LINEAR INEQUALITIES AND QUADRATIC FORMS

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1. Introduction. There are known criteria for a quadratic form to be positive definite, and criteria for a system of linear inequalities to have a solution. In this paper the two problems are shown to be related. The principal theorem is Theorem 5.1.

2. Definitions and Notation. We will consider a quadratic form

$$Z(x) \equiv \sum_1^n a_{ij}x_ix_j, \quad \text{with } a_{ij} = a_{ji} ,$$

and ask whether it is positive in the first orthant, i.e., whether it is positive for non-negative values of the x_i.

If $Z(x) > 0$ for $x \geqq 0$, we call it *conditionally definite* and if $Z(x) \geqq$ for $x \geqq 0$, we call it *conditionally semi-definite.* (Since we will only be concerned with positive definiteness, we will omit the word "positive" throughout the paper.) Finally, if $Z(x) \geqq 0$ when $x \geqq 0$ and $Z(x) > 0$ when $x > 0$, we call $Z(x)$ *conditionally almost-definite.*

As a matter of notation, we recall that $Ax \geq 0$ or $x \geq 0$ means that at least one component of the vector in question is positive.

In discussing $Z(x)$ we shall have occasion to refer to the form obtained by setting $x_{k_1}, x_{k_2}, \cdots, x_{k_s}$ equal to zero, that is, the form

$$\sum_{i,j \neq k_1, \cdots, k_s} a_{ij}x_ix_{ij} .$$

We shall call this a principal minor of $Z(x)$ and denote it $Z_{k_1 \cdots k_s}(x)$. In referring to the corresponding matrix, $A^{k_4 \cdots k_s}$ we will assume x has the appropriate number of components when we write $A^{k_1 \cdots k_s}x$.

3. Quadratic forms in the first orthant. We first prove a theorem which is not strictly necessary but may be some intrinsic interest. It concerns the game whose matrix is $A = (a_{ij})$ and whose value is v. (For completeness we remind the reader of the following definition of the value v of a game with matrix $B = (b_{ij})$, $i = 1, \cdots, m$; $j = 1, \cdots, n$. Let X be the set of vectors $x = (x_1, \cdots, x_m)$ with $x_1 \geqq 0$ and $\sum_1^m x_i = 1$; Y the set of $y = (y_1, \cdots, y_n)$ with $y_j \geqq 0$ and $\sum_1^n y_j = 1$. Then it can be shown that

$$\max_{x \in X} \min_{y \in Y} \sum b_{ij}x_iy_j = \min_{y \in Y} \max_{x \in X} \sum b_{ij}x_ix_j ,$$

and this quantity is called the value of the game with matrix B).

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THEOREM 3.1. *Suppose each principal minor of $Z(x)$ is conditionally definite. Then in order that $Z(x)$ be conditionally definite, it is necessary and sufficient that $v > 0$.*

Proof. Suppose $v \leqq 0$. Then there is a $y \geqq 0$ with $Ay \leqq 0$. But

$$Z(y) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} y_j \right) y_i \leqq 0.$$

This shows the necessity.

Suppose, now, that $v > 0$. Then there is a vector $\bar{x}$ with $\bar{x} \geqq 0$ and $A\bar{x} > 0$. Every vector $x \geqq 0$ can be written as a convex combination of $k\bar{x}$, $k > 0$, and some vector x', with $x' \geqq 0$ and x' in one of the coordinate planes. That is, for any $x \geqq 0$, $x = \lambda k \bar{x} + (1 - \lambda)x'$, $k > 0$ and $0 \leqq \lambda \leqq 1$.

We note the fact that, for any u, v,

$$\begin{aligned} Z[\lambda u + (1 - \lambda)v] &= \sum a_{ij}[\lambda u_i + (1 - \lambda)v_i][\lambda u_j + (1 - \lambda)v_j] \\ &= \lambda^2 \sum a_{ij} u_i u_j + 2\lambda(1 - \lambda) \sum a_{ij} u_i v_j + (1 - \lambda)^2 \sum a_{ij} v_i v_j \,. \end{aligned}$$

Thus,

$$(1) \quad Z[\lambda u + (1 - \lambda)v] = \lambda^2 Z(u) + (1 - \lambda)^2 Z(v) + 2\lambda(1 - \lambda) \sum a_{ij} u_i v_j \,.$$

Applying

$$(1) \quad \text{to } x = \lambda k \bar{x} + (1 - \lambda)x',\ Z(x) = \lambda^2 k^2 Z(\bar{x}) + (1 - \lambda)^2 Z(x') + 2\lambda(1 - \lambda)k \sum a_{ij} \bar{x}_i x'_j \,.$$

Since every principal minor of $Z(x)$ is conditionally difinite, $Z(x') > 0$. Since $\sum a_{ij} \bar{x}_i > 0$, $j = 1, \cdots, n$, $Z(\bar{x}) > 0$ and $\sum a_{ij} \bar{x}_i x'_j > 0$. Therefore, $Z(x) > 0$ for $x > 0$ and the sufficiency is proved.

We can state the following theorem, the proof of which is almost identical with the proof of Theorem 3.1.

THEOREM 3.2. *If each principal minor of $Z(x)$ is conditionally semi-definite, then $Z(x)$ is conditionally semi-definite if and only if $v \geqq 0$.*

For symmetry we state the foregoing as theorems on systems of linear inequalities.

THEOREM 3.3. *Suppose each principal minor of $Z(x)$ is conditionally definite. Then the system $Ax > 0$, $x \geqq 0$ has solutions if and only if $Z(x)$ is conditionally definite.*

THEOREM 3.4. *Suppose each principal minor of $Z(x)$ is conditionally semi-definite. Then the system $Ax \geqq 0$, $x \geqq 0$ has solutions if and only if $Z(x)$ is conditionally semi-definite.*

These theorems raise the question of the relation between the form

$Z(x)$ and the system $Ax \geq 0, x \geq 0$. The following theorem answers it.

THEOREM 3.5. *Suppose every principal minor of $Z(x)$ is conditionally semi-definite. Then the system $Ax \geq 0, x \geq 0$ has solutions if and only if $Z(x)$ is conditionally almost-definite.*

Proof. Suppose $Ax \geq 0, x \geq 0$ is consistent and let $\bar{x}$ be a solution. As in the proof of Theorem 3.1, represent any $x > 0$ by $x = \lambda k\bar{x} + (1 - \lambda)x'$, where λ, k, and x' have the same significance as before. Then

$$Z(x) = \lambda^2 k^2 Z(\bar{x}) + (1 - \lambda)^2 Z(x') + 2\lambda(1 - \lambda)k \sum a_{ij}\bar{x}_i x'j .$$

Now if $\bar{x} > 0, Z(\bar{x}) > 0$, and $Z(x)$ will be if $\lambda > 0$, that is, if $x > 0$. On the other hand, if for every i for which $\sum a_{ij}\bar{x}_j > 0$ it happens that $\bar{x}_i = 0, Z(\bar{x}) = 0$. However, if $x > 0$ then $x'_i > 0$ if $\bar{x}_i = 0$, and thus

$$\sum a_{ij}\bar{x}_i x'_j > 0.$$

Thus in any case $Z(x) > 0$ if $x > 0$.

Now suppose $Z(x)$ conditionally almost-definite. Consider the convex hull, A^*, of the row vectors of A. If this contains a vector in the first orthant, then the system $Ax > 0, x > 0$ has solutions.

If A^* does not intersect the first orthant in any non-zero vector, then A^* and the first orthant can be strictly separated by a hyperplane through the origin. One normal to this hyperplane, y, will lie interior to the first orthant.

Thus $Ay \leqq 0$ and since $y > 0, Z(y) \leqq 0$, contrary to the hypothesis that $Z(x)$ is conditionary almost-definite. Thus the theorem is proved.

4. Further development of Section 3. In the five theorems of § 3, it is natural to try to replace the hypotheses concerning the principal minors of $Z(x)$ by some condition relating more directly to linear inequalities.

It is not difficult to verify that a quadratic form in two variables, $ax^2 + 2bxy + cy^2$, is conditionally definite if and only if $a > 0, c > 0$ and either $b^2 < ac$ or $b > 0$. This is equivalent to the statement that

(1) $ax > 0$
(2) $cy > 0$
(3) $ax + by > 0, bx + cy > 0$

all have non-negative solutions. Proceeding by induction, we can state the following theorem.

THEOREM 4.1. *A necessary and sufficient condition that $Z(x) = \sum a_{ij}x_i x_j$ be conditionally definite is that for each principal minor $A^{k_1}, \cdots {}^{k_r}$ of A, the system $A^{k_1}, \cdots {}^{k_r} x > 0$ $x \geqq 0$ be consistent.*

Clearly, Theorems 3.4 and 3.5 can be restated in this way but we forbear doing so here.

5. Positive Definite Forms. It is clear that to see whether a form $\sum a_{ij}x_ix_j$ is positive in the orthant where $x_{i_1}, x_{i_2}, \cdots, x_{i_r}$ are negative or zero, and the other unknowns positive or zero, we have only to multiply the i_1th, i_2th, $\cdots$, i_rth rows and columns of $A \equiv (a_{ij})$ by -1 and inquire whether the resulting form is conditionally definite. We may call the form $\sum b_{ij}x_ix_j$ obtained in this way a symmetric transform of $\sum a_{ij}x_kx_j$. Thus a quadratic form is positive definite if and only if every symmetric transform is conditionally definite.

THEOREM 5.1. *A quadratic form $\sum a_{ij}x_ix_j$ is positive definite if and only if every system $Bx > 0, x \geqq 0$ is consistent where B is a symmetric transform of a principal minor of A.*

6. Linear Inequalities. Let B be any $m \times n$ matrix and $C = BB^T$. In [1] it was shown that $Bx \geq 0$ has solutions if and only if $Cy \geq 0$, $y \geqq 0$ does. It can be shown that $Bx > 0$ has solutions if and only if $Cy > 0, y \geqq 0$ does.

Using these results, plus the foregoing discussion, we can summarize as follows:

THEOREM 6.1. *The system $Bx > 0$ is consistent if and only if the form $\sum c_{ij}y_iy_j$ is conditionally definite.*

THEOREM 6.2. *The system $Bx \geq 0$ is consistent if and only it $\sum c_{ij}y_iy_j$ is conditionally almost-definite.*

T. S. Motzkin in [2] has given a condition for a quadratic form to be conditionally semi-definite, the condition involving the signs of various determinants. No other discussion of this question is known to the writer.

REFERENCES

1. J. W. Gaddum, *A theorem on convex cones with applications to linear inequalities*, Proc. Amer. Math. Soc. Vol. **3**, No. 6, pp. 957–960.
2. National Bureau of Standards Report 1818 (1952), 11–12.

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