A GENERAL SOLUTION FOR A CLASS OF APPROXIMATION PROBLEMS

ANNETTE SINCLAIR

1. Introduction. This paper generalizes a class of theorems showing the existence of an approximating function which may be required to satisfy certain auxiliary conditions.

Various theorems in analytic function theory which prove the existence of a function fulfilling specified conditions in an open set R have been proved by using a method of the following type. The set R is covered by an increasing sequence of sets $\{R_i\}$. Then the existence of a convergent sequence of functions $\{f_i(z)\}$ is shown such that each $f_n(z)$ behaves properly in R_n and such that $\{f_i(z)\}$ converges to a function satisfying the required conditions everywhere in R. Examples of theorems in which such a method of proof can be applied are furnished by the Mittag-Leffler Theorem, the Carleman Approximation Theorem [1], some rate of growth theorems proved by P. W. Ketchum [2], and the author's generalization of Runge's Theorem [5]. W. Kaplan considered certain problems of this type and remarked, [1], that Brelot has pointed out that this type of proof is valid for approximation to a function $Q(x_1, x_2 \cdots, x_n)$ continuous for all $(x_1, x_2, \cdots, x_n)$ by a function $u(x_1, x_2, \cdots, x_{n+1})$ harmonic for all $(x_1, x_2, \cdots, x_{n+1})$.

The present paper attempts to give an abstract solution for this general class of problems. Examples are also given of some new results obtainable by applying Theorem 1 and fundamental approximation theorems.

In Theorem 3 approximation by an analytic function is considered on a point set S consisting of an infinite number of circular discs tangent on the real axis. It is shown that a function $w(z)$ analytic at interior points of S and continuous on the closure of any finite number of the circular regions—hence, continuous at their points of tangency—can be approximated by an integral function $f(z)$. Moreover, $f(z)$ can be chosen so that the approximation is stronger than uniform approximation—so that corresponding to any $\{\varepsilon_i\}$ there exists $f(z)$ such that

$$|f(z) - w(z)| < \varepsilon_i \text{ on } S_i,$$

where S_i is the ith. circular region.

Theorem 2 combines some previously obtained results [5] by requiring that certain auxiliary conditions be satisfied simultaneously.

Received October 25, 1956.

Theorem 1 can sometimes also be used to show that when an approximation is known to be impossible in the infinite case that analogous results cannot hold in the finite case. An example of this usage is given.

In Part II a topological abstraction is made of Theorem 1 in which the sets of functions are topologized and the system so obtained interpreted as an inverse mapping system. It is then shown that Theorem 1 can be regarded as a special case of Theorem 5.

PART I

2. Fundamental theorem. Let R be an open subset of a topological space. A sequence of sets $\{R_i\}$ which satisfy the following conditions will be called an *increasing sequence of R-covering sets.*

(1) $R_i \subset R$;

(2) $\overline{R}_i$ *is interior to* R_{i+1};

(3) $\bigcup_{i=1}^{\infty} R_i = R$.

$W_1 \cup W_2 \cup \cdots$ such that $W_k \cap W_m = \phi$ for $k \neq m$ is said to be a *decomposition* of a set S if $S = W_1 \cup W_2 \cup \cdots$. An *R-covering sequence* $\{R_i\}$ for R *and a decomposition* $W_1 \cup W_2 \cup \cdots$ of a set $S \subset R$ are said to *correspond* if, for every n, $W_n \subset R_n$, but $W_{n+1} \cap R_n = \phi$.

For a given set $S \subset R$ suppose that an increasing sequence $\{R_i\}$ of R-covering sets and a decomposition $W_1 \cup W_2 \cup \cdots$ of S correspond. Let there be defined classes of functions $\mathscr{W}_n$ and $\mathscr{R}_n$ transforming W_n and R_n respectively into the complex plane, $n = 1, 2, \cdots$. Suppose that each function of $\mathscr{R}_n$ defines a function of $\mathscr{R}_{n-1}$, $n = 2, 3, \cdots$.

THEOREM 1. *Let* $S, R, R_n, W_n, \mathscr{R}_n$, *and* $\mathscr{W}_n$, $n = 1, 2, \cdots$, *be defined as above. Suppose that*

(1) *If* $\{g_i(X)\}$ *is a sequence of functions of* $\mathscr{R}_{n+1}$ *which converges on* R_{n+1} *and uniformly on any closed subset of* R_{n+1}, $\lim_{i\to\infty} g_i(X)$ *defines a function of* $\mathscr{R}_n$;

and (2) *Any function defined on* R_n *by an arbitrary function of the class* $\mathscr{R}_n$ *and on* W_{n+1} *by a function of* $\mathscr{W}_{n+1}$ *can be uniformly approximated arbitrarily closely on* $R_n \cup W_{n+1}$ *by a function of* $\mathscr{R}_{n+1}$, $n = 0, 1, 2, \cdots$ (*where* R_0 *is the null set.*).

Let $w(X)$ *be a function defined on* S *in such a way as to determine a function of* $\mathscr{W}_i$ *for each* i. *Then, corresponding to any* $\{\varepsilon_i\}$, *there exists* $r(X)$ *defined on* R *which determines a function of* $\mathscr{R}_n$ *for each* n *such that*

$$| r(X) - w(X) | < \varepsilon_i \text{ when } X \in W_i, \qquad i = 1, 2, \cdots,$$

Proof. Suppose $\{\varepsilon_i\}$ and $w(X)$ preassigned. When n is taken as 0, (2) implies the existence of $r_1(X)$ of $\mathscr{R}_1$ such that

$$|r_1(X) - w(X)| < \varepsilon_1/2^2 \qquad \text{when } X \in W_1 .$$

In general, for $n = 1, 2, \cdots$, choose $r_n(X)$ of $\mathscr{R}_n$ so that

$$|r_n(X) - r_{n-1}(X)| < \frac{\varepsilon^{(n)}}{2^{n+1}} \qquad \text{on } R_{n-1}$$

and

$$|r_n(X) - w(X)| < \frac{\varepsilon^{(n)}}{2^{n+1}} \qquad \text{on } W_n ,$$

where $\varepsilon^{(n)} = \min_\infty \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$.

Since $\{r_i(X)\}_{i=k+1}$ converges in R_{k+1}, for arbitrary k, and uniformly on any closed subset of R_{k+1}, it follows from (1) that $\lim_{i\to\infty} r_i(X) = r(X)$ defines a function of $\mathscr{R}_k$ for $k = 1, 2, \cdots$.

It remains to show that $r(X)$ satisfies the assigned approximation conditions. For any k, there exists $m > k$ so that

$$|r(X) - r_m(X)| < \frac{\varepsilon^{(k)}}{2^k} \qquad \text{when } X \in \overline{R}_k .$$

Now

$$\begin{aligned} |r(X) - w(X)| &\le |r(X) - r_m(X)| + |r_m(X) - w(X)| \\ \le |r(X) - r_m(X)| &+ \sum_{j=k+1}^{m} |r_j(X) - r_{j-1}(X)| + |r_k(X) - w(X)| \\ < \frac{\varepsilon^{(k)}}{2^k} &+ \sum_{j=k+1}^{m} \frac{\varepsilon^{(j)}}{2^{j+1}} + \frac{\varepsilon^{(k)}}{2^{k+1}} \qquad \text{when } X \in W_k \subset R_k . \end{aligned}$$

Thus, $|r(X) - w(X)| < \varepsilon^{(k)} \leqq \varepsilon_k$ when $X \in W_k$. This completes the proof of the theorem.

3. Applications to specific problems. In Theorem 2 we consider approximation on a Q-set of the complex plane having an infinite number of components. A set S is a *Q-set* if its component are closed and its set of *sequential limit points* lie in $C(S)$, the complement of S. (*A sequential limit point of* S is a limit point of a set of points chosen one from each component of S. We note incidentally that a Q-set in the complex plane has at most a denumerable number of components and that its set of sequential limit points may separate the plane [5].)

A set in $C(S)$ is called a *$B^*(S)$-set* if it contains the set B of sequential limit points of S and exactly one point of each component $I_k(S)$ of $C(S)$ such that $I_k(S) \cap B = \phi$.

The author has shown [5] that if S is any Q-set of the complex plane and B^* any $B^*(S)$-set there exists an increasing sequence $\{R_i\}$ of closed covering sets for $C(B)$ such that

(1) If S_t is any component of S and if $S_t \cap R_n \neq \phi$, then $S_t \subset R_n$;

(2) If $I_r(R_n, S)$ is any component of $C(R_n \cup S)$,

$$I_r(R_n, S) \cap B^* \neq \phi .$$

When we set $W_i = S \cap R_i \cap C(R_{i-1})$, we obtain a decomposition $W_1 \cup W_2 \cup \cdots$ of S which corresponds to the increasing sequence $\{R_i\}$ of covering sets for $C(B)$.

A function is *meromorphic on a set* if it is single-valued and analytic in a neighborhood of each point of the set except for poles.

THEOREM 2. *Suppose S is a Q-set, B its set of sequential limit points, and B^* any $B^*(S)$-set. Let $\{R_i\}$ be an increasing sequence of covering sets for $C(B)$ as described above which determines the corresponding decomposition $W_1 \cup W_2 \cup \cdots$ of S. Suppose $w(z)$ is meromorphic on S and that I denotes the set of points of S at which $w(z)$ has poles. Then, corresponding to any sequence $\{\varepsilon_i\}$ of positive constants, there exists $r(z)$ meromorphic in $C(B)$ and analytic in $C(B^* \cup I)$ such that*

$$|r(z) - w(z)| < \varepsilon_i \text{ when } z \in (W_i - I), \qquad i = 1, 2, \cdots$$

It can be required that

(1) *The poles of $r(z)$ at points of I have the same principal parts as $w(z)$;*

and (2) *If K is an isolated interior subset of S such that $K \cap I = \phi$, $r(z)$ can be chosen so that $r(k) = w(k)$ at each point k of K. If B^* has no limit point on S, $r(z)$ can be required to have the same multiplicities at points of K as $w(z)$.*

Proof. Define $\mathscr{W}_i$ as the set of those functions meromorphic on W_i and analytic on $(W_i - I)$ which have poles with the same principal parts as $w(z)$ on $(I \cap W_i)$ and k-points with the same multiplicities as $w(z)$ on $(K \cap W_i)$. In $\mathscr{R}_i$ include those functions meromorphic on R_i and analytic in $R_i - (I \cup B^*)$ which have poles with the same principal parts as $w(z)$ on $(I \cap R_i)$ and k-points with the same multiplicities as $w(z)$ on $(K \cap R_i)$, also those functions which are identically constant on a component of R_i which contains no point of I.

Suppose $\{g_i(z)\}$ is a sequence of functions of $\mathscr{R}_{n+1}$ which converges in R_{n+1} and uniformly on any closed subset of R_{n+1} (where any points of $(I \cup B^*)$ are deleted from a closed subset which contains them). Then $\lim g_i(z)$ is meromorphic on R_n and analytic in $R_n - (I \cup B^*)$ with poles and k-points identical with those of $w(z)$ at points of I and K, except that $\lim g_i(z)$ may be identically constant on a component of R_n which contains no points of I. Thus, $\lim g_i(z) \in \mathscr{R}_n$ and (1) of Theorem 1 is satisfied.

Before applying Theorem 1 it remains to show that for any $g(z)$ of $\mathscr{R}_n$ and $v(z)$ of $\mathscr{W}_{n+1}$, corresponding to arbitrary $\varepsilon > 0$, there exists $f(z)$ of $\mathscr{R}_{n+1}$ such that

$$|f(z) - g(z)| < \varepsilon \text{ when } z \in R_n,$$
$$\text{and } |f(z) - v(z)| < \varepsilon \text{ when } z \in W_{n+1}.$$

This follows from Walsh's generalization of Runge's Theorem [7, p. 15] and from another theorem of Walsh [7, p. 313] after it is noted that a finite number of poles on $R_n \cup W_{n+1}$ cause no real difficulty. Just apply the general Mittag-Leffler Theorem [4] to show the existence of a function $h(z)$ meromorphic in $C(B)$ whose poles coincide with those of $g(z)$ and $v(z)$ on R_n and W_{n+1} respectively with the same principal parts. Then define $F(z) = \begin{cases} g(z) - h(z) \text{ on } R_n. \\ v(z) - h(z) \text{ on } W_{n+1}. \end{cases}$

Since $F(z)$ is analytic on $R_n \cup W_{n+1}$, by Walsh's generalization of Runge's Theorem [7, p. 15], there exists a rational function $k(z)$ whose poles lie in B^* such that $|F(z) - k(z)| < \varepsilon$ when $z \in (R_n \cup W_{n+1})$. Another theorem by Walsh [7, p. 313] implies that $k(z)$ can be chosen so that $k(z) = F(z)$ at points of K and so that $k(z)$ has the same multiplicities at these points as $F(z)$. Set $f(z) = h(z) + k(z)$.

Now $f(z)$ is meromorphic on R_{n+1} and its poles on R_{n+1} lie at points of $I \cup (R_{n+1} \cap B^*)$ with those on I having the same principal parts as $h(z)$, hence as $g(z)$ or $v(z)$, and so the same as $w(z)$. Also

$$\begin{aligned} |g(z) - f(z)| &= |[g(z) - h(z)] - k(z)| \\ &= |F(z) - k(z)| < \varepsilon \text{ when } z \in R_n \end{aligned}$$

and similarly $|v(z) - f(z)| = |F(z) - k(z)| < \varepsilon$ when $z \in W_{n+1}$. Since $k(z) = F(z)$ at points of K,

$$\begin{aligned} f(z) &= h(z) + k(z) = h(z) + F(z) = h(z) + g(z) - h(z) \\ &= g(z) \text{ on } R_n \cap K \end{aligned}$$

and, similarly, $v(z)$ on $W_{n+1} \cap K$.

This completes the proof that the hypothesis of Theorem 1 is satisfied. Hence, by Theorem 1, there is a function $r(z)$ defined on $C(B)$ (where ∞ is allowed as a functional value) which determines a function of $\mathscr{R}_n$ for each n such that

$$|r(z) - w(z)| < \varepsilon_i \text{ when } z \in W_i, \qquad i = 1, 2, \cdots .$$

Thus, $r(z)$ is meromorphic in $C(B)$, analytic in $C(I \cup B^*)$, and has poles and k-points of $w(z)$ on S as specified and also satisfies the required approximation condition. (*In* general, $r(z)$ is not identically constant on a component of $C(B)$.)

In Theorem 3 S consists of circular discs tangent on the real axis. More precisely, let $W_i = \{z/|z - i| \leq 1/2\}$, except that $z = i - 1/2$ is deleted, and define S as $\bigcup_{i=1}^{\infty} W_i$. Set $R_i = \{z/|z| \leq i + 1/2\}$ and let R be the finite plane. Then $\{R_i\}$ and the decomposition $W_1 \cup W_2 \cup \cdots$ of S correspond.

THEOREM 3. *Suppose S defined as in the preceding paragraph. Let $w(z)$ be any function analytic at interior points of S and continuous on the boundary except at infinity. Then, corresponding to any $\{\varepsilon_i\}$, there exists an integral function $r(z)$, such that $|r(z) - w(z)| < \varepsilon_i$ when $z \in W_i$ and $r(i + 1/2) = w(i + 1/2), i = 1, 2, \cdots$.*

Proof. Let $\mathscr{R}_i$ be the set of all functions $f(z)$ analytic on R_i such that $f(k + 1/2) = w(k + 1/2)$ for $k = 1, 2, \cdots, i$. Let $\mathscr{W}_i$ be the set of all functions $f(z)$ analytic at interior points of W_i and continuous on W_i such that $f(i + 1/2) = w(i + 1/2)$ and $\lim_{z \to i} f(z - 1/2) = w(i - 1/2)$, $(z - 1/2) \in W_i \; i = 1, 2, \cdots$.

If $\{g_i(z)\}$, where $g_i(z)$ is a member of $\mathscr{R}_{n+1}$, converges on R_{n+1}, uniformly on any closed subset of R_{n+1}, $\lim g_i(z)$ gives a function of $\mathscr{R}_n$,

By a theorem of Walsh [7, p. 47] a function $g(z)$ analytic interior to and continuous on a closed set C which does not separate the plane and which is bounded by a finite number of Jordan curves, as is the case if $C = R_n \cup W_{n+1}$, can be uniformly approximated on C by a polynomial $p(z)$. Then by another theorem of Walsh [7, p. 310], $p(z)$ can be chosen so that $p(k + 1/2) = g(k + 1/2)$, $k = 1, 2, \cdots, n + 1$. If $g(k + 1/2) = w(k + 1/2)$ then $p(z) \in \mathscr{R}_{n+1}$. Thus, the hypothesis of Theorem 1 is satisfied and the required conclusion follows.

The next theorem is an extension of the Carleman approximation theorem in that values of ahe approximating function are preassigned at certain points.

THEOREM 4. (*Carleman Approximation Theorem*). *Let $w(x)$ be a continuous complex-valued function of x for $-\infty < x < \infty$. Then, corresponding to any $\{\varepsilon_i\}$, there exists an integral function $f(z)$ such that*

$$|f(x) - w(x)| < \varepsilon_i \text{ when } i - 1 < |x| \leq i, \qquad i = 1, 2, \cdots,$$

and such that $f(i) = w(i), i = \pm 1, \pm 2, \cdots$.

Proof. The proof is like that of Theorem 3 when W_i is defined as $\{x/i - 1 < |x| \leq i\}$ (except that W_1 also includes the origin); R_i as $\{z/|z| \leq i\}$; R as the finite plane; $\mathscr{W}_i$ as those functions continuous on W_i such that $f(\pm i) = w(\pm i)$ and $\lim_{x \to (\pm i \mp 1)} f(x) = w(\pm i \mp 1)$; and $\mathscr{R}_i$ as those functions $f(z)$ analytic on R_i such that $f(k) = w(k)$, $k = \pm 1, \pm 2, \cdots, \pm i$.

Theorem 1 can sometimes be used to show that certain requirements on the approximating function cannot, in general, be made, even when the approximation is on a set having only a finite number of components. Next an application of this type is indicated.

When approximating a function analytic and simple on each component of a closed set C by a function $f(z)$ analytic in a preassigned finite region D containing C, one cannot, in general, require that $f(z)$ be simple in D. To verify this we consider a Q-set S whose components are simply connected and which has infinity as its only sequential limit point. Suppose $\{R_i\}$ is an increasing sequence of R-covering sets, as described for Theorem 2, which gives the corresponding decomposition $W_1 \cup W_2 \cup \cdots$ of S. Let $\mathscr{R}_i$ (and $\mathscr{W}_i$) consist of all functions analytic and simple on $R_i(W_i)$, also all constants. We note that (1) of Theorem 1 is satisfied [6, p, 203]. If (2) were also satisfied, Theorem 1 would imply that arbitrary $w(z)$ simple on S could be approximated on S by a function simple in the whole finite plane. Since $w(z)$ can be chosen so that $f(z)$ would necessarily have an essential singularity at ∞, this does not hold. We conclude that (2) is not, in general, satisfied.

Theorems 2, 3, and 4 and the illustration just stated are examples of some of the applications which can be made of Theorem 1.

PART II

4. Topological abstraction of Theorem 1. Theorem 1 can be interpreted as a density result for a Cartesian product space. The author's original version treated the $\mathscr{R}_i$'s of Theorem 1 with the respective topologies induced by the metrics

$$d_i(f, g) = \sup_{x \in R_i} |f(X) - g(X)|$$

as a *nested sequence of spaces*. The interpretation given in Theorem 5 as an inverse mapping system was suggested by Prof. Hans Samelson of the University of Michigan. In addition to having the advantage of conforming to convention, this formulation applies to classes of functions other than analytic functions.

If $\{W_i\}$ is any sequence of topological spaces, W^∞ denotes the Cartesian product space $W_1 \times W_2 \times \cdots$. We shall be concerned with the *box topology* for W^∞ in which a neighborhood of $w = (w_1, w_2, \cdots)$ is defined as $\mathrm{N}_{w_1}(W_1) \times \mathrm{N}_{w_2}(W_2) \times \cdots$.

If $\{R_i\}_{i=1}^{\infty}$ is a denumerable system of T_2-spaces and if for $n = 2,3,\cdots$, there is defined a continuous transformation $\prod_{n-1}^{n}$ of R_n into R_{n-1}, the system $\sum = \{R_i, \prod^i\}$ of the R_i's and $\prod$'s is an *inverse mapping system*, [3, p. 31]. The subset R of $R^\infty = R_1 \times R_2 \times \cdots$ of all those points $x = \{x_i\}$ such that $\prod_i^{i+1} x_{i+1} = x_i$ is called the *limit space of the inverse mapping system* $\sum$.

In Theorem 5 we suppose that $R_1, R_2, \cdots$ are given sets and that for each i, and arbitrary points $p, q \in R_i$, there is defined a metric $d_i(p, q)$, where ∞ is allowed as a possible value. Then R_i with the neighborhood system induced by $d_i(p, q)$ is a T_2-space. If, for $i = 2,3,\cdots$, a transformation $\prod_{i-1}^{i}$ of R_i into R_{i-1} is defined which is a contraction (that is, $d_{i-1}(\prod_{i-1}^{i} p, \prod_{i-1}^{i} q) \leq d_i(p, q)$), then the $\prod$'s are continuous and $\{R_i, \prod_{i-1}^{i}\}$ is an inverse mapping system.

Before stating Theorem 5 we note that the R_i's of this theorem are analogous to the $\mathscr{R}_i$'s and the W_i's to the $\mathscr{W}_i$'s of Theorem 1.

THEOREM 5. *Let* $\{W_i\}_{i=1}^{\infty}$ *be a system of topological spaces and let* $\{R_i, \prod^i\}$ *be an inverse mapping system as described in the preceding paragraphs. Suppose that for each* i *there is defined a continuous transformation* f_i *which maps* R_i *into* W_i. *Suppose also that the following conditions are satisfied:*

(1) *If* $\{p_j^{(n)}\}_{j=1}^{\infty}$ *is a Cauchy sequence in* R_n, *its image* $\{\prod_{n-1}^{n} p_j^{(n)}\}_{j=1}^{\infty}$ *is convergent in* R_{n-1};

(2) $f_1(R_1)$ *is dense in* W_1 *and, when* $n > 1$, $\prod_{n-1}^{n} \times f_n(R_n)$ *is dense in* $R_{n-1} \times W_n$.

Then under the transformation $\{\times f_i\}$ *the image of the limit space* R *of the inverse mapping system* $\sum$ *is dense in* W^∞ *by the box topology.*

Proof. Let $w = (w_1, w_2, \cdots)$ be any point of W^∞ and let $N_w = N_{w_1}(W_1) \times N_{w_2}(W_2) \times \cdots$ be an arbitrary neighborhood of w.

Since $f_1(R_1)$ is dense in W_1, there is a point $r_1 \in R_1$ such that $f_1(r_1) \in N_{w_1}(W_1)$. There exists $N_{r_1}(R_1) \subset N_{r_1}^{(1)}(R_1)$, where $N_{r_1}^{(\alpha)} = \{p \in R_1 / d_1(p, r_1) < \alpha\}$. Since f_1 is continuous and R_1 is regular, we can suppose $N_{r_1}(R_1)$ chosen so that $f_1(\bar{N}_{r_1}) \subset N_{w_1}(W_1)$.

In general, since $\prod_{n-1}^{n} \times f_n(R_n)$ is dense in $R_{n-1} \times W_n$, there exists $r_n \in R_n$ so that $\prod_{n-1}^{n} \times f_n(r_n) \in N_{r_{n-1}}(R_{n-1}) \times N_{w_n}(W_n)$. There exists $N_{r_n}(R_n) \subset N_{r_n}^{(1/2^{n-1})}(R_n)$. Since f_n and $\prod_{n-1}^{n}$ are continuous and R_n is regular, we can suppose $N_{r_n}(R_n)$ chosen so that $N_{r_n}(R_n) \subset N_{r_n}^{(1/2^{n-1})}(R_n)$ and so that

$$\prod_{n-1}^{n} \times f_n(\bar{N}_{r_n}) \subset N_{r_{n-1}}(R_{n-1}) \times N_{w_n}(W_n) .$$

The sequence $\{\prod_n^{n+i} r_{n+i}\}_{i=0}^{\infty}$, where $\prod_n^{n+i} = \prod_n^{n+1} \cdots \prod_{n+i-1}^{n+i}$, is a Cauchy sequence in R_n. For, corresponding to any $\varepsilon > 0$, there exists $m > n$ so that $1/(2^{m-2}) < \varepsilon$; then

$$d_n(\prod_n^{m+k} r_{m+k}, \prod_n^m r_m) \leq \sum_{i=0}^{k-1} d_n(\prod_n^{m+i+1} r_{m+i+1}, \prod_n^{m+i} r_{m+i})$$
$$\leq \sum_{i=0}^{k-1} d_{m+i}(\prod_{m+i}^{m+i+1} r_{m+i+1}, r_{m+i}) < \sum_{i=0}^{k-1} \frac{1}{2^{m+i-1}} < \frac{1}{2^{m-2}} < \varepsilon .$$

The first inequality follows from the triangle inequality, the second from the fact that the $\prod$'s are contractions, and the third holds since $\prod_{m+i}^{m+i+1} r_{m+i+1} \in N_{r_{m+i}}(R_{m+1}) \subset N_{r_{m+i}}^{((1/2)m+i-1)}$.

By (1) the image of the Cauchy sequence above is convergent in R_{n-1}. Hence, we let $r^{(n-1)}$ denote $\lim_{i\to\infty} \prod_{n-1}^{n+i} r_{n+i}$ in R_{n-1}. Now $r^{(n-1)} \in \bar{N}_{r_{n-1}}(R_{n-1})$ and $f_{n-1}(\bar{N}_{r_{n-1}}(R_{n-1})) \subset N_{w_{n-1}}(W_{n-1})$. Hence, $f_{n-1} r^{(n-1)} \in N_{w_{n-1}}(W_{n-1})$.

To complete the proof of the theorem it is sufficient to show that $\{r^{(i-1)}\}_{i=2}^{\infty}$ belongs to the limit space, that is, that $r^{(n-2)} = \prod_{n-2}^{n-1} r^{(n-1)}$ for $n = 3, 4, \cdots$. Since $\{\prod_{n-1}^{n+i} r_{n+i}\}$ converges to $r^{(n-1)}$ in R_{n-i}, corresponding to any $\delta > 0$, there exists k such that $i > k$ implies $d_{n-1}(\prod_{n-1}^{n+i} r_{n+1}, r^{(n-1)}) < \delta$. Then, since the π's are contractions, $d_{n-2}(\prod_{n-2}^{n+i} r_{n+i}, \prod_{n-2}^{n-1} r^{(n-1)}) < \delta$ for all $i > k$. Now $\lim \prod_{n-2}^{n+i} r_{n+i}$ is unique in R_{n-2}, and so $\prod_{n-2}^{n-1} r^{(n-1)} = \lim_{i\to\infty} \prod_{n-2}^{n+i} r_{n+i} = r^{(n-2)}$. This completes the proof of Theorem 5.

If a function of $\mathscr{R}_n$ defines a function of $\mathscr{W}_n$, $n = 1, 2, \cdots$, Theorem 1 can be obtained from Theorem 5. Since each function of $\mathscr{R}_n$ in Theorem 1 defines a function of $\mathscr{R}_{n-1}$ (and in the case just specified, also $\mathscr{W}_n$), transformations $\prod_{n-1}^n$ and f_n are determined of $\mathscr{R}_n$ into $\mathscr{R}_{n-1}$ and $\mathscr{W}_n$. Let us define a metric $d_n(f, g)$ for each $\mathscr{R}_n$ (also $\mathscr{W}_n$) as $\sup_{X \in R_n (\text{or } W_n)} |f(X) - g(X)|$. Thus, T_2-topologies are determined for $\mathscr{R}_n$ and $\mathscr{W}_n$ respectively. If $f, g \in \mathscr{R}_n$ then,

$$\sup_{X \in R_{n-j} \text{ or } W_n} |f(X) - g(X)| \leq \sup_{X \in R_n} |f(X) - g(X)|$$

and so $\prod_{n-1}^n$ and f_n are contractions and hence continuous. We note that $\{\mathscr{R}_i, \prod_{i-1}^i\}$ is an inverse mapping system $\sum$. The hypotheses (1) and (2) of Theorem 1 correspond to (1) and (2) of Theorem 5. By Theorem 5 the image of the limit space $\mathscr{R}$ of the inverse mapping system $\sum$ is dense in $\mathscr{W}^\infty$. This is just a statement that corresponding to any function $w(X)$ which defines a point w of $\mathscr{W}^\infty$ and to any $\{\varepsilon_i\}$ there exists $r(X)$ which determines a function of $\mathscr{R}_n$ for each n such that $|r(X) - w(X)| < \varepsilon_i$ when $X \in W_i$. In this way Theorem 1 can be regarded as a special case of Theorem 5.

References

1. W. Kaplan, *Approximation by entire functions*, Mich. Math. J. **3** (1955–6), 43–52.
2. P. W. Ketchum, *On the possible rate of growth of an analytic function*, Trans. Amer. Math. Soc. **49** (1941), 211–228.
3. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, New York, 1942.
4. G. Mittag-Leffler, *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*, Acta. Math. **4** (1884), 1–79.
5. A. Sinclair, *Generalization of Runge's theorem to approximation by analytic functions*, Trans. Amer. Math. Soc. **72** (1952), 148–164.
6. W. J. Thron, *The theory of functions of a complex variable*, John Wiley and Sons, Inc., 1953.
7. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1935.

Southern Illinois University
Carbondale, Illinois