ON A COMMUTATIVE EXTENSION OF A COMMUTATIVE BANACH ALGEBRA

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Let A be a commutative Banach algebra without identity such that

(1.a) there exists an approximate identity (i.e. there exists a net $\{u_\alpha\} \subset A$, so that $\| u_\alpha \| = 1$ and $u_\alpha x \to x$ for all $x \in A$);

(1.b) if $\hat{A}$ designates Gelfand's representation of A [3], and M the space of regular maximal ideals of A, then the boundary of M with respect to $\hat{A}$, is equal to M[1].

Let $\mathscr{L}(A)$ be the algebra of all bounded linear operators on A; the mapping $x \to T_x$ of A into $\mathscr{L}(A)$, where $T_x y = xy$, $y \in A$, is isomorphic and isometric (by (1.a)) onto a subalgebra $\tilde{A}$ of $\mathscr{L}(A)$,

Let $\mathscr{A}$ be the set of those operators $T \in \mathscr{L}(A)$ which commute with all $T_x \in \tilde{A}$, that is such that

$$(1) \qquad T(xy) = (Tx)y = x(Ty), \qquad x, y \in A.$$

LEMMA (i). *For all* $T \in \mathscr{A}$, *we have* $T = \lim T_{Tu_\alpha}$, *the limit being considered in the strong operator topology.*

(ii) $\mathscr{A}$ *is the closure of* $\tilde{A}$ *in the strong operator topology.*

(iii) $\mathscr{A}$ *is the largest commutative subalgebra of* $\mathscr{L}(A)$ *which contains* $\tilde{A}$.

(iv) $\tilde{A}$ *is an ideal in* $\mathscr{A}$.

Proof. From (1) and (1.a), it follows that

$$T_{Tu_\alpha} y = Tu_\alpha \cdot y = T(u_\alpha y) \to Ty$$

for all $T \in \mathscr{A}$ and $y \in A$, hence (i) is proved. (ii) results from (i). Concerning (iii), it is enough to prove that $\mathscr{A}$ is commutative; or, by (i) and (1)

$$T_1 T_2 x = \lim T_{T_1 u_\alpha} T_2 x = T_2 \lim T_{T_1 u_\alpha} x = T_2 T_1 x, \qquad T_1, T_2 \in \mathscr{A}, \ x \in A.$$

If $T \in \mathscr{A}$ and $x, y \in A$, then

$$TT_x y = T(xy) = (Tx)y = T_{Tx} y,$$

hence

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[1] For example this condition is satisfied if $\mathscr{A}$ is regular or selfadjoint, see [3, p. 81].

$$TT_x = T_xT = T_{Tx}, \tag{2}$$

whence (iv) follows.

Now, let $\mathscr{M}$ be the space of the maximal ideals of $\mathscr{A}$. We can pass to the main result of our note.

THEOREM 1. *There is a homeomorphism $m \to \tilde{m}$ of M, on an open subset $\tilde{M}$ of $\mathscr{M}$, such that for all $m \in M$, and $x \in A$,*

$$\hat{T}_x(\tilde{m}) = \hat{x}(m)\,;$$

if $\tilde{m}_0 \notin \tilde{M}$ then $\hat{T}_x(\tilde{m}_0) = 0$.

Proof. Observe that by (1.b) and by a theorem of Neumark [4][2] to every $m \in M$ there corresponds an $\tilde{m} \in \mathscr{M}$ such that $\hat{x}(m) = \hat{T}_x(\tilde{m})$ for all $x \in A$. We shall show that $\tilde{m}$ is uniquely determined. If $\hat{T}_x(\tilde{m}_1) = \hat{x}(m) = \hat{T}_x(\tilde{m}_2)$ for all $x \in A$, then by (2)

$$\begin{aligned}\hat{T}(\tilde{m}_1)\hat{x}(m) &= \hat{T}(\tilde{m}_1)\hat{T}_x(\tilde{m}_1) = \widehat{TT}_x(\tilde{m}_1) = \hat{T}_{Tx}(\tilde{m}_1) = \hat{T}x(m)\\ &= \hat{T}_{Tx}(\tilde{m}_2) = \widehat{TT}_x(\tilde{m}_2) = \hat{T}(\tilde{m}_2)\hat{x}(m)\,,\end{aligned}$$

where $x \in A$ and $T \in \mathscr{A}$ are arbitrary. Choose $x \in \mathscr{A}$ such that $\hat{x}(m) \neq 0$; then $\hat{T}(\tilde{m}_1) = \hat{T}(\tilde{m}_2)$ for all $T \in \mathscr{A}$; hence $\tilde{m}_1 = \tilde{m}_2$.

Let $\hat{T}_x(\tilde{m}_0) \not\equiv 0$; then the homomorphism $x \to \hat{T}_x(\tilde{m}_0)$ has as kernel a regular maximal ideal m_0 of A, and from $\hat{x}(m_0) = \hat{T}_x(\tilde{m}_0)$ it follows that $\tilde{m}_0 \in \tilde{M}$. Thus, if $\tilde{m}_0 \notin \tilde{M}_0$, then necessarily $\hat{T}_x(\tilde{m}_0) \equiv 0$. This result shows also that $\tilde{M}$ is open in $\mathscr{M}$. In fact, if $\tilde{m}_0 \in \tilde{M}$, there exists an $x \in A$ such that $\hat{T}_x(\tilde{m}_0) \neq 0$; but then $\hat{T}_x(\tilde{m}) \neq 0$ in a neighborhood V of $\tilde{m}_0$; hence $V \subset \tilde{M}$.

The mapping $\tilde{m} \to m$ being evidently continuous, it remains to prove the continuity of the direct mapping $m \to \tilde{m}$. It is enough to show that the topology of $\tilde{M} \subset \mathscr{M}$ is the weak topology generated on $\tilde{M}$ by the functions $\hat{T}_x(\tilde{m})$, $x \in A$; this results from Theorem 5 G of [3], because that the functions $\hat{T}_x(\tilde{m})$ are continuous on $\tilde{M}$, vanish at infinity (with respect to $\tilde{M}$), separate the points of $\tilde{M}$ and do not all vanish at any point of $\tilde{M}$. (These facts are direct consequences of the preceding results).

In this manner, M can be considered identical with $\tilde{M}$; in what follows we consider $M \subset \mathscr{M}$ and $\hat{T}_x(m) = \hat{x}(m)$.

From now on, we suppose that A is *semi-simple*. Then we have the following

[2] In fact we use a slight extension of the Theorem 3, p. 195.

COROLLARY. (i) *If* $\hat{T}_1(m) = \hat{T}_2(m)$ *for* $m \in M$ *then* $T_1 = T_2$ (ii) $\mathscr{A}$ *is semi-simple.*

Proof. (ii) results from (i), and (i) results from the relation

$$\widehat{T_1 x}(m) = \widehat{T_1 T_x}(m) = \hat{T}_1(m)\hat{T}_x(m) = \hat{T}_2(m)\hat{T}_x(m) = \widehat{T_2 T_x}(m) = \widehat{Tx}(m)\,;$$

A being semi-simple, we conclude that $T_1 x = T_2 x$ for all $x \in A$, that is $T_1 = T_2$.

THEOREM 2. *A function* f *defined on* M *is a factor function of* $\hat{A}$ (that is $f\hat{x} = \hat{y} \in \hat{A}$ for all $\hat{x} \in \hat{A}$) *if and only if there is a* $T \in \mathscr{A}$, *such that* $f(m) = \hat{T}(m)$, $m \in M$.

Proof. If $f(m) = \hat{T}(m)$ then by (2)

$$f(m)\hat{x}(m) = \hat{T}(m)\hat{x}(m) = \widehat{TT_x}(m) = \hat{T}_{Tx}(m) = \widehat{Tx}(m) \in \hat{A}\,.$$

Conversely, if f is a factor function of $\hat{A}$, then the operator T_f defined by $T_f x = y$ where $\hat{y} = f\hat{x}$ is a linear closed operator defined on A, since A is semi-simple. Hence T_f is bounded. But $f\hat{x}\hat{y} = \hat{x}f\hat{y}$, so that $T_f \in \mathscr{A}$. Thus for all $m \in M$ we have

$$\hat{T}_f(m)\hat{x}(m) = \hat{T}_f(m)\hat{T}_x(m) = \widehat{T_f x}(m) = \hat{y}(m) = f(m)\hat{x}(m)\,,$$

for arbitrary $x \in A$. It follows that $\hat{T}_f(m) = f(m)$.

To understand the sense of these results, let us consider the case $A = L^1(G)$ where G is a locally compact abelian group which is not discrete. Let $M'(G)$ be the algebra of all bounded complex measures on G. Then, if $T_\mu x = \mu * x$, $x \in L^1(G)$ then T_μ is a linear bounded operator on A, and the mapping $\mu \to T_\mu$ is isomorphic and isometric on $M^1(G)$ into $\mathscr{A}$ [1]. Observing that $M = \hat{G}$ one may see easily that

$$\hat{T}_\mu(m) = \int_G \overline{(m, s)}d\mu(s)\,. \tag{3}$$

THEOREM 3. $\mathscr{A}$ *is isomorphic and isometric with* $M'(G)$.

Proof. It remains to show that for every $T \in \mathscr{A}$, there is a $\mu \in M^1(G)$ such that $T = T_\mu$. For the measures $\{\mu_\alpha\}$, where $d\mu_\alpha(s) = Tu_\alpha(\gamma)ds$, we have $\|\mu_\alpha\| \leqq \|T\|$. But the sphere of radius $\|T\|$ of $M'(G)$ (considered as the conjugate space of $K(G)$ or $C(G \cup \{\infty\})$) is weakly compact. Hence there is a $\mu \in M^1(G)$, which is a weak cluster point of $\{\mu_\alpha\}$. Consequently, by Lemma (i),

$$\hat{T}(m) = \lim \hat{T}u_\alpha(m) = \lim \int_G \overline{(m, s)}Tu_\alpha(s)ds = \int_G \overline{(m, s)}d\mu(s) = \hat{T}_\mu(m)\,.$$

By Corollary (i) we conclude that $T = T_{\mu}$.

Let us give some known corollaries of these results. From Theorems 1 and 3, we may obtain directly that every maximal ideal of $M^1(G)$ which does not contain $L^1(G)$ corresponds to a character of the group G, a fact established by H. Cartan and R. Godement [1]. In the same manner, Theorems 2,3 and (3) show that every factor function for the Fourier transform is the Fourier transform of a bounded measure (both the definition of a factor function and this result in the special case of the additive group of the real numbers are due to E. Hille [2] ; the extension to the general case of a locally compact abelian group was done by R.S. Edwards, Pacific J. Math. 1953 and independently by I. Cuculescu).

References

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