ON THE ISOMETRIES OF CERTAIN FUNCTION-SPACES

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1. Introduction. In Chapter 11 of his well-known book [1], S. Banach has given theorems characterizing the linear, norm-preserving operators on the spaces L_p and l_p, where $1 \leqq p < \infty$ and $p \neq 2$. The proofs are not given completely and the theorems are stated in less than full generality. The first purpose of this paper is to supply a new proof for a somewhat more general theorem ; besides being set in an arbitrary (σ-finite) measure space, this theorem applies to values of $p < 1$. The preliminaries in § 2 turn up one interesting fact (Theorem 2.2) as a bonus.

The second purpose is generalization ; there are other spaces besides L_p where a norm, metric, or something like it is defined in terms of an integral

$$I[f] = \int \Phi(|f(x)|)\,d\mu \,, \tag{1.1}$$

and the method we use on L_p spaces can be applied to some of these others as well. The conclusions are that like the L_p case, isometries come from non-singular transformations of the underlying measure space, but unlike L_p, not all such transformations give isometries.

2. Some inequalities. The first lemma and theorem serve as preparation for the generalization, as well as the L_p theorem.

LEMMA 2.1. *Let $\Phi(t)$ be a continuous, strictly increasing function defined for $t \geqq 0$, with $\Phi(0) = 0$, and let z and w be complex numbers. If $\Phi(\sqrt{t})$ is a convex function of t, then*

$$\Phi(|z+w|) + \Phi(|z-w|) \geqq 2\Phi(|z|) + 2\Phi(|w|) \,, \tag{2.1}$$

while if $\Phi(\sqrt{t})$ is concave the reverse inequality is true. Provided the convexity or concavity is strict, equality holds if and only if $zw = 0$.

Proof. Since $\Phi(\sqrt{t})$ is convex, Theorem 92 of [7] gives

$$\Phi^{-1}\left\{ \frac{\Phi(|z+w|) + \Phi(|z-w|)}{2} \right\} \geqq \left\{ \frac{|z+w|^2 + |z-w|^2}{2} \right\}^{1/2} = \{|z|^2 + |w|^2\}^{1/2} \,. \tag{2.2}$$

But the convexity of $\Phi(\sqrt{t})$ implies that $t^2/\Phi(t)$ is decreasing, strictly if

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the convexity is strict. This is the hypothesis of Theorem 105 of [7], which asserts that

(2.3) $$\{|z|^2 + |w|^2\}^{1/2} \geqq \Phi^{-1}\{\Phi(|z|) + \Phi(|w|)\} .$$

Combining (2.2) and (2.3) yields (2.1), since Φ^{-1} is an increasing function. In case $\Phi(\sqrt{t})$ is strictly convex, we also obtain from Theorem 105 the fact that (2.3) (and hence (2.1)) is strict, unless z or w is zero. The case where $\Phi(\sqrt{t})$ is concave may be treated similarly using the same two theorems; inequalities (2.2) and (2.3) are both reversed.

Let (X, F, μ) be a measure space; we will always assume that $X \in F$ and that μ is σ-finite. Given $\Phi(t)$, a functional I on measurable functions is defined by (1.1). The set of functions $f(x)$ such that $I[f] < \infty$ will be denoted L_Φ; in general, L_Φ need not be a linear space. In case $\Phi(t) = t^p$ for some $p > 0$, L_Φ in the space L_p and $I[f] = \|f\|_p^p$.

THEOREM 2.1. *Let $\Phi(t)$ be a continuous, strictly increasing function* (for $t \geqq 0$) *with $\Phi(0) = 0$ and $\Phi(\sqrt{t})$ convex, and suppose that $f(x) + g(x)$ and $f(x) - g(x)$ belong to L_Φ. Then*

(2.4) $$I[f + g] + I[f - g] \geqq 2I[f] + 2I[g] .$$

If $\Phi(\sqrt{t})$ is concave and $f(x)$ and $g(x)$ belong to L_Φ, the reverse inequality to (2.4) *is true. If the convexity or concavity of $\Phi(\sqrt{t})$ is strict, equality holds in* (2.4) *if, and only if, $f(x)g(x) = 0$ almost everywhere.*

Proof. (2.4) may be written

$$\int_x \{\Phi(|f(x) + g(x)|) + \Phi(|f(x) - g(x)|) - 2\Phi(|f(x)|) - 2\Phi(|g(x)|)\} d\mu \geqq 0.$$

This holds if $\Phi(\sqrt{t})$ is convex, because by the lemma the integrand is non-negative. Equality can occur only when the integrand is zero almost everywhere; if $\Phi(\sqrt{t})$ is strictly convex this means that for almost all x either $f(x) = 0$ or $g(x) = 0$. The case where $\Phi(\sqrt{t})$ is concave is similar.

REMARK. Theorem 2.1 is equally true for spaces at real or complex functions; this will also be the case for the main Theorems 3.1 and 4.1, but won't be mentioned explicitly.

COROLLARY 2.1.[1] *If $f(x)$ and $g(x)$ belong to L_p, $p \geqq 2$, then*

[1] Inequality (2.5) was used by Clarkson in [**4**]. He did not discuss the condition for equality, which was not needed for his application (but is for ours). [**3**] is also closely related

$$\|f+g\|_p^p + \|f-g\|_p^p \geqq 2\|f\|_p^p + 2\|g\|_p^p . \tag{2.5}$$

If $0 < p \leqq 2$, *the reverse inequality holds. In either case, if* $p \neq 2$, *equality occurs if, and only if,* $f(x)\, g(x) = 0$ *almost everywhere.*

This corollary has an immediate application to a question raised by Boas, who showed in [2] that the spaces L_p and H_p are isomorphic for $1 < p < \infty$.[2] The question is whether an isometric mapping of H_p onto L_p is possible when $p \neq 2$.

THEOREM 2.2. *Provided* $0 < p < \infty$ *and* $p \neq 2$, *there is no isometric linear mapping of* H_p *onto* L_p.

Proof. In H_p, a function not identically zero must be different from zero almost everywhere. Hence by Corollary 2.1, the equality can never hold in (2.5) unless $\|f\|_p = 0$ or $\|g\|_p = 0$. But in L_p there are pairs of nonnull functions for which equality holds. Since the occurrence of equality must be preserved by a linear isometric mapping, no such mapping can take H_p onto L_p.

3. The isometries of L_p spaces. A "regular set isomorphism" of the measure space (X, F, μ) will mean a mapping T of F into itself, defined modulo sets of measure zero, satisfying

$$T(X - A) = TX - TA \tag{3.1}$$

$$T\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} TA_n \text{ for disjoint } A_n \tag{3.2}$$

$$\mu(TA) = 0 \text{ if, and only if, } \mu(A) = 0 \tag{3.3}$$

for all sets A, A_n belonging to F. A regular set isomorphism induces a linear transformation (also to be denoted T) on the set of measurable functions, which is characterized by $T\varphi_A = \varphi_{TA}$, where φ_A is the characteristic function of the set A.[3]

THEOREM 3.1. *Let* U *be a linear operator on a space* L_p *for some positive* $p \neq 2$, *such that*

$$\|Uf\|_p = \|f\|_p \quad \textit{for all } f(x) \in L_p . \tag{3.4}$$

Then there exists a regular set-isomorphism T *and a function* $h(x)$ *such that* U *is given by*

[2] Here L_p is formed with X the circumference of the unit circle and μ normalized Lebesgue measure; see for instance **[10]** for background on the H_p spaces. Only one fact is needed in the proof of Theorem 2.2.

[3] This process is described in detail in **[5]**, pp. 453 and 454; the assumption made there that the set mapping is measure-preserving can be replaced by (3.3).

$$Uf(x) = h(x)Tf(x)\ ; \tag{3.5}$$

if a measure μ^* *is defined by* $\mu^*(A) = \mu(T^{-1}A)$, *then*

$$|h(x)|^p = \frac{d\mu^*}{d\mu} \text{ a.e. } on\ TX\ . \tag{3.6}$$

Conversely, for any regular set-isomorphism T *and any* $h(x)$ *satisfying* (3.6), *the operator* U *defined by* (3.5) *satisfies* (3.4).

Proof. We will carry out the proof under the assumption that $\mu(X) < \infty$; the extension to the σ-finite case is straightforward. Suppose that (3.4) holds, and define a set mapping by

$$TA = \{x : U\varphi_A(x) \neq 0\}\ . \tag{3.7}$$

Now if A and B are disjoint sets,

$$\|\varphi_A + \varphi_B\|_p^p + \|\varphi_A - \varphi_B\|_p^p = 2\|\varphi_A\|_p^p + 2\|\varphi_B\|_p^p\ .$$

By (3.4) the same relation holds for $U\varphi_A$ and $U\varphi_B$, and so by Corollary 2.1 we conclude that $U\varphi_A$ and $U\varphi_B$ have almost disjoint support, or that $\mu(TA \cap TB) = 0$. Consequently $T(A \cup B) = TA \cup TB$ to within a set of measure zero, and the extension to denumerable sums follows from the continuity of U. In particular, except for sets of measure zero, $TX = TA \cup T(X - A)$ and the latter are disjoint, so that $T(X - A) = TX - TA$. Thus the mapping T satisfies (3.1) and (3.2) ; (3.3) is obvious in view of (3.4) so that T is a regular set-isomorphism.

Since $\mu(X) < \infty$, $\varphi_X \in L_p$ and we can let $h(x) = U\varphi_X(x)$. (In the σ-finite case $h(x)$ would have to be defined piecemeal.) For any set $A \in F$, $h(x) = U\varphi_A(x) + U\varphi_{X-A}(x)$. But the two functions on the right have (almost) disjoint support, so that $U\varphi_A(x)$ agrees with $h(x)$ almost everywhere that the former is not zero. Hence

$$U\varphi_E(x) = h(x)\varphi_{TE}(x) = h(x)T\varphi_E(x) \text{ a.e.} \tag{3.8}$$

By (3.8) and the linearity of U, (3.5) holds for any simple function, and since such functions are dense in L_p and U is continuous, (3.5) is valid in general.

It remains to identify $|h(x)|^p$. From (3.4) and (3.5) we have

$$\mu(A) = \|\varphi_A\|_p^p = \|U\varphi_A\|_p^p = \int_X |h(x)|^p \varphi_{TA}^p(x) d\mu = \int_{TA} |h(x)|^p d\mu\ . \tag{3.9}$$

But by (3.3) the measure $\mu^*(A) = \mu(T^{-1}A)$, (defined for sets in the range of the mapping T), is absolutely continuous with respect to μ, and so

$$\mu(A) = \mu^*(TA) = \int_{TA} \frac{d\mu^*}{d\mu} d\mu\ . \tag{3.10}$$

Comparison of (3.9) and (3.10) together with the uniqueness of the Radon-Nikodym derivative proves (3.6). The converse statement can easily be verified.

COROLLARY 3.1. *Suppose that U is a linear transformation of functions measurable on (X, F, μ) which preserves the L_p norm for two different positive values of p. Then there exists a measure-preserving set-isomorphism T such that*

$$Uf(x) = h(x)Tf(x) ,$$

where $|h(x)| = 1$ almost everywhere on TX. It follows that U is a norm-preserving operator on L_p for all p.

Proof. One of the two values of p, say p_1, is different from 2, so the theorem applies. It follows that (whether $p_2 = 2$ or not)

$$|h(x)|^{p_1} = \frac{d\mu^*}{d\mu} = |h(x)|^{p_2} \text{ a.e. } ,$$

and so $|h(x)| = 1$ or 0. By (3.3), $|h(x)| = 1$ a.e. on TX, which implies that T is measure-preserving.

REMARK. Presumably it most often happens in cases of interest that an invertible "regular set-isomorphism" is generated by an essentially one-to-one onto, measurability-preserving, non-singular point mapping. It is easy to see that if the measure space is discrete, this is always so; much wider conditions on the measure-space are known under which it is so for all *measure-preserving* transformations.[4] If such a theorem were available which applied to all regular set-isomorphisms, Theorem 3.1 could be sharpened. As it is, the corollary can be improved if (X, F, μ) has "sufficiently many measure-preserving transformations" (see [6]) by replacing the set mapping T by a point mapping. Similar remarks apply to the results of the next section.

4. Generalization. In this section we shall consider functionals $I[f]$ defined by (1.1) with various functions $\Phi(t)$ other than t^p. We assume hereafter that $\Phi(t)$ is continuous and strictly increasing, with $\Phi(0) = 0$ and $\Phi(1) = 1$.

DEFINITION. A positive number a will be called a "multiplier" of $\Phi(t)$ provided $\Phi(at) = \Phi(a)\Phi(t)$ for all $t \geqq 0$. The set of all multipliers

[4] In particular, von Neumann showed in [9] that if X is a closed region in R_n and μ is equivalent to Lebesgue measure, a measure-preserving set transformation can be obtained from a point mapping. Further results on this problem are contained in [6].

will be denoted by M.

It is not hard to show that M is a group under multiplication which contains all its non-zero limit points; with the aid of this and some well known facts we obtain

LEMMA 4.1. *M contains all positive numbers if, and only if, $\Phi(t) = t^p$ for some p; otherwise M consists of all the integral powers of some positive number $a_0 \neq 1$, or else of the identity above.*

As an example of a well-behaved function with a discrete set of multipliers we mention $\Phi(t) = t^p \exp \sin \log t$; $a_0 = e^{2\pi}$ and the function is convex for large p. There is, however, a quite general sufficient condition ensuring that $M = \{1\}$:

LEMMA 4.2. *Suppose that $\Phi(t)$ is of regular variation[5] at $t = 0$ or $t = \infty$. Then either $\Phi(t) = t^p$ or $M = \{1\}$.*

Proof. If a number $a \neq 1$ is a muliplier of Φ, it follows from the definition that

$$\Phi(a^n) = \Phi^n(a) \text{ for all integers } n .$$

Suppose $\Phi(t) = t^p L(t)$ where $\lim_{t\to\infty} L(ct)/L(t) = 1$ for all $c > 0$. (The case of regular variation at $t = 0$ is entirely similar.) Combining these things gives

$$L(a^n) = \left\{\frac{\Phi(a)}{a^p}\right\}^n .$$

It follows from this and the defining property of a slowly-varying function that $\Phi(a) = a^p$ and that $L(a^n) = 1$ for all n. Now for any value of t,

$$\Phi(ta^n) = t^p a^{np} L(ta^n) .$$

But using the fact that a^n is a multiplier and the value of $\Phi(a^n)$,

$$\Phi(ta^n) = \Phi(t)\Phi(a^n) = a^{np} t^p L(t) .$$

Hence $L(ta^n) = L(t)$ for all n, and together with the fact that $L(a^n) = 1$ this implies $L(t) = 1$, and so $\Phi(t) = t^p$.

THEOREM 4.1. *Suppose that $\Phi(\sqrt{t})$ is either strictly convex or strictly concave, and that U is a linear operator on the space L_Φ over*

[5] $\Phi(t)$ is of regular variation at ∞ if $\lim_{t\to\infty} \Phi(ct)/\Phi(t)$ exists for all $c > 0$; this implies that $\Phi(t) = t^p L(t)$ for some p, where $L(t)$ is a slowly-varying function (i.e., $L(ct)/L(t) \to 1$ for all $c > 0$). The case $t = 0$ is defined similarly. These ideas are due to Karamata **[7]**.

(X, F, μ) such that

$$I[Uf] = I[f] \text{ for all } f(x) \in L_\Phi . \tag{4.1}$$

Then there exists a regular set-isomorphism T and a function $h(x)$ such that U is given by

$$Uf(x) = h(x)Tf(x) ; \tag{4.2}$$

if μ^* is the measure $\mu(T^{-1}A)$, $h(x)$ must satisfy

$$\Phi(|h(x)|) = \frac{d\mu^*}{d\mu} \text{ and } |h(x)| \in M \text{ a.e. on } TX . \tag{4.3}$$

Conversely, if T is a regular set-isomorphism such that there exist functions satisfying (4.3), *and $h(x)$ is such a function, then U defined by* (4.2) *is an isometry. If in addition to the other hypotheses $\Phi(t)$ is of regular variation at either $t = 0$ or $t = \infty$, but is not a power of t, then T must be measure-preserving and $|h(x)| = 1$ a.e. on TX.*

Proof. As before we assume for simplicity that $\mu(X) < \infty$. Suppose that (4.1) holds, and define

$$TA = \{x : U\varphi_A(x) \neq 0\} . \tag{4.4}$$

The fact that T maps disjoint sets into almost disjoint sets follows from (4.1) and Theorem 2.1 ; thereafter the proof of (4.2) is the same as that of the corresponding part of Theorem 3.1.

From (4.1) and (4.2) we have (since $\Phi(1) = 1$)

$$\mu(A) = I[\varphi_A] = I[U\varphi_A] = \int_X \Phi(|h(x)\varphi_{TA}(x)|)d\mu = \int_{TA} \Phi(|h(x)|)d\mu . \tag{4.4}$$

But (as before)

$$\mu(A) = \mu^*(TA) = \int_{TA} \frac{d\mu^*}{d\mu} d\mu , \tag{4.5}$$

and comparison proves the first part of (4.3). Replacing $\varphi_A(x)$ in (4.4) by $t\varphi_A(x), t > 0$, we obtain

$$\Phi(t)\mu(A) = \int_{TA} \Phi(t|h(x)|)\, d\mu .$$

Comparing this with (4.5) gives

$$\frac{\Phi(t|h(x)|)}{\Phi(t)} = \frac{d\mu^*}{d\mu} = \Phi(|h(x)|) \text{ a.e. on } TX ,$$

and the second part of (4.3) follows. Conversely, provided (4.3) holds, it is easy to verify that (4.2) gives an isometry. The last assertion of the theorem follows immediately from Lemma 4.2.

EXAMPLES. If $\Phi(t) = t/(1+t)$ and the measure space is chosen appropriately, L_Φ becomes the space S or s [**1**, pp. 9-10]; for any measure space and this choice of $\Phi(t)$, $\rho(f, g) = I[f-g]$ is a metric. From the above theorem,[6] the only isometries of these spaces are those induced by measure-preserving transformations of the underlying measure space. Somewhat more generally, any function $\Phi(t)$ satisfying our other assumptions which is concave must also be subadditive, so that $\rho(f, g) = I[f-g]$ is a metric; since $\Phi(t)$ concave implies $\Phi(\sqrt{t})$ strictly concave, Theorem 4.1 applies.

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[6] The fact that $\Phi(1) \neq 1$ causes no difficulty; U is still an isometry and the theorem applies to the space $L_{\Phi/\Phi(1)}$ with the measure $\Phi(1)\mu$.