MODULUS OF A BOUNDARY COMPONENT

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§1. PRELIMINARIES AND SUMMARY

1.1 Preliminary definitions. Let R be an open Riemann surface, and let $\{G_n\}$ $(n = 1, 2, \cdots)$ be an infinite sequence of subregions of R such that:

(a) the relative boundary of each G_n is compact,

(b) $G_n \supset G_{n+1}$, and

(c) $\bigcap_{n=1}^{\infty} \overline{G}_n = 0$.

$\{G_n\}$ is said to define a *boundary component* γ of R in the sense of Kerékjártó [6] and Stoilow [16]. Here two sequences of subregions $\{G_n\}$ and $\{G'_n\}$ are considered to be equivalent and to define the same γ if each region G_n includes a region G'_m. That this is a proper equivalence relation follows immediately.

Let γ be a boundary component of R, and let S be a subregion of R. If there exists a defining sequence $\{G_n\}$ of γ with $G_{n_0} = S$, for some n_0, we call S a *neighborhood of* γ. Throughout this paper we shall consider only neighborhoods S of γ such that the relative boundary of S is a closed analytic Jordan curve γ_0.

By an *exhaustion* of R, we mean an infinite sequence $\{R_n\}$ $(n = 1, 2, \cdots)$ of subregions of R as follows (see [16]):

(1) each R_n is compact relative to R and the relative boundary β_n of R_n consists of a finite number of closed analytic Jordan curves β_{ni},

(2) $R_n \subset R_{n+1}$,

(3) $\bigcup^{\infty} R_n = R$, and

(4) each connected component S_{ni} of $R - \overline{R}_n$ is non-compact (relative to R) and its boundary consists of a single curve β_{ni}.

Each set $R - \overline{R}_n$ is said to be a *boundary neighborhood* of R. It is easy to see that, for any boundary component γ of R, there exists a single connected component S_{ni} which is a neighborhood of γ.

A property is said to be a *boundary property* (respectively a *γ-property*) if the following is true. If a Riemann surface R has the property then every Riemann surface R' which admits a conformal mapping from a boundary neighborhood of R' (a neighborhood of γ', where γ' is a boundary

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component of R') onto a boundary neighborhood of R (a neighborhood of γ) has the property.

Let u be a harmonic function on a subregion S of R. We shall denote by $\bar{u}$ the conjugate harmonic function of u and by $D(u; S)$ the Dirichlet integral of u over S.

1.2. Capacity of a boundary component. Let γ be a boundary component of an open Riemann surface R, P_0 a point of R, and $K_z\colon |z| \leqq 1$ a fixed parametric disc on R with $z = 0$ corresponding to P_0. Let $\{R_n\}$ be an exhaustion of R with $P_0 \in R_1$, and let γ_n denote the curve β_{ni} which separates γ from P_0. This means that γ_n separates a neighborhood of γ from P_0.

We consider the class $\{t\}_\gamma$ of single-valued functions on R which satisfy the following conditions:

(1.1) each t is harmonic on $R - P_0$ and has the form

$$t = \log |z| + h(z)$$

in K_z, where h is harmonic and $h(0) = 0$.

$$\int_{\gamma n} d\bar{t} = 2\pi \text{ and } \int_{\beta n i \neq \gamma n} d\bar{t} = 0\,, \qquad \text{for all } n\,, \tag{1.2}$$

where γ_n and β_{ni} are described in the positive sense with respect to R_n.

We further consider the corresponding class $\{t\}_{\gamma n}$ on R_n, and we denote by t_n the function of this class with $t_n = k_n$ on γ_n and $t_n = k_{ni}$ on $\beta_{ni} \neq \gamma_n$, where k_n and k_{ni} are real numbers.

The following theorem due to Sario is proved in [14] (see also Savage [15]). Let $t \in \{t\}_\gamma$, and let

$$I(t) = \lim \frac{1}{2\pi} \int_{\beta n} t d\bar{t}\ .$$

THEOREM 1. *The sequence of functions $\{t_n\}$ is compact. Let t_γ denote a limit function of $\{t_n\}$. Then we have the following conclusions:*

$$t_\gamma \in \{t\}_\gamma \textit{ and, for any } t,\ \min I(t) = I(t_\gamma)\ . \tag{1.3}$$

$$I(t) = I(t_\gamma) + D(t - t_\gamma; R)\ . \tag{1.4}$$

$$k_n \leqq k_{n+1} \textit{ and } I(t_\gamma) = \lim k_n \equiv k_\gamma\ . \tag{1.5}$$

By (1.4), for $k_\gamma < \infty$, the minimizing function t_γ is unique. t_γ is called the *capacity function* of R for γ, and the quantity $c_\gamma = e^{-k\gamma}$ is called the *capacity* of γ (with respect to K_z). Let $z' = az + \cdots, a \neq 0$, be a new local parameter in the neighborhood of P_0, and let c'_γ denote the capacity of γ with respect to this local parameter. It follows, from the definition of the capacity, that

(1.6) $$c_\gamma = |a| c'_\gamma .$$

Hence, the condition $c_\gamma = 0$ is independent of the local parameter which is used in the neighborhood of P_0. Using Green's formula, it is easy to see that this condition is also independent of P_0. A boundary component γ is called *weak* if it has a capacity $c_\gamma = 0$. The class of Riemann surfaces for which all γ are weak is denoted by C_γ. The boundary of a Riemann surface R belonging to C_γ is called *absolutely disconnected* [14, 15].

1.3. Summary. Let R be an open Riemann surface, γ a boundary component of R, S a neighborhood of γ, and γ_0 the relative boundary of S. The present paper deals with a conformal invariant of S which is denoted by $\mu(S; \gamma_0, \gamma)$ (or, simply, for fixed S, by μ_γ) and is called the *modulus of S for γ_0 and γ* (the *modulus of γ*).

In §2 harmonic functions u on S with $u = 0$ on γ_0 and satisfying conditions (2.3) are considered, and a theorem is proved which establishes the existence of a minimizing function $u_\gamma = u(z; S; \gamma_0, \gamma)$ for the Dirichlet integral $D(u; S)$. The modulus is defined by setting $\mu_\gamma = D(u_\gamma; S)$. The notion of a parabolic boundary component is defined by the condition $\mu_\gamma = \infty$, and a theorem is proved which shows the equivalence of parabolicity and weakness.

In §3 measurable conformal metrics are considered. An important minimal property of the conformal metric $\rho_\gamma = |\mathrm{grad}\, u_\gamma|$ corresponding to a result of Wolontis [17] and Strebel [18] is proved, which connects μ_γ with the extremal length of a certain family of curves on S. As an application, a characterization of a parabolic boundary component is obtained in terms of conformal metrics. Another characterization of a parabolic boundary component is given by means of the divergence of a modular series $\sum \mu(E_n; \gamma_{n-1}, \gamma_n)$. The sufficient part of this theorem implies the modular criterion of Savage [15]. A theorem shows the equivalence of perimeter in Ahlfors and Beurling's sense and capacity in Sario's sense.

Section 4 deals with the class M_γ of Riemann surfaces for which all γ are parabolic in the case of a finite genus. The conformal mapping properties of u_γ and t_γ are discussed, and, for planar Riemann surfaces, the equalities $O_{SB} = M_\gamma = O_{SD}$ [1, 14] are proved. Finally a theorem is proved which shows the connection between M_γ and the class of Riemann surfaces for which the continuation is topologically unique, or which do not possess essential continuations.

§2. HARMONIC FUNCTIONS AND MODULUS

2.1. Moduli of a compact subregion. Let S_0 denote a relatively compact subregion of a Riemann surface R. We assume that the boundary of S_0 is a set $\gamma_0 \cup \alpha_0$, where γ_0 is a closed analytic Jordan curve and α_0 consists of a finite number of closed analytic Jordan curves $\alpha_{01}, \cdots, \alpha_{0k}$ ($k \geqq 1$). We assign to each $\alpha_{0i} (i = 1, \cdots, k)$ as positive orientation the positive sense with respect to S_0 and to γ_0 the sense for which γ_0 and α_0 are homologous.

If u is a harmonic function on S_0 then we denote the conjugate period of u around α_{0i} by $p_i(u)$. This is defined by the integral $\int_{\alpha'_{0i}} d\bar{u}$, where α'_{0i} is any closed Jordan curve on S_0 such that α_{0i} and α'_{0i} are homologous. If u is harmonic on $S_0 \cup \alpha_{0i}$ then clearly $p_i(u) = \int_{\alpha_{0i}} d\bar{u}$. The period vector $(p_1(u), \cdots, p_k(u))$ will be denoted by $p(u)$.

LEMMA 1. *There is a harmonic function* $u_0 = u(z; S_0; \gamma_0, k_{01})$ *on* S_0 *satisfying the following conditions*:

(a) $u_0 = 0$ *on* γ_0 *and* $u_0 = \mu_{0i} = const.$ *on* $\alpha_{0i} (i = 1, \cdots, k)$,

(b) $p(u_0) = (1, 0, \cdots, 0)$.

(c) $0 < u_0(z) < \mu_{01}$ *on* S_0 *and on the boundary curves* $\alpha_{02}, \cdots, \alpha_{0k}$.

Proof. Denote the harmonic measure of α_{0i} with respect to S_0 by ω_i, and consider the function

$$u(z) = \sum_{i=1}^{k} \mu_i \omega_i(z) , \tag{2.1}$$

where μ_i are arbitrary real numbers. Clearly, this function is harmonic on $\bar{S}_0 = S_0 \cup \gamma_0 \cup \alpha_0$. Setting $a_{ij} = p_i(\omega_i)$, we obtain

$$p_i(u) = \int_{\alpha_{0i}} d\bar{u} = \sum_{j=1}^{k} a_{ij} \mu_j .$$

We assert that this linear mapping of the k-dimensional cartesian space into itself is one-to-one. In fact, from Green's formula

$$D(u) \equiv D(u; S_0 = \sum_{i=1}^{k} \int_{\alpha_{0i}} u d\bar{u} = \sum_{i=1}^{k} \mu_i p_i(u) ,$$

we see that the condition $p_i(u) = 0$, for all i, implies $D(u) = 0$, that is $u \equiv 0$ (since $u = 0$ on γ_0) and consequently $\mu_i = 0$, for all i, which proves our assertion. Hence we deduce in particular that the above linear mapping is onto, i.e., for any p, there is a function $u = \sum \mu_i \omega_i(z)$ such that $p(u) = p$. Let u_0 denote the function (1.1) corresponding to $p_0 = (1, 0, \cdots, 0)$. This is clearly the unique bounded harmonic function on S_0 satisfying (a) and (b).

Now denote the maximum and the minimum of u_0 on the boundary of S_0 by M_0 and m_0 respectively. From the maximum principle, we have

$m_0 < u_0(z) < M_0$ on S_0. It follows that $\partial u_0/\partial n \leqq 0$ on each boundary curve $\gamma(M_0)$ on which $u_0(z) = M_0$. Here $\partial/\partial n$ denotes the derivative in the direction of the interior normal. Since u_0 is not constant and $\partial u_0/\partial n$ is continuous, there exists a subarc of $\gamma(M_0)$ on which $\partial u_0/\partial n < 0$ and therefore

$$\int_{\gamma(M_0)} d\bar{u}_0 = -\int_{\gamma(M_0)} \frac{\partial u_0}{\partial n} |dz| > 0 ,$$

where $\gamma(M_0)$ is described in the positive sense with respect to S_0. This and condition (b) implies that $\gamma(M_0)$ coincides necessarily with α_{01}, whence $M_0 = \mu_{01}$ and this maximum is attained only on α_{01}. Similarly, it can be proved that $m_0 = 0$ and that this minimum is attained only on γ_0 This completes the proof of Lemma 1.

LEMMA 2. *The function u_0 gives the minimum of $D(u)$,*

$$\min D(u) = D(u_0) ,$$

in the class of all harmonic functions u on S_0 with $u = 0$ on γ_0 and $p(u) = (1, 0, \cdots, 0)$.

Proof. Clearly, the function u_0 belongs to the class of admissible functions and, by Green's formula,

$$D(u_0) = \sum_{i=1}^{k} \mu_{0i} p_i(u_0) = \mu_{01} < \infty .$$

Let u be any admissible function with $D(u) < \infty$. Setting $u - u_0 = h$, we have

$$D(u) = D(u_0) + D(h) + 2D(u_0, h) ,$$

where $D(u_0, h) = D(u_0, h; S_0)$ is the mixed Dirichlet integral of u_0 and h over S_0. We shall show that $D(u_0, h) = 0$. If u is harmonic on $\overline{S}_0$ then Green's formula gives immediately

$$D(u_0, h) = \int_{\alpha_0} u_0 d\bar{h} = \sum_{i=1}^{k} \mu_{0i}\, p_i(h) = 0$$

since, for all i, $p_i(h) = p_i(u) - p_i(u_0) = 0$. If the above assumption is not true, we consider the open set $S_0(\varepsilon) = S_0 - \cup_{i=1}^{k} E_{0i}(\varepsilon)$, where ε is a positive number, sufficiently small, and $E_{0i}(\varepsilon)$ is the set (of points of S_0 for which) $\mu_{0i} - \varepsilon \leqq u_0(z) \leqq \mu_{0i} + \varepsilon$. The boundary of $S_0(\varepsilon)$ consists only of level lines of u_0. On the other hand each level line $c(\mu): u_0(z) = \mu$ $(0 < \mu < \mu_{01}$, $\mu \neq \mu_{0i}$, $i = 1, \cdots, k)$ is a dividing cycle on S_0 (that is, $c(\mu)$ is homologous with a sum of α_{0i}) and therefore $\int_{c(\mu)} d\bar{h} = 0$. Hence, Green's formula gives again $D(u_0, h; S_0(\varepsilon)) = 0$ and, as $\varepsilon \to 0$, $D(u_0, h) = 0$. We conclude finally that

$$D(u) = D(u_0) + D(u - u_0)\ , \tag{2.2}$$

which proves our lemma.

The uniqueness of the minimizing function u_0 is an immediate consequence of (2.2). For, if $D(u) = D(u_0)$, we conclude from (2.2) that $D(u - u_0) = 0$, that is $u \equiv u_0$, since $u - u_0 = 0$ on γ_0.

The function $u_0 = u(z; S; \gamma_0, \alpha_{01})$ will be called the *extremal function* of S_0 for γ_0 and α_{01}. The quantity $\mu_{01} = D(u_0)$ will be called the *modulus* of S_0 for γ_0 and α_{01} and denoted generally by $\mu(S_0; \gamma_0, \alpha_{01})$.

2.2. Modulus of a boundary component. Let us consider a boundary component γ of an open Riemann surface R, and let S be a given neighborhood of γ. Let γ_0 be the relative boundary of S (see 1.1). An exhaustion of S is a sequence $\{S_n\}$ $(n = 1, 2, \cdots)$ of subregions of R such that: (1) S_n is a relatively compact subregion of R and the relative boundary of S_n is a set $\gamma_0 \cup \alpha_n$, where $\gamma_0 \cap \alpha_n = 0$ and α_n consists of a finite number of closed analytic Jordan curves α_{ni}, (2) $S_n \subset S_{n+1}$, (3) $\bigcup_{n+1}^{\infty} S_n = S$, and (4) each connected component of $S - S_n$ is non-compact and its relative boundary consists of a single α_{ni}. We assign to each α_{ni} as positive orientation the positive sense with respect to S_n and to γ_0 the sense for which γ_0 and α_n are homologous.

Let γ_n be the curve α_{ni} which separates γ from γ_0, and let $\{n\}_\gamma$ be the class of all harmonic functions u on S with $u = 0$ on γ_0 and

$$\int_{\gamma_n} d\bar{u} = 1 \text{ and } \int_{\alpha_{ni} \neq \gamma_n} d\bar{u} = 0\ , \tag{2.3}$$

for all n. It is easy to see, using Green's formula, that conditions (2.3) are independent of the particular exhaustion which is used.

THEOREM 2. *In $\{u\}_\gamma$ there exists a function u_γ with the property*

$$\min D(u; S) = D(u_\gamma; S)\ .$$

Moreover, for any u,

$$D(u; S) = D(u_\gamma; S) + D(u - u_\gamma; S)\ . \tag{2.4}$$

Proof. Denote by u_n the extremal function of S_n for γ_0 and γ_n, and put $\mu_n = D(\mu_n; S_n) =$ value of u_n on γ_n; μ_n is the modulus of S_n for γ_0 and γ_n.

Since the restriction of u_{n+1} to S_n satisfies the condition of Lemma 2 (where S_0 is replaced by S_n and α_{01} by γ_n), we have

$$\mu_n = D(u_n; S_n) \leqq D(u_{n+1}; S_n) \leqq D(u_{n+1}; S_{n+1}) = \mu_{n+1}\ .$$

Similarly, we see that $\mu_n \leqq \mu_\gamma$, where μ_γ is the greatest lower bound of

$D(u;S)$ for u *in* $\{u\}_\gamma$. Thus, $\lim_{n\to\infty}\mu_n$ exists and we have

$$\lim_{n\to\infty}\mu_n \leqq \mu_\gamma .$$

For a fixed N, let s be the bounded harmonic function on S_N with $s=0$ on γ_0 and $s=d$ on α_N, where d is a constant value determined by $\int_{\alpha_N} d\bar{s}=1$. From Green's formula $\int_{\alpha_N} u_n d\bar{s} - s d\bar{u}_n) = 0$ and the boundary behavior of u_n and s, we obtain

$$\int_{\alpha_N} u_n d\bar{s} = d ,$$

for all $n \geqq N$, whence $\min_{\alpha_N} u_n \leqq d$. It follows from Harnack's principle that the sequence $\{u_n\}$ is compact. A subsequence, say again $\{u_n\}$, converges, uniformly on each S_N, to a function u_γ. Obviously this function belongs to $\{u\}_\gamma$, so that

$$\mu_\gamma \leqq D(u_\gamma; S) .$$

On the other hand, the lower semicontinuity of the Dirichlet integral gives

$$D(u_\gamma; S) \leqq \lim D(u_n; S_n) = \lim \mu_n .$$

From the three preceding inequalites we conclude that

$$D(u_\gamma; S) = \lim \mu_n = \mu_\gamma ,$$

which proves the first assertion of Theorem 2.

Let us now prove equality (2.4), for any u in $\{u\}_\gamma$. This is evident if $D(u;S)=\infty$. Suppose $D(u;S)<\infty$, and put $u-u_\gamma=h$. For any real number ε, $u_\gamma+\varepsilon h \in \{u\}_\gamma$, and therefore

$$D(u_\gamma+\varepsilon h) = D(u_\gamma) + 2\varepsilon D(u_\gamma, h) + \varepsilon^2 D(h) \geqq D(u_\gamma) .$$

Since $D(u_\gamma+\varepsilon h)<\infty$, this is possible only if $D(u_\gamma, h)=0$, so that, as $\varepsilon=1$, we obtain (2.4).

As in Lemma 2, the uniqueness of the minimizing function u_γ in the case $\mu_\gamma<\infty$ is an immediate consequence of (2.4).

The function u_γ will be called the *extremal function* of S for γ_0 and γ and denoted generally by $u(z; S; \gamma_0, \gamma)$. The conformal invariant $\mu_\gamma = D(u_\gamma, S)$ will be called the *modulus* of S for γ_0 and γ or, simply, for fixed S, the modulus of γ. It will be denoted generally by $\mu(S; \gamma_0, \gamma)$.

2.3. Parabolic boundary components. Let γ be a boundary component of an open Riemann surface R. Consider any two neighborhoods S and S' of γ, and denote by γ_0 and γ'_0 the relative boundaries of S and

S' respectively. Set $u(z; S; \gamma_0, \gamma) = u_\gamma$, $u(z; S'; \gamma'_0, \gamma) = u'_\gamma$, $\mu(S; \gamma_0, \gamma) = \mu_\gamma$, $\mu(S'; \gamma'_0, \gamma) = \mu'_\gamma$.

LEMMA 3. *The moduli μ_γ and μ'_γ are simultaneously finite or infinite.*

Proof. Suppose first $S \subset S'$, and let $\{S'_n\}$ be an exhaustion of S'. The regions $S_n = S \cap S'_n$ give, for n sufficiently large, an exhaustion of S. Set $u(z; \gamma_0, \gamma_n) = u_n$, $u(z; S'_n; \gamma'_0, \gamma_n) = u'_n$, $\mu(S_n; \gamma_0, \gamma_n) = \mu_n$, $\mu(S'_n; \gamma'_0, \gamma_n) = \mu'_n$.
From Green's formula

$$\int_{\alpha_n \cup \gamma_0^{-1}} (u'_n d\bar{u}_n - u_n d\bar{u}'_n) = 0 ,$$

it follows

$$\mu'_n - \mu_n = \int_{\gamma_0} u'_n d\bar{u}_n .$$

Hence, as $n \to \infty$, we obtain

$$\mu'_\gamma - \mu_\gamma = \int_{\gamma_0} u'_\gamma d\bar{u}_\gamma .$$

This proves our lemma in the particular case $S \subset S'$.

Let us now consider the general case, and construct a third neighborhood S'' of γ such that $S'' \subset S \cap S'$. Let γ''_0 denote the relative boundary of S'', and put $\mu(S''; \gamma''_0, \gamma) = \mu''_\gamma$. As before, μ_γ and μ''_γ are simultaneously finite or infinite. The same is valid for μ'_γ and μ''_γ and consequently for μ_γ and μ'_γ, which completes the proof of Lemma 3.

A boundary component γ of R is called *parabolic* if $\mu_\gamma = \infty$ and *hyperbolic* if $\mu_\gamma < \infty$. From Lemma 3, this condition is independent of the neighborhood S which is used, i.e. the parabolicity of a γ is a γ-property of R. The class of all Riemann surfaces for which all boundary components are parabolic will be denoted by M_γ. The property $R \in M_\gamma$ (or $R \notin M_\gamma$) is a boundary property of R.

Consider now the capacity function t_γ of R for γ with respect to a fixed parametric disc $|z| \leqq 1$. Let λ denote a positive number which is sufficiently small such that the level line $c(\lambda): t_\gamma(z) = \log\lambda$ is a closed Jordan curve and the set $t_\gamma(z) \leqq \log\lambda$ is compact. The set $S(\lambda): t_\gamma(z) > \log\lambda$ is then a neighborhood of γ. Put $u(z; S(\lambda); c(\lambda), \gamma) = u_{\gamma,\lambda}$, $\mu(S(\lambda); c(\lambda), \gamma) = \mu_{\gamma,\lambda}$.

LEMMA 4. *If λ satisfies the above conditions, then*

$$t_\gamma(z) - \log \lambda = 2\pi u_{\gamma,\lambda}(z) , \tag{2.5}$$

and

$$k_\gamma - \log\lambda = 2\pi\mu_{\gamma,\lambda} . \tag{2.6}$$

Proof. Consider an exhaustion $\{R_n\}$ of R as in 2.1. The regions $S_n(\lambda) = R_n \wedge S(\lambda)$ give, for n sufficiently large, an exhaustion of $S(\lambda)$. Set $u(z; S_n(\lambda); c(\lambda), \gamma_n) = u_{n,\lambda}$, $\mu(S_n(\lambda); c(\lambda), \gamma_n) = \mu_{n,\lambda}$, $t_\gamma - 2\pi u_{\gamma,\lambda} = h$, $t_n - 2\pi u_{n\pi} = h_n$, where t_n is the function on R_n defined in 1.2. From Green's formula, we have

$$D(h_n ; S_n(\lambda)) = \int_{\beta n} h_n d\bar{h}_n - \int_{c(\lambda)} h_n d\bar{h} = - \int_{c(\lambda)} h_n d\bar{h}_n ,$$

since h_n = const. on β_{ni} and $\int_{\beta_{ni}} d\bar{h}_n = 0$, for all β_{ni}. Hence, by the lower semicontinuity of the Dirichlet integral,

$$D(h; S(\lambda)) \leqq -\int_{c(\lambda)} h d\bar{h} = 0 ,$$

since h = const. $= \log\lambda$ on $c(\lambda)$ and $\int_{c(\lambda)} d\bar{h} = 0$. We conclude that $h \equiv \log\lambda$, which proves (2.5).

Now apply Green's formula on $S_n(\lambda)$ to $u_{n,\lambda}$ and t_n. We obtain

$$k_n - 2\pi\mu_{n,\lambda} = \int_{c(\lambda)} t_n d\bar{u}_{n,\lambda} ,$$

whence, as $n \to \infty$,

$$k_\gamma - 2\pi\mu_{\gamma,\lambda} = \int_{c(\lambda)} t_\gamma d\bar{u}_{\gamma,\lambda} = \log\lambda ,$$

which completes the proof of Lemma 4.

THEOREM 3. *A boundary component γ of R is parabolic if and only if it has a vanishing capacity.*

Proof. This is evident from Lemmas 3 and 4.

COROLLARY. $M_\gamma = C_\gamma$.

§3 MODULUS AND CONFORMAL METRICS

3.1. Definitions. Consider a non-negative function $\rho(z)$ which is defined on each parametric disc $K_z : |z| \leqq 1$ of a subregion S of R and satisfies

$$\rho(z) = \rho(z') \left| \frac{dz'}{dz} \right|$$

at corresponding points z, z' of any two overlapping K_z and $K_{z'}$. We say that ρ is a conformal metric on S. We define the ρ-length of any cycle c (finite set of closed Jordan curves) on S by the lower Darboux integral (see [4])

$$l(\rho; c) = \int_c \rho(z)|dz| .$$

A conformal metric ρ is said to be measurable on S if its restriction to any parametric disc is measurable in Lebesgue's sense. If ρ is a measurable conformal metric on S, we define the ρ-area of S by the Lebesgue integral

$$\mathrm{A}(\rho; S) = \int_S \rho^2(z) d\sigma_z ,$$

where σ_z is the Lebesgue measure on K_z. A measurable conformal metric ρ defined on S is said to be A-bounded on S if $A(\rho; S) < \infty$.

3.2. **Extremal conformal metrics.** Consider first the relatively compact subregion S_0 of 2.1. We prove the following

LEMMA 5. *The conformal metric* $\rho_0 = |\operatorname{grad} u_0|$ *gives the minimum of* $A(\rho; S_0)$,

$$\min A(\rho; S_0) = A(\rho_0; S_0) , \tag{3.1}$$

in the class of all conformal metrics satisfying $l(\rho; c) \geqq 1$, *for all dividing cycles* c *on* S_0 *which separate* α_{01} *from* γ_0.

Moreover, for any admissible ρ,

$$A(\rho; S_0) \geqq A(\rho_0; S_0) + A(\rho - \rho_0; S_0) . \tag{3.2}$$

Proof. Clearly the conformal metric ρ_0 satisfies the condition of the lemma, and $A(\rho_0; S_0) = D(u_0; S_0) = \mu_{01} < \infty$. Let ρ be any admissible conformal metric on S_0 with $A(\rho; S_0) < \infty$.

We evaluate the integral

$$\int_{S_0} \rho(z)\rho_0(z) d\sigma_z .$$

Take $w_0 = u_0 + i\bar{u}_0$ for the local parameter on S_0, so that $\rho_0(w_0) \equiv 1$. Denote the level line $u_0(z) = \mu$ $(0 \leqq \mu \leqq \mu_{01}$; see Lemma 1) by $c(\mu)$. From Fubini's theorem,

$$\int_{S_0} \rho(z)\rho_0(z) d\sigma_z = \int_0^{\mu_{01}} d\mu \int_{c(\mu)} \rho(w_0) d\bar{u}_0 .$$

Here the integral $\int_{c(\mu)} \rho(w_0) d\bar{u}_0$ exists almost everywhere, for μ on the closed interval $[0, \mu_{01}]$. But $c(\mu)$ is, for any $\mu \neq \mu_{0i}$, a dividing cycle on S_0 which separates α_{01} from γ_0 and therefore, almost everywhere,

$$\int_{c(\mu)} \rho(w_0) d\bar{u}_0 = \int_{c(\mu)} \rho(z) |dz| \geqq \int_{c(\mu)} \rho(z) |dz| \geqq 1$$

From the two preceding relations it follows that

$$\int_{S_0} \rho(z) \rho_0(z) d\sigma_z \geqq \mu_{01} .$$

Now put $\rho = \rho_0 + (\rho - \rho_0)$ in $A(\rho; S_0)$; we obtain

$$A(\rho; S_0) = \mu_{01} + A(\rho - \rho_0; S_0) + 2\int_{S_0} \rho\rho_0 d\sigma - 2\mu_{01}$$

and, from the preceding inequality, we conclude finally that

$$A(\rho; S_0) \geqq \mu_{01} + A(\rho - \rho_0; S_0) ,$$

which proves our lemma.

Clearly the admissible conformal metric which minimizes $A(\rho; S_0)$ is unique. For, if $A(\rho; S_0) = A(\rho_0; S_0) = \mu_{01} < \infty$, we deduce from (3.2) that $A(\rho - \rho_0; S_0) = 0$, i.e. $\rho = \rho_0$ almost everywhere on S_0.

Now let γ be a boundary of R, and let S be a given neighborhood of γ. Let $\{\rho\}_\gamma$ denote that class of all measurable conformal metrics defined on S which satisfy the condition

$$l(\rho; c) \geqq 1 , \tag{3.3}$$

for all dividing cycles c which separate γ from γ_0. If $u \in \{u\}_\gamma$, then obviously $|\text{grad} u| \in \{\rho\}_\gamma$. This is valid, in particular, for the conformal metric $\rho_\gamma = |\text{grad} u_\gamma|$. The ρ_γ-area of S is $A(\rho_\gamma; S) = D(u_\gamma; S) = \mu_\gamma$.

THEOREM 4. *In $\{\rho\}_\gamma$ the conformal metric $\rho_\gamma = |\text{grad} u_\gamma|$ gives the minimum of $A(\rho; S)$:*

$$\min A(\rho; S) = A(\rho_\gamma; S) . \tag{3.4}$$

Moreover, for any ρ,

$$A(\rho; S) \geqq A(\rho_\gamma; S) + A(\rho - \rho_\gamma; S) . \tag{3.5}$$

Proof. If $A(\rho; S) = \infty$, (3.5) is evident. Assume now that there exists in $\{\rho\}_\gamma$ a conformal metric ρ which is A-bounded.

Set $|\text{grad} u_n| = \rho_n$ (see 2.2). Since $A(\rho; S) \geqq A(\rho; S_n)$, we conclude from Lemma 5 that

$$A(\rho; S) \geqq \mu_n + A(\rho - \rho_n; S_n)$$

As $n \to \infty$, Fatou's Lemma gives immediately

$$A(\rho; S) \geqq \mu_\gamma + \liminf A(\rho - \rho_n; S_n) \geqq \mu_\gamma + A(\rho - \rho_\gamma; S)\ ,$$

which proves (3.5) and the theorem.

As in Lemma 5, the uniqueness of the minimizing conformal metric ρ_γ in the case $\mu_\gamma < \infty$ is an immediate consequence of (3.5).

By Theorem 4, the quantity $\lambda_\gamma = \mu_\gamma^{-1}$ is equal to the extremal length of the family of all dividing cycles c on S separating γ from γ_0 ([1], [5]).

3.3. Parabolic boundary components. We return to the condition $\mu_\gamma = \infty$ studied in 2.2.

THEOREM 5. *A boundary component γ of R is parabolic if and only if, for any neighborhood S of γ and for any A-bounded conformal metric ρ on S, there exists a dividing cycle separating γ from γ_0 with an arbitrarily small ρ-length.*

Proof. If $\mu_\gamma < \infty$, the conformal metric ρ_γ is A-bounded and satisfies $l(\rho; c) \geqq 1$, for all dividing cycles separating γ from γ_0. Conversely, if there is an A-bounded conformal metric ρ on S satisfying $l(\rho; c) \geqq \varepsilon > 0$, for all dividing cycles c separating γ from γ_0, the conformal metric $\rho^* = (1/\varepsilon)\rho$ is A-bounded and belongs to $\{\rho\}_\gamma$. Therefore, by Theorem 4, $\mu_\gamma < \infty$.

THEOREM 6. *Suppose R is imbedded in a larger Riemann surface R^*. If a boundary component γ of R or a part of γ realized on R^* contains a continuum γ^*, then γ is hyperbolic.*

Proof. Let $K^* : |z^*| \leqq 1$ denote a parametric disc on R^* for which $K^* \cap \gamma^*$ contains a continuum, say again γ^*. Since γ^* is a boundary continuum of R, there exists a disc $\overline{R}_0 \subset K^* \cap R$. In K^* let $Q = aba'b'$ be a rectangle such that its side a is completely interior to R_0 and its opposite sides b, b' have common points with γ^*.

Set $R - \overline{R}_0 = S$. We define a conformal metric ρ_0 on S by setting $\rho_0(z^*) = 1$ on $Q \cap S$ and $\rho_0 = 0$ otherwise. Clearly ρ_0 is A-bounded and satisfies $l(\rho_0; c) \geqq l_0 > 0$, where l_0 is the length of a in K^* and c is any dividing cycle separating γ from γ_0. Hence, by Theorem 5, γ is not parabolic.

Let S be a given neighborhood of a boundary component γ of R, and let $\{S_n\}$ be an exhaustion of S as in 2.2. Let E_n denote the connected component of $S_n - S_{n-1}$ whose boundary includes γ_{n-1} and γ_n. We assert that

$$\mu(S;\gamma_0,\gamma) \geqq \sum_{n=1}^{\infty} \mu(E_n;\gamma_{n-1},\gamma_n) \ . \tag{3.6}$$

In fact, since the restriction of ρ_γ to E_n is admissible in Lemma 5 (where S_0 is replaced by E_n, γ_0 and α_{01} by γ_{n-1} and γ_n respectively), we conclude that $A(\rho_\gamma; E_n) \geqq \mu(E_n;\gamma_{n-1},\gamma_n)$. Therefore, $\mu(S;\gamma_0,\gamma) \geqq \sum_{n=1}^{\infty} A(\rho_\gamma; E_n) \geqq \sum_{n=1}^{\infty} \mu(E_n;\gamma_{n-1},\gamma_n)$, which proves (3.6).

Similarly, it may be proved that

$$\mu(S;\gamma_0,\gamma) \geqq \mu(E_1;\gamma_0,\gamma_1) + \mu(S^*_1;\gamma_1,\gamma) \ , \tag{3.7}$$

where S^*_1 is the connected component of $S - \overline{S}_1$ whose relative boundary is γ_1.

THEOREM 7. *A boundary component γ of R is parabolic if and only if there exists an exhaustion of S for which*

$$\sum_{n=1}^{\infty} \mu(E_n;\gamma_{n-1},\gamma_n) = \infty \ . \tag{3.8}$$

Proof. By (3.6), the condition (3.8) is sufficient for the parabolicity of γ.

Conversely, assume that γ is parabolic, and let $\{S_n\}$ be a given exhaustion of S. Since

$$\lim_{n\to\infty} \mu(S_n;\gamma_0,\gamma_n) = \mu(S;\gamma_0,\gamma) = \infty \ ,$$

we can choose $n_1 \geqq 1$ such that $\mu(S_n;\gamma_0,\gamma_{n_1}) \geqq 1$. Let $S^*_{n_1}$ denote the connected component of $S - \overline{S}_{n_1}$ whose relative boundary is γ_{n_1}. $S^*_{n_1}$ is a neighborhood of γ. Since γ is parabolic, we have

$$\lim_{n\to\infty} \mu(S^*_{n_1,n};\gamma_{n_1},\gamma_n) = \mu(S^*_{n_1};\gamma_{n_1},\gamma) = \infty \ ,$$

where $S^*_{n_1,n} = S^*_{n_1} \cap S_n$. Therefore, we can choose $n_2 > n_1$ such that $\mu(S^*_{n_1,n_2};\gamma_{n_1},\gamma_{n_2}) \geqq 1$. Continuing this procedure, we obtain an exhaustion $\{S_{n_k}\}$ $(k = 1, 2, \cdots)$ of S, which satisfies condition (3.8). Thus Theorem 7 is established.

3.4. Perimeter and capacity. Let $|z| \leqq r_0$ be a fixed parametric disc on R, and let $S(r)$ denote the complement of the disc $|z| \leqq r$ $(0 < r \leqq r_0)$ with respect to R. Set $\mu(S(r); |z| = r, \gamma) = \mu_{\gamma,r}$. By (3.7), for $r' < r$,

$$\mu_{\gamma,r'} \leqq \frac{1}{2\pi} \log \frac{r}{r'} + \mu_{\gamma,r}$$

or

$$-2\pi\mu_{\gamma,r'} - \log r' \leqq -2\pi\mu_{\gamma,r} - \log r \ .$$

Therefore,

$$\pi_\gamma = \lim_{r\to 0} \frac{1}{r} e^{-2\pi\mu_{\gamma,r}}$$

exists. According to Ahlfors and Beurling [1], we call π_γ perimeter of γ with respect to the fixed parametric dics $|z| \leqq r_0$. Let $z' = \lambda(z) = az + \cdots, a \neq 0$, be a new local parameter in the neighborhood of the point $P_0 \in R$ corresponding to $z = 0$, and let π'_γ denote the perimeter of γ with respect to the parametric disc $|z'| \leqq r'_0$. Set $|z| = r$ and $|z'| = r'$. For corresponding r and r' by $z' = \lambda(z)$, we have

$$|a|r(1 - \varepsilon_r) \leqq r' \leqq |a|r(1 + \varepsilon_r) \ ,$$

where ε_r is a positive function of r and $\varepsilon_r \to 0$, as $r \to 0$. It follows, from the conformal invariance and the monotony of modulus, that

$$\pi_\gamma = |a|\, \pi'_\gamma \ . \tag{3.9}$$

We now prove the following.

THEOREM 8. *If the perimeter π_γ and the capacity c_r are defined with respect to the same parametric disc $|z| \leqq r_0$, then $\pi_\gamma = c_\gamma$.*

Proof. From (1.6) and (3.9), it is sufficient to prove the required equality for a particular parametric disc of the point P_0. We choose this parametric disc, say again $|z| \geqq r_0$, such that $t_\gamma \equiv \log|z|$ on $|z| \leqq r_0$. Then, by (2.6), we conclude immediately that

$$\pi_\gamma = \lim_{\lambda\to 0} \frac{1}{\lambda} e^{-2\pi\mu_{\gamma,\lambda}} = e^{-k_\gamma} = c_\gamma \ ,$$

which proves our theorem.

COROLLARY. *If P_γ denote the class of Riemann surfaces defined by $\pi_\gamma = 0$, for all γ, then $P_\gamma = c_\gamma = M_\gamma$.*

§ 4. RIEMANN SURFACES OF FINITE GENUS

4.1. Planar subregions. Let γ be a boundary component of an open Riemann surface R, and suppose that γ is hyperbolic and possesses a neighborhood S which is planar.

Set, as usually, $u(z; S; \gamma_0, \gamma) = u_\gamma$, $\mu(S; \gamma_0, \gamma) = \mu_\gamma$, and consider the function $w = F_\gamma(z)$ defined by

(4.1) $$F_\gamma(z) = \exp 2\pi(u_\gamma(z) + i\bar{u}_\gamma(z))$$

Consider an exhaustion $\{S_n\}$ of S as in 2.2. Since S is planar, the homology group $H^1(S)$ is generated from the boundary curves α_{ni} of $S_n(n = 1, 2, \cdots)$, and we conclude by (2.3) that F_γ is single-valued. We now prove the following [7]:

THEOREM 9. *The function $w = F_\gamma(z)$ maps the region S univalently onto the annulus*

$$A_{0,\mu_\gamma} : 1 < |w| < e^{2\pi\mu_\gamma}$$

slit along a set of circular arcs around the origin. Here the boundary circumferences $|w| = 1$ and $|w|e^{2\pi\mu_\gamma}$ correspond to γ_0 and γ respectively. The total area of the slits vanishes.

Proof. We define the function $w = F_n(z)$ on S_n by

$$F_n(z) = \exp 2\pi(u_n(z) + i\bar{u}(z)) ,$$

where $u_n = u(z; S_n; \gamma_0, \gamma_n)$. As before, we see that F_n is single-valued, for all n.

The function $w = F_n(z)$ gives a one-to-one conformal mapping of S_n onto the covering surface $S_{n,w} = (S_n, w = F_n(z))$. By the definition of u_n, $|F_n(z)|$ assumes constant values on the boundary curves of S_n and satisfies on S_n:

$$1 < |F_n(z)| < e^{2\pi\mu_n} .$$

It follows that $S_{n,w}$ is an unlimited covering surface of the annulus A_{0,μ_n} slit along a finite number of circular arcs. On the other hand, evaluate the ρ_0-area of $S_{n,w}$, where

$$\rho_0(w) = \frac{1}{2\pi|w|} = \frac{1}{2\pi}\left|\frac{d}{dw}\log w\right| .$$

Since, for $w = F_n(z)$,

$$\rho_n(z) = |\mathrm{grad}\, u_n(z)| = \frac{1}{2\pi}\left|\frac{d}{dz}\log w\right| = \rho_0(w)\left|\frac{dw}{dz}\right| ,$$

we obtain

$$A(\rho_0; S_{n,w}) = A(\rho_n; S_n) = \mu_n .$$

This is equal to the ρ_0- area of the annulus A_{0,μ_n}. It follows that the covering surface $S_{n,w}$ consists necessarily of a single sheet, that is the function F_n is univalent. Since $F_n \to F_\gamma$ uniformly on each S_N, F_γ is also univalent.

Let us now consider the image $S_w = F_\gamma(z)$. Denote the connected components of the boundary of S_w which correspond to γ_0 and γ by γ_w^0 and γ_w respectively. Clearly γ_w^0 is the circumference $|w| = 1$. Further, since $\mu_n \leqq \mu_\gamma$, for all n, S_w is included in the annulus A_{0,μ_γ}. As before, the ρ_0-area of S_w is

$$A(\rho_0; S_w) = A(\rho_\gamma; S) = \mu_\gamma ,$$

since

$$\rho_\gamma(z) = \rho_0(w)\left|\frac{dw}{dz}\right| \qquad (w = F_\gamma(z)) .$$

This is equal to the ρ_0-area of the annulus A_{0,μ_γ}. Accordingly, the complements of S_w with respect to A_{0,μ_γ} has a (logarithmic and Euclidian) vanishing area.

Assume finally that the set $A_{0,\mu_\gamma} - S_w$ possesses a connected component γ^*_w which is not a point or a circular arc around the origin. Construct two circumferences $|w| = r_i$ $(i = 1, 2; r < r_1 < r_2 < e^{2\pi\mu_\gamma})$ having common points with γ^*_w, and consider a point w_0 in the annulus $r_1 < |w| < r_2$. Let K_ε be the disc $|w - w_0| \leqq \varepsilon$. Obviously, for ε sufficiently small, the conformal metric ρ_ε, defined by $\rho_\varepsilon = 0$ on K_ε and $\rho_\varepsilon(w) = \rho_0(w)$ on $S_w - K_\varepsilon$, satisfies the condition (3.3), for all dividing cycles c on S_w separating γ_w from γ_w^0. This contradicts Theorem 4, since $A(\rho_\varepsilon; S_w) < A(\rho_0; S_w) = \mu_\gamma$. Therefore, the continuum γ^*_w does not exist. In particular, γ_w coincides with $|w| = e^{2\pi\mu_\gamma}$. Theorem 9 is completely proved.

4.2. Planar Riemann surfaces. Suppose now that R itself is planar. Let $|z| \leqq 1$ be a fixed parametric disc on R, γ a hyperbolic boundary component of R, and $c_\gamma > 0$ the capacity of γ with respect to $|z| \leqq 1$. Consider the function $w = T_\gamma(z)$ defined by

$$T_\gamma(z) = c_\gamma \exp(t_\gamma(z) + i\bar{t}_\gamma(z)) .$$

By Lemma 4 and Theorem 9, we have the following [14]:

THEOREM 10. *The function $w = T_\gamma(z)$ is univalent and single-valued on R and maps R onto the unit circle slit along a set of circular arcs of vanishing total area. The boundary component γ is mapped into the unit circumference.*

Let SB (SD) be the class of univalent single-valued analytic functions having a bounded modulus (a finite Dirichlet integral), and let $O_{SB}(O_{SD})$ be the class of Riemann surfaces with no functions belonging to $SB(SD)$.

THEOREM 11. [1, 14] *For planar Riemann surfaces,* (4.2) $$O_{SB} = M_\gamma = O_{SB} .$$

Proof. Assume first that the planar surface R possesses a hyperbolic boundary component γ. Then, the function T_γ of Theorem 10 obviously belongs to the class SB and SD.

Conversely, suppose that there exists on R a function $w = T(z)$ which belongs to the class SB or SD. In both cases, the image $R_w = T(R)$ has a finite Euclidian area. Let $K_\varepsilon : |w - w_0| \leqq \varepsilon$ be a disc which is completely included in R_w. Denote by γ_w the connected component of the boundary of R_w which separates $w = 0$ from $w = \infty$ or contains $w = \infty$, The conformal metric $\rho(w) = 1/2\pi\varepsilon$ is clearly A-boundary on $R_w - K_\varepsilon$ and satisfies condition (3.3), for all dividing cycles on $R_w - K_\varepsilon$ which separate γ_w from $|w - w_0| = \varepsilon$. We conclude that the boundary component γ of R which corresponds to γ_w is hyperbolic.

4.3. Riemann surfaces of finite genus. A *continuation* of a Riemann surface R is defined by (1) another Riemann surface R' and (2) a one-to-one conformal mapping $T: R \to R'$, $T(R) \subset R'$, [**2, 4, 8, 9, 11, 12**]. If R' is a compact Riemann surface, the continuation is called *compact*. If $R' - T(R)$ contains interior points, the continuation is called *essential* [**9, 12**].

Let R be a Riemann surface of finite genus. We say that the continuation of R is *topologically unique* if, for any two compact continuations $T_\nu : R \to R'_\nu (\nu = 1, 2)$ of R, there exists a topological mapping $h^*_{12} = R'_1 \to R'_2$, $h^*_{12}(R'_1) = R'_2$, with $h^*_{12}\, T_1(R) = h_{12}$, where $h_{12} = T_2 T_1^{-1}$. If, in addition, h^*_{12} is always a conformal mapping, the continuation of R is said to be *conformally unique*.

Let O_{AD} denote the class of Riemann surfaces with no non-constant single-valued analytic functions having a finite Dirichlet integral. It is well known that the continuation of a Riemann surface R of finite genus is conformally unique if and only if $R \in O_{AD}$ [**1, 8, 12**]. We now prove the following

THEOREM 12. *For Riemann surfaces of finite genus, the following conditions are equivalent*:

(1) $R \in M_\gamma$

(2) *The continuation of R is topologically unique.*

(3) *R does not possess an essential continuation.*

Proof. (1) $\to$ (2). If $R \in M_\gamma$ and $T_\nu : R \to R'_\nu$ $(\nu = 1, 2)$ are compact continuations of R, then, by Theorem 6, the sets $\beta_\nu = R'_\nu - T_\nu(R)$ are totally disconnected. Set $T_2 T_1^{-1} = h_{12}$. We define a topological mapping h^*_{12} of R'_1 onto R'_2 as follows. First, set $h^*_{12}(P_1) = h_{12}(P_1)$, for any $P_1 \in T_1(R)$. Now let $P_1 \in \beta_1$. Since β_1 is totally disconnected, there is a fundamental sequence $\{U_n\}$ of neighborhoods of P_1 such that the open sets $V_n = U_n \cap T_1(R)$ are connected. Set $E(P_1) = \cap_n \overline{h_{12}(V_n)}$. Clearly this is a closed and connected set. On the other hand, $E(P_1) \subset \beta_2$ and, since β_2 is totally disconnected $E(P_1)$ contains a single point P_2. Set $h^*_{12}(P_1) = P_2$. It is easy to see that h^*_{12} is a topological mapping between R'_1 and R'_2.

(2) → (3). If R possesses an essential continuation $T_1: R \to R'_1$, we may construct in an evident manner another compact continuation $T_2: R \to R'_2$ of R such that R'_1 and R'_2 have different genera.

(3) → (1). Assume that $R \notin M_\gamma$, i.e. R possesses some boundary component γ which is hyperbolic. Let S be a neighborhood of γ. We have $\mu_\gamma < \infty$. By Theorem 9, there is a one-to-one conformal mapping of S into the finite annulus $1 < |w| < e^{2\pi\mu_\gamma}$. Let K_w denote the set $|w| > 1$. Clearly the Riemann surface $R' = (R - S) \cup K_w$ defines an essential continuation of R, and therefore (3) → (1). Thus, Theorem 12 is established

COROLLARY 1. *For Riemann surfaces of finite genus, we have* $O_{AD} \subset M_\gamma$.

Note that by a theorem of Ahlfors and Beurling [1] this inclusion is strict.

COROLLARY 2. *Let* $R \in M_\gamma - O_{AD}$ *and of finite genus. Then there exist two compact continuations* $T_\nu: R \to R'_\nu$ $(\nu = 1, 2)$ *of* R *such that the corresponding topogical mapping* h^*_{12} *is not a conformal mapping.*

In particular, we conclude from Corollary 2 that there exist Pompeiu functions which are univalent (see [3], [10], and [16]).

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