(z) = x(\lambda_0) \tag{4.6.4}$$

has infinitely many solutions $z \in U$. If we put $\lambda = \lambda_0$ in (4.6.2), the resulting function h has infinitely many zeros in U.

This proves the first assertion of the lemma. The second is an immediate consequence, for if $a_1, \cdots, a_n$ are preassigned in U, pick points $b_1, \cdots, b_n$ in U such that $h(b_1) = \cdots = h(b_n) = 0$, and apply Lemma 4.3.

4.7. *Proof of Theorem* 10. If ϕ is continuous, T_ϕ is continuous in the weak*-topology of H_1. Since S is weak*-compact, $|T_\phi|$ attains its maximum on S, so that S_ϕ is not empty; S_ϕ is weak*-compact since it is the set of all $f \in S$ at which $T_\phi(f) = ||T_\phi||$. Hence (a) holds.

If there were functions in S_ϕ with arbitrarily many zeros in U, then, by Lemma 4.3, S_ϕ would contain functions $f_m (m = 1, 2, 3, \cdots)$ with

$$f_m\left(\frac{1}{2}\right) = f_m\left(\frac{1}{3}\right) = \cdots = f_m\left(\frac{1}{m}\right) = 0 \; . \tag{4.7.1}$$

By Lemma 1.6, S_ϕ is compact in the topology of pointwise convergence in U. Thus $\{f_m\}$ must have a limit point in S_ϕ, but this is impossible since any limit point g would satisfy

$$g\left(\frac{1}{m}\right) = 0 \qquad (m = 2, 3, 4, \cdots) \tag{4.7.1}$$

and thus be identically 0.

It now follows from Lemma 4.6 that M_f is a finite Blaschke product, for every $f \in S_\phi$. Hence (b) holds.

For the rest of this proof, n will denote the smallest non-negative integer for which (b) holds. *Our proof of* (c) *will use the fact that* (b) *holds, but will not make any direct use of the continuity of* ϕ.

We shall first prove that there is a unique $h \in S_\phi$ which has a zero of multiplicity n at $z = 0$ (the existence of such an $h \in S_\phi$ follows from our choice of n and Lemma 4.3). Assume that there are two such functions, h_1 and h_2. Define

$$g_i(z) = z^{-n} h_i(z) \qquad (i = 1, 2\,;\; z \in U) \; . \tag{4.7.2}$$

Since $h_i \in S_\phi$, $\arg h_1 = \arg h_2$ a.e. on C, so that $\arg g_1 = \arg g_2$ a.e. on C, and thus $S^{g_1} = S^{g_2}$. Since $g_1 \neq g_2$, Theorem 9 shows that S^{g_1} contains a function g_3 with $g_3(0) = 0$. If

$$h_3(z) = z^n g_3(z) \qquad (z \in U) \; , \tag{4.7.3}$$

then $h_3 \in S_\phi$ (since $\arg h_3 = \arg h_1$ a.e. on C), and h_3 has a zero of multiplicity at least $n+1$ at $z=0$. This contradicts our choice of n.

Thus S_ϕ contains a unique function h with a zero of multiplicity n at $z=0$. Define the function g by

$$g(z) = z^{-n}h(z) \qquad (z \in U) . \tag{4.7.4}$$

S^g consists of g alone, for if some $g_4 \neq g$ were in S^g, the function h_4 defined by

$$h_4(z) = z^n g_4(z) \qquad (z \in U) \tag{4.7.5}$$

would be in S_Φ, would have a zero of multiplicity n at $z=0$, and would be distinct from h. By Theorem 8(b), g is therefore a strong outer function, and it is clear from the foregoing that g is the only such function which satisfies (4.1.6), with $a_1 = \cdots = a_n = 0$.

The remaining assertion of the theorem is now an immediate consequence of Lemma 4.3.

4.8. *Proof of Theorem* 11. If ϕ is analytic in $|z|>R$, for some $R<1$, the conclusions of Theorem 10 are of course valid. Let g be the strong outer function whose existence is guaranteed by Theorem 10. Then if f with $\|f\|=1$ is of the form (4.1.7), with $\lambda>0$ and $|a_i| \leqq 1$, f will be in S_ϕ because of Lemma 4.3.

It remains to be proved that every $f \in S_\phi$ is of the form (4.1.7). Our argument now is similar to that of Section 8.4 of [7]. Let ψ be the function which is related to ϕ as in § 4.2. Since $T_\psi = T_\phi$, $\psi(e^{i\theta})$ and $\phi(e^{i\theta})$ differ by the boundary values of a bounded analytic function in U, and since ϕ is analytic in $|z| > R$, ψ is bounded and analytic in $R_1 < |z| < 1$, for any R_1 which satisfies $R < R_1 < 1$.

Our discussion in § 4.2 shows that a function $f \in H_1$ with $\|f\|=1$ is in $S_\psi = S_\phi$ if and only if

$$f(e^{i\theta})\psi(e^{i\theta}) \geqq 0 \qquad \text{(a.e. on } C) . \tag{4.8.1}$$

For any such f, define F_f by

$$F_f(z) = f(z)\psi(z) \qquad (R_1 < |z| < 1) . \tag{4.8.2}$$

F_f is analytic in $R_1<|z|<1$, has real boundary values a.e. on C, and satisfies

$$\lim_{r\to 1}\int_{-\pi}^{\pi} |F_f(re^{i\theta}) - F_f(e^{i\theta})|d\theta = 0 \; ; \tag{4.8.3}$$

hence it can be extended, by the Schwarz reflection principle, so as to be analytic in $R_1<|z<R_1^{-1}$.

Let g be the strong outer function chosen earlier, and define h by

$h(z) = z^n g(z)$. Then $h \in S_\phi$. Since $F_f = f\psi$ and $F_h = h\psi$, the function $f/h = F_f/F_h$ is meromorphic in $|z| < R_1^{-1}$; it is real and non-negative on C, except possibly for poles, since f and h are both in S_ϕ. Hence f/h is of the form (4.2.4), i.e.,

$$f(z) = \lambda z^n g(z) \prod_{i=1}^{m} \frac{(z-a_i)(1-\bar{a}_i z)}{(z-b_i)(1-\bar{b}_i z)} \qquad (z \in U) \tag{4.8.4}$$

for some $\lambda > 0$ and $|a_i| \leqq 1$, $|b_i| \leqq 1$.

We can assume that none of the a_i is equal to any b_j. Then, since f is analytic in U, either all b_i are 0 and $m \leqq n$, or some b_i satisfies $|b_i| = 1$. This last alternative would contradict the fact that g is a strong outer function.

Thus each $f \in S_\phi$ is of the form (4.1.7), and the proof is complete.

4.9. Remarks. The hypothesis of Theorem 11 can be (apparently) weakened; instead of assuming that ϕ is analytic in $|z| > R$ for some $R < 1$, it is enough to assume that ϕ is analytic in $R < |z| < 1$ (and, of course, bounded on C). Indeed, any such ϕ is of the form $\phi = \phi_1 + \phi_2$, where ϕ_1 is bounded and analytic in U, $\phi_1(0) = 0$, and ϕ_2 is analytic in $|z| > R$. Thus $T_\phi = T_{\phi_2}$, and we are back in the situation covered by Theorem 11.

Our second remark is that the functionals T_ϕ with ϕ analytic in $|z| > R (R < 1)$ can be neatly characterized in terms of functional analysis. If T_ϕ is of this type, suppose $f_n \in H_1$ and f_n converges to f, uniformly on compact subsets of U; we do not assume that $\{\|f_n\|\}$ is bounded; then $T_\phi(f_n) \to T_\phi(f)$. This is a consequence of the Cauchy integral theorem: if $R < r < 1$, then

$$\int_{-\pi}^{\pi} f_n(e^{i\theta})\phi(e^{i\theta})\,d\theta = r\int_{-\pi}^{\pi} f_n(re^{i\theta})\phi(re^{i\theta})\,d\theta .$$

But the converse of this is also true:

4.10. *If T is a linear functional on H_1 with the property that $T(f_n) \to T(f)$ whenever $f_n \to f$ uniformly on compact subsets of U, then there exists a function ϕ, analytic in $|z| > R$ for some $R < 1$, such that $T = T_\phi$.*

Proof. Let E be the linear space of all continuous functions on U, with the topology of uniform convergence on compact subsets. Since E is locally convex and since T is continuous on the linear subspace H_1 of E, the generalized Hahn-Banach theorem [2, Corollary 1, p. 111] allows us to extend T to a continuous linear functional on E, and this functional is of the form

$$T(f) = \int f d\mu \qquad (f \in E) , \tag{4.10.1}$$

where μ is a measure with compact support in U [3, Proposition 11, p. 73].

Choose $R < 1$, such that the support of μ lies in $|z| < R$. If

$$\phi(w) = w\int \frac{d\mu(z)}{w-z} , \tag{4.10.2}$$

then ϕ is analytic in $|w|>R$, and for all $f \in H_1$ we have

$$\begin{aligned} T(f) &= \int\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{i\theta})}{e^{i\theta}-z}e^{i\theta}d\theta\right\}d\mu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\theta})\left\{\int\frac{d\mu(z)}{e^{i\theta}-z}\right\}e^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\theta})\phi(e^{i\theta})d\theta = T_\phi(f) . \end{aligned}$$

V. APPENDIX

Let H_∞ be the class of all functions which are analytic and bounded in U, and let A consist of all uniformly continuous analytic functions in U (so that every $f \in A$ can be extended continuously to the closed unit disc). With the norm

$$\|f\|_\infty = \sup_{z\in U} |f(z)| ,$$

both H_∞ and A are Banach spaces. The extreme points of their unit spheres have been identified jointly by Arens, Buck, Carleson, Hoffman, and Royden, during the Princeton Conference on Functions of a Complex Variable in September, 1957. For the sake of completeness, we include the result in the present paper.

THEOREM 12. *Let X be H_∞ or A; suppose $f \in X$ and $\|f\|_\infty = 1$. Then f is an extreme point of the unit sphere of X if and only if*

$$\int_{-\pi}^{\pi} \log \{1 - |f(e^{i\theta})|\}\, d\theta = -\infty . \tag{*}$$

Proof. Suppose (*) holds. If $g \in X$ and $\|f+g\|_\infty = \|f-g\|_\infty = 1$,

$$|g(z)|^2 \leqq 1 - |f(z)|^2 \leqq 2\{1 - |f(z)|\} \qquad (z \in U) .$$

Consequently, $\int_{-\pi}^{\pi} \log |g(e^{i\theta})|\, d\theta = -\infty$, and this implies that g is identically 0. Hence f is an extreme point.

Conversely, suppose (*) is false and $X = H_\infty$. Define

$$g_1(z) = \exp\left\{\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log [1-|f(e^{i\theta})|]d\theta\right\} \qquad (z \in U) .$$

Then $g_1(z) \neq 0$, $|g_1(z)| < 1$, and $|g_1(e^{i\theta})| \leqq 1 - |f(e^{i\theta})|$ a.e. on C. It follows that $\|f+g_1\|_\infty \leqq 1$ and $\|f-g_1\|_\infty \leqq 1$, so that f is not an extreme point.

Finally, if $X=A$ and (*) is false, let E be the set of all $e^{i\theta} \in C$ at which $|f(e^{i\theta})|=1$. There is then a continuous function ϕ on C such that (i) $0 \leqq \phi(e^{i\theta}) \leqq 1-|f(e^{i\theta})|$, (ii) $\int_{-\pi}^{\pi} \log \phi(e^{i\theta})\, d\theta > -\infty$, (iii) on every closed subarc of C which is disjoint from E, ϕ has a bounded derivative. Put

$$g_2(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \phi(e^{i\theta})\, d\theta\right\} \qquad (z \in U) .$$

Then $g_2 \in A$ (by (iii)), and g_2 has all the properties of g_1 which were used in the preceding paragraph. It follows, as above, that f is not an extreme point.

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