DISTAL TRANSFORMATION GROUPS

ROBERT ELLIS

Let X be a topological space and G a group of homeomorphisms of X onto itself. Then G is said to be *distal* if given any three points x, y, z in X and any filter $\mathscr{F}$ on G, then $x\mathscr{F}\to z$ and $y\mathscr{F}\to z$ implies that $x = y$. The above definition of distal is a topological variant of the one given in [2]; the two notions coincide when the underlying space X is compact.

This paper deals with two topics in the study of distal transformation groups. First, a recursive characterization of these groups is given in a general setting, and second it is shown that under suitable restrictions on X and G, distal is a property strong enough to imply equicontinuity of G. In order to make this statement precise a few definitions are needed. For a complete discussion of the following notions, the reader is referred to [2].

Let a, b be functions of X into X and let $x \in X$. Then xa will denote the image of x under a, and ab the composite function first a then b. Under the operation of composition X^X is a semigroup such that the maps $b\to ab$ $(b \in X^X)$ are continuous for all $a \in X^X$, and the maps $b\to ba$ $(b \in X^X)$ are continuous for all continuous functions a of X into X. The group G may be regarded as a subset of X^X and its closure T formed. One may also consider S the closure of G in the topology of uniform convergence on X. When X is compact, S is a topological group of homeomorphisms of X onto X but is in general not compact, whereas T is compact but is in general not a group. Hence in studying T instead of S the emphasis is on the algebraic rather than the topological structure.

A subset A of G is said to be *syndetic* if there exists a compact subset K of G such that $AK=G$. (If no topology is specified for G, then it is assumed to be provided with the discrete topology.) A point $x \in X$ is an *almost periodic point with respect to* G if given any neighborhood U of x, there exists a syndetic subset A of G such that $xA=[xa \mid a \in A]\subset U$. If every point of X is an almost periodic point with respect to G, then G is said to be *pointwise almost periodic.*

Let I be a set with cardinal number $\alpha>0$. Then each $g \in G$ induces a homeomorphism $(x_i \mid i \in I)\to(x_ig \mid i \in I)$ of X^α onto X^α which will also be referred to as g. Under this identification G becomes a group of

Received March 14, 1958. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF 18 (600) - 1116. Reproduction in whole or in part is permitted for any purpose of the United States Governmennt.

homeomorphisms of X^{α} onto X^{α}.

The characterization mentioned in the second paragraph is that if $\overline{xG}$ is compact for all $x \in X$ and X is Hausdorff, then G is distal if and only if G is pointwise almost periodic on X^{α} for all cardinals $\alpha > 0$.

The following lemma is probably well-known but the proof is included for the sake of completeness. For references to the literature see [3].

LEMMA 1. *Let S be a compact Hausdorff space with a semigroup structure such that the maps $s \to ts$ $(s \in S)$ are continuous for all $t \in S$. Then there exists an idempotent $u \in S$.*

Proof. Let $\mathscr{E}$ denote the class of non-null compact subsets E of S such that $E^2 \subset E$. Then $\mathscr{E} \neq \phi$ since $S \in \mathscr{E}$. If $\mathscr{E}$ is ordered by inclusion, an application of Zorn's lemma shows that there is a minimal element A in $\mathscr{E}$. If $r \in A$, then rA is a non-null compact subset of S such that $rA \in \mathscr{E}$ and $rA \subset A$. Hence $rA = A$ since A is minimal. Thus there exists $p \in A$ with $rp = r$. Define $L = [a \mid a \in A$ and $ra = r]$. Then $p \in L$, and L is a compact subset of A. Moreover k, $1 \in L$ imply that $rk1 = r1 = r$; that is $L^2 \subset L$. Thus $L \in \mathscr{E}$ and so $L = A$. Hence $r \in L$; that is $r^2 = r$. The proof is completed.

THEOREM 1. *Let X be a Hausdorff space and G a group of homeomorphisms of X onto X such that $\overline{xG}$ is compact for all $x \in X$. Then the following statements are pairwise equivalent.*

(1) *The closure T of G in X^X is a compact group.*

(2) *For every cardinal $\alpha > 0$, G is pointwise almost periodic on X^{α}.*

(3) *There exists a cardinal $\alpha > 1$ such that G is pointwise almost periodic on X^{α}.*

(4) *The group G is distal.*

Proof. (1) implies (2). Let α be a cardinal > 0, and let I be a set of cardinal α. Let $x = (x_i \mid i \in I) \in X^I$ and U a neighborhood of x. Then there exists a finite subset J of I and open subsets V_i $(i \in J)$ of X such that $x \in W = \times(W_i \mid i \in I) \subset U$ where $W_i = V_i$ $(i \in J)$ and $W_i = X$ $(i \in I - J)$. Let $N = [t \mid t \in T$ and $x_i t \in V_i$ $(i \in J)]$. Then N is an open neighborhood of the identity e of T. Let $t \in T$. Since the map $r \to rs$ $(r \in T)$ of T onto T is a homeomorphism for all $s \in T$, $t^{-1}N$ is a non-null open subset of T. Hence there exists $g \in G$ such that $g \in t^{-1}N$; that is $t \in Ng^{-1} \subset NG$. Thus $T \subset NG$, and so $T \subset NK$ for some finite subset K of G. Since G is a subgroup of T and $K \subset G$, $G \subset (N \cap G)K$. Thus $A = N \cap G$ is a syndetic subset of G with $xA \subset U$.

That (2) implies (3) is clear.

(3) implies (4). Let x, y, $z \in X$ and let $\mathscr{F}$ be a filter on G such

that $x\mathscr{F}\to z$ and $y\mathscr{F}\to z$. Let $\mathfrak{a}$ be a cardinal >1, I a set of cardinal $\mathfrak{a}$, and i and j two distinct elements of I. Let $w=(w_k|k\in I)\in X^{\mathfrak{a}}$ such that $w_k=x$, $k\neq j$ and $w_j=y$. Then $w\mathscr{F}\to u=(u_k|k\in I)$ where $u_k=z(k\in I)$. Hence $u\in \overline{wG}\subset \times(\overline{w_kG}|k\in I)$. Thus $u\in\overline{wG}$ which is a compact set on which G is pointwise almost periodic. Therefore by [2 ; 4.07] $w\in\overline{uG}$. Consequently there exists a filter $\mathscr{G}$ on G such that $u\mathscr{G}\to w$; that is $z\mathscr{G}\to x$ and $z\mathscr{G}\to y$. Thus $x=y$.

(4) implies (1). Since $T\subset \times(\overline{xG}|x\in X)$, T is a compact subset of X^X. That $T^2\subset T$ follows directly from the definition of T and the fact that the maps $t\to st\,(t\in T)$ and $t\to tg\,(t\in T)$ of T into T are continuous for all $s\in T$ and $g\in G$. It remains to be shown that given $t\in T$ then it is invertible and that $t^{-1}\in T$.

To this end let $t\in T$. Then tT is a compact subset of T such that $(tT)(tT)\subset tTT\subset tT$. Hence by Lemma 1 there exists $u\in tT$ such that $u^2=u$. Let $x\in X$ and $\mathscr{F}$ a filter on G such that $\mathscr{F}\to u$. Let $y=xu$. Then $x\mathscr{F}\to xu=y$, and $y\mathscr{F}\to yu=xu^2=xu=y$. Hence $x=y$ since G is assumed distal. Thus $xu=x\,(x\in X)$; that is $u=e$ the identity of T.

Since $e\in tT$, there exists $s\in T$ such that $ts=e$. A similar argument applied to s instead of t produces $r\in T$ with $sr=e$. Hence $t=te=tsr=r$; in other words $ts=st=e$. The proof is completed.

REMARK. Let X be a Hausdorff space, and let G be a distal group of homeomorphisms of X onto X such that $\overline{xG}$ is compact for all $x\in X$. Then G is pointwise almost periodic on X.

A topological group G is said to be *generative* provided that G is abelian and is generated by some compact neighborhood of the identity. The remainder of this paper will be concerned with the transformation group (X, G, π) where X is a Hausdorff space and the group G is generative.

THEOREM 2. *Let X be locally compact zero-dimensional, let G be distal, and let $\overline{xG}$ be compact for all $x\in X$. Then G is equicontinuous.*

Proof. By Theorem 1. G is pointwise almost periodic on $X\times X$. Hence G is locally weakly almost periodic on $X\times X$ [2 ; 7.07], and so $[\overline{(x, y)G}|x, y\in X]$ is a star closed decomposition of $X\times X$[2 ; 4.16]. Let $x\in X$ and α an index on X. Then α is a neighborhood of $\overline{(x, x)G}$, and therefore there exists a neighborhood V of x such that $(V\times V)G\subset\alpha$; that is G is equicontinuous at x. The proof is completed.

The group G is said to be *regularly almost periodic at the point* $x\in X$ if given any neighborhood U of x there exists a syndetic subgroup H of G with $xH\subset U$. If G is regularly almost periodic at x for all

$x \in X$, then G is called *pointwise regularly almost periodic on* X.

THEOREM 3. *Let G be distal and let $\overline{xG}$ be compact zero-dimensional for all $x \in X$. Then G is pointwise regularly almost periodic on X.*

Proof. Let $x \in X$, U a neighberhood of x, and consider the action of G on the invariant subset $Y=\overline{xG}$ of X. Let $G_x=[g \mid g \in G \text{ and } xg=x]$. Then $xhg=xgh=xh(h \in G, g \in G_x)$. Hence by continuity $yg=y(y \in Y, g \in G_x)$. For $k \in K=G/G_x$ and $y \in Y$ set $yk=yg$ where $k=gG_x$. Then K may be regarded as a group of homeomorphisms of Y onto Y such that $\overline{xK}=Y$. By Theorem 2, K is equicontinuous, therefore $T=$closure of K in Y^Y is a group of homeomorphisms of Y onto Y. Hence T is a topological group.

Let $t, s \in T$ such that $xt=xs$. Then since all the maps involved are continuous and K is commutative, $xkt=xtk=xsk=xks\,(k \in K)$, hence $yt=ys(y \in Y)$, i.e. $t=s$. Consequently, the map $t \to xt(t \in T)$ of T onto Y is continuous and one-to-one, hence a homeomorphism. Thus T is compact zero-dimensional.

Now let $V=U \cap Y$. Then $N=[t \mid t \in T \text{ and } xt \in V]$ is a neighborhood of the identity of T. Hence there is an open closed invariant subgroup L of T with $L \subset N$. Since L is open, there exists a finite subset F of K with $T=LF$. Set $M=K \cap L$. Then $K=MF$ and M is a syndetic subgroup of K, such that $xM \subset V$. Consequently H, the inverse image of M under the projection of K onto G/G_x is the required syndetic subgroup of G. The proof if completed.

THEOREM 4. *Let X be locally compact metric, let G be distal, let $\overline{xG}$ be compact zero-dimensional for all $x \in X$, and suppose G contains only countably many subgroups. Then the set of points R at which G is equicontinuous is a residual subset of X.*

As an example of the type of group being considered in Theorem 4, let f be a homeomorphism of X onto X and set $G=[f^n \mid n=0, \pm 1, \cdots]$.

Proof. Let $[H_n \mid n=1, 2, \cdots]$ be the set of syndetic subgroups of G, and let α be a metric on X. For m, n positive integers set $E(n, m)=[x \mid xH_n \subset S(x, 1/m)]$ where $S(x, 1/m)=[y \mid \alpha(x, y) \leqq 1/m]$, Then $E(n, m)$ is a closed subset of X for all positive integers n, m, and $\cup[E(n, m) \mid n=1, \cdots]=X$ by Theorem 3. Hence $E(m)=\cup[\text{int } E(n, m) \mid n=1, \cdots]$ is an everywhere dense open subset of X. Let $E=\cap[E(m) \mid m=1, \cdots]$. Then E is a residual subset of X. Moreover, from the definition of E, it follows that given any neighborhood U of $x \in E$ there exist a neighborhood V of x and a syndetic subgroup A of G such that $VA \subset U$. Assume U compact and let K be a compact subset of G such that $AK=G$. Then

$(V\times V)G=(V\times V)AK\subset(U\times U)K\subset(U\times U)G$ shows that $\overline{(V\times V)G}$ is compact and that $\cap[\overline{(V\times V)G}|V$ a neighborhood of x contained in $U]=$ $\cap[(V\times V)G|V$ a neighborhood of x contained in $U]$. The proof that G is equicontinuous at x is now completed as in Lemma 1 [1]. Thus $E\subset R$.

The theorems in the second part of the paper suggest the following problems:

(1) Can the assumption that G is generative be dropped in any of these theorems?

(2) To what extent can the condition of zero-dimensionality be relaxed in Theorem 2?

The example [1] of a ring of concentric circles rotating at different rates about their common center shows that zero-dimensionality must be replaced by some other condition i.e. cannot be dispensed with entirely even if X is compact. It is conjectured that a sufficient condition would be that X be minimal under G; that is that $\overline{xG}=X$ for all $x\in X$. If this were true then in the general case where all that is assumed is that $\overline{xG}$ is compact for all $x\in X$, the group G would be distal if and only if G is an equicontinuous family of maps of $\overline{xG}$ onto $\overline{xG}$ for all $x\in X$.

The notion of distal was considered by Hilbert see [4] in an attempt to give a topological characterization of the concept of a rigid group of motions. According to the above conjecture and Theorem 1 this would be adequate if X were compact and there existed a point $x\in X$ with $\overline{xG}=X$.

References

1. W. H. Gottschalk, *Characterization of almost periodic transformation groups*, Proc. Amer. Math. Soc. **7**(1956), 709–712.

2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloquium Publications, **36**(1955).

3. A. D. Wallace, *The structure of topological semigrous*. Bull Amer. Math. Soc. vol **61** No. 2 1955.

4. L. Zippin, *Transformation groups*, Lectures in Topology, The University of Michigan Conference of 1940.

University of Pennsylvania