MINIMAL COVERINGS OF PAIRS BY TRIPLES

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1. Introduction. Let F be a finite set with n members, $n \geqq 3$. An *F-covering of pairs by triples*, which we abbreviate F-copt, is a set S of triples of distinct members of F which has the property that each pair of distinct members of F is contained in at least one member of S. If n is a positive integer, $n \geqq 3$, then an n-copt is an F-copt for the set $F = \{1, 2, \cdots, n\}$. We assume throughout that $n \geqq 3$.

For any finite set A, let $C(A)$ denote the number of members of A. An F-copt S is *minimal* if $C(S) \leqq C(S')$ for every F-copt S'. If $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$, then a minimal n-copt S turns out to be *exact* in the sense that each pair is contained in exactly one member of S. Such exact coverings are called *Steiner triple systems*. The existence of Steiner triple systems for all n (of form $6h + 1$ or $6h + 3$) was proved by M. Reiss [2] in 1859.

Let S be a minimal n-copt and let $C(S) = \mu(n)$. The main result of this paper is obtained in §2, where we determine $\mu(n)$ explicitly for $n \geqq 3$. In § 3 we discuss certain properties of minimal n-copts, and give several methods for constructing minimal n-copts.

2. Determination of $\mu(n)$. Let S be a minimal n-copt. For each integer i, $1 \leqq i \leqq n$, we define $\alpha(i)$ to be the number of members of S that contain i. Then

$$\sum_{i=1}^{n} \alpha(i) = 3 \cdot C(S) .$$

Since i must appear in members of S with $n - 1$ other numbers we have $\alpha(i) \geqq [n/2]$. ($[x]$ is the largest integer which is not greater than x.) Thus,

$$(1) \qquad \mu(n) = C(S) \geqq \frac{n}{3}\left[\frac{n}{2}\right] .$$

Since $(n/3)\,[n/2]$ may not be an integer, we define $\varphi(n)$ to be the least integer which is not less than $(n/3)\,[n/2]$. It is easy to compute

$$(2) \qquad \varphi(n) = \begin{cases} n^2/6 \text{ if } n = 6k , \\ n(n-1)/6 \text{ if } n = 6k + 1 \text{ or } n = 6k + 3 , \\ (n^2 + 2)/6 \text{ if } n = 6k + 2 \text{ or } n = 6k + 4 , \\ (n^2 - n + 4)/6 \text{ if } n = 6k + 5. \end{cases}$$

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We may clearly improve (1) to

$$\mu(n) = C(S) \geqq \varphi(n) . \tag{3}$$

Our main theorem proves that in (3) equality holds for every n.

Let A, B and C be pairwise disjoint sets, each having the same number n of members. A *tricover* for the system (A, B, C) is a set K of triples (x, y, z), $x \in A$, $y \in B$, $z \in C$ such that each pair uv, u and v in different ones of A, B, C, is contained in exactly one member of K.

LEMMA 1. *If n is a positive integer and A, B, C are pairwise disjoint sets each of which has n members, then a tricover K for (A, B, C) exists. Moreover, if $a \in A$, $b \in B$ and $c \in C$, then K may be chosen so that $(a, b, c) \in K$.*

Proof. Let the members of A, B, C be respectively

$$a_1, a_2, \cdots, a_n ; \qquad b_1, b_2, \cdots, b_n ; \qquad c_1, c_2 \cdots, c_n ,$$

where $a_1 = a$, $b_1 = b$, $c_1 = c$. We define K to be the set of all triples (a_i, b_j, c_k) for which $k \equiv i + j - 1 \pmod{n}$, $1 \leqq i, j, k \leqq n$. The set K obviously has the desired properties.

REMARK. Any tricover for (A, B, C) must have n^2 members.

LEMMA 2. *Let A, B, C be pairwise disjoint sets, each having n members. Let p be an integer such that $0 < p \leqq n/2$. Let $A^* \subset A$, $B^* \subset B$, $C^* \subset C$ be sets, each of which has p members and let K^* be a tricover for (A^*, B^*, C^*). Then there exists a tricover K for (A, B, C) such that $K^* \subset K$.*

Proof. Let

$$A = \{a_1, a_2, \cdots, a_n\} ,$$
$$B = \{b_1, b_2, \cdots, b_n\} ,$$
$$C = \{c_1, c_2, \cdots, c_n\} .$$

We can assume that

$$A^* = \{a_1, a_2, \cdots, a_p\} ,$$
$$B^* = \{b_1, b_2, \cdots, b_p\} ,$$
$$C^* = \{c_1, c_2, \cdots, c_p\} .$$

For $1 \leqq i, j \leqq p$, let m_{ij}^* be the unique integer k such that $(a_i, b_j, c_k) \in K^*$. Clearly $1 \leqq m_{ij}^* \leqq p$ and the square array (m_{ij}^*) is a Latin square of order p. It follows from a theorem of Marshall Hall [1] that there exists a Latin square (m_{ij}), $1 \leqq i, j \leqq n$, such that $m_{ij} = m_{ij}^*$,

$1 \leqq i, j \leqq p$. Let

$$K = \{(a_i, b_j, c_{m_{ij}}) \mid 1 \leqq i, j \leqq n\} .$$

The set K is the desired tricover.

In order to produce an inductive proof of our main theorem, it is convenient to restrict ourselves to a special type of minimal n-copt for the case $n \equiv 5 \pmod 6$. Also, for $n \equiv 3 \pmod 6$, there is a special type of minimal n-copt whose existence we wish to establish, and it is possible to include this result in our main theorem. For these reasons we introduce the notion of "admissible F-copt."

An F-copt S is *admissible* if $C(S) = \varphi(n)$, $n = C(F)$, and:
(1) $n \equiv 0, 1, 2$, or $4 \pmod 6$;
(2) $n \equiv 3 \pmod 6$ and S contains a set of pairwise disjoint triples whose union is F; or
(3) $n \equiv 5 \pmod 6$ and S contains four elements of the form (a, b, x), (a, b, y), (a, b, z), (x, y, z).

THEOREM 1. *If n is a positive integer, $n \geqq 3$, then there exists an admissible n-copt.*

Proof. Our proof is by induction on n. However, it is necessary to prove independently that there are admissible n-copts for $n = 3, 5, 7, 9, 11, 13$, and 15. We accomplish this by exhibiting such admissible n-copts.

$n = 3$

(1,2,3)

$n = 5$

(1,2,3)
(1,2,4)
(1,2,5)
(3,4,5)

$n = 7$

(1,2,3)
(1,4,5)
(1,6,7)
(2,4,6)
(2,5,7)
(3,4,7)
(3,5,6)

$n = 9$

(1,2,3)	(2,4,9)
(4,5,6)	(2,5,8)
(7,8,9)	(2,6,7)
(1,4,7)	(3,4,8)
(1,5,9)	(3,5,7)
(1,6,8)	(3,6,9)

$n = 11$

(1, 2, 3)	(3, 6,10)
(1, 2, 4)	(3, 7, 9)
(1, 2, 5)	(3, 8,11)
(3, 4, 5)	(4, 6,11)
(1, 6, 7)	(4, 7, 8)
(1, 8, 9)	(4, 9,10)
(1,10,11)	(5, 6, 9)
(2, 6, 8)	(5, 7,11)
(2, 7,10)	(5, 8,10)
(2, 9,11)	

$n = 13$

(1, 2, 3)	(3, 6,12)
(1, 4, 5)	(3, 8,13)
(1, 6,13)	(3, 9,10)
(1, 7, 8)	(4, 6, 7)
(1, 9,12)	(4, 8, 9)
(1,10,11)	(4,12,13)
(2, 4,10)	(5, 8,11)
(2, 5, 6)	(5, 9,13)
(2, 7, 9)	(5,10,12)
(2. 8,12)	(6, 8,10)
(2,11,13)	(6, 9,11)
(3, 4,11)	(7,10,13)
(3, 5, 7)	(7,11,12)

$n = 15$

(1, 2, 3)	(2, 6, 8)	(3,12,14)	(6, 9,14)
(1, 4,14)	(2, 7,14)	(4, 5, 6)	(6,12,13)
(1, 5, 9)	(2, 9,11)	(4, 8,13)	(7, 8, 9)
(1, 6,10)	(2,10,15)	(4, 9,10)	(7,10,13)
(1, 7,12)	(3, 4, 7)	(4,11,15)	(8,11,14)
(1, 8,15)	(3, 5,11)	(5, 7,15)	(9,12,15)
(1,11,13)	(3, 6,15)	(5, 8,12)	(10,11,12)
(2, 4,12)	(3, 8,10)	(5,10,14)	(13,14,15)
(2, 5,13)	(3, 9,13)	(6, 7.11)	

Our proof now divides into six cases. In Case r, $0 \leqq r \leqq 5$, we assume that $n \equiv r \pmod 6$, that $n > 3$ and that there exist admissible m-copts for $3 \leqq m < n$. We then show that these assumptions imply that there exists an admissible n-copt.

Case 0. Let S_1 be an admissible $(n-1)$-copt having $(1, 2, 3)$, $(1, 2, 4)$, and $(1, 2, 5)$ as three of its members. If we delete $(1, 2, 3)$ from S_1 and add

$$(1, 3, n), (2, 3, n), (4, 5, n), (6, 7, n), \cdots, (n-2, n-1, n),$$

we obtain a set S of triples which is an n-copt. Since S_1 has

$$[(n-1)^2 - (n-1) + 4]/6 = (n^2 - 3n + 6)/6$$

members, S has

$$(n^2 - 3n + 6)/6 - 1 + n/2 = n^2/6 = \varphi(n)$$

members.

Case 1. We have exhibited admissible n-copts for $n = 7$ and $n = 13$. Therefore we may assume $n = 6h + 1, h > 2$.

We consider two subcases.

Subcase i. Either $h \equiv 0$ or $h \equiv 1 \pmod 3$. Then there exists k such that $2h + 1 = 6k + 1$ or $2h + 1 = 6k + 3$.

Let

$$A_1 = \{1, \cdots, 2h, n\}$$
$$A_2 = \{2h + 1, \cdots, 4h, n\}$$
$$A_3 = \{4h + 1, \cdots, 6h, n\}$$

and let S_j be an admissible A_j-copt for $j = 1, 2, 3$. Let T be a tricover for $(\{1, \cdots, 2h\}, \{2h + 1, \cdots, 4h\}, \{4h + 1, \cdots, 6h\})$. We now define $S = S_1 \cup S_2 \cup S_3 \cup T$. It is easy to verify that S is an n-copt, and that S has

$$3 \cdot \frac{(2h+1)2h}{6} + (2h)^2 = \frac{n(n-1)}{6} = \varphi(n)$$

members.

Subcase ii. $h \equiv 2 \pmod 3$. In this case there exists k such that $2h + 1 = 6k + 5$. We define A_1, A_2, A_3 as above. Now, for $j = 0, 1, 2$, we let S_{j+1} be an admissible A_{j+1}-copt such that S_{j+1} contains a subset R_{j+1} whose members are:

$$\begin{gathered}(2jh + 1, 2jh + 2, 2jh + 3)\\ (2jh + 1, 2jh + 2, 2jh + 4)\\ (2jh + 1, 2jh + 2, n)\\ (2jh + 3, 2jh + 4, n)\,.\end{gathered}$$

Let T be a tricover for $(\{1, \cdots, 4\}, \{2h + 1, \cdots, 2h + 4\}, \{4h + 1, \cdots, 4h + 4\})$, and let T^* be a tricover for $(\{1, \cdots, 2h\}, \{2h + 1, \cdots, 4h\}, \{4h + 1, \cdots, 6h\})$ that is an extension of T. Since $h \geqq 5$, the existence of such a tricover follows from Lemma 2. We next take an admissible copt U for

$$\{1, \cdots, 4, 2h + 1, \cdots, 2h + 4, 4h + 1, \cdots, 4h + 4, n\}\,.$$

Finally, we define

$$S = (S_1 - R_1) \cup (S_2 - R_2) \cup (S_3 - R_3) \cup (T^* - T) \cup U\,.$$

It is easy to check that S is an n-copt. The number of member of S is

$$3 \cdot \left[\frac{(2h + 1)^2 - (2h + 1) + 4}{6} - 4\right] + \left[(2h)^2 - 16\right] + 26$$
$$= 6h^2 + h = \frac{n(n - 1)}{6}.$$

Thus, S is admissible.

Case 2. Let S_1 be an admissible $(n - 1)$-copt. We define S to be the set of triples obtained by adding to S_1 the triples

$$(1, 2, n), (3, 4, n), \cdots, (n - 3, n - 2, n), (n - 2, n - 1, n)\,.$$

Then, S is an n-copt and S has

$$\frac{(n - 1)(n - 2)}{6} + \frac{n}{2} = \frac{n^2 + 2}{6}$$

members. Thus S is admissible.

Case 3. There exists h such that $n = 6h + 3$. Since we have listed admissible n-copts for $n = 3, 9, 15$, we may assume $h > 2$. We consider two subcases.

Subcase i. $h \equiv 0$ or $h \equiv 1 \pmod 3$. In this case there exists k such that $2h + 1 = 6k + 1$ or $2h + 1 = 6k + 3$. Let S_1 be an admissible $(2h+1)$–copt. For each triple $(a, b, c) \in S_1$ we choose a tricover for $(\{3a-2, 3a-1, 3a\}, \{3b-2, 3b-1, 3b\}, \{3c-2, 3c-1, 3c\})$. The union of all such tricovers, together with the triples $(1, 2, 3), (4, 5, 6), \cdots, (n-2, n-1, n)$ is an n-copt S. The number of members of S is

$$9 \cdot \frac{(2h+1) \cdot 2h}{6} + (2h + 1) = (2h + 1)(3h + 1) = \frac{n(n-1)}{6} .$$

If follows that S is admissible.

Subcase ii. $h \equiv 2 \pmod 3$. In this case there exists k such that $2h + 1 = 6k + 5$. We choose an admissible $(2h + 1)$-copt S_1 that contains the triples $(1, 2, 3), (1, 2, 4), (1, 2, 5), (3, 4, 5)$. If (a, b, c) is any other member of S_1, we choose a tricover for $(\{3a - 2, 3a - 1, 3a\}, \{3b - 2, 3b - 1, 3b\}, \{3c - 2, 3c - 1, 3c\})$. Let S_2 be the 15-copt exhibited at the beginning of our proof. We now define S to be the set whose members are the members of S_2, the members of the chosen tricovers, and the triples $(16, 17, 18), \cdots, (n - 2, n - 1, n)$. S is an n-copt, and the number of members of S is

$$35 + 9\left[\frac{(2h+1)^2 - (2h+1) + 4}{6} - 4\right] + \frac{n-15}{3} = \frac{n(n-1)}{6} .$$

Since S has $(1, 2, 3), (4, 5, 6), \cdots, (n - 2, n - 1, n)$ as members, S is admissible.

Case 4. For this case, the construction is exactly the same as in Case 2.

Case 5. We first observe that numbers of the form $6h + 5$, h a non-negative integer, form the same set as numbers of the form $3s - 4$, s an odd integer and $s > 1$. We have listed an admissible 5-copt, and an admissible 11-copt. Thus, we may assume $n = 6h + 5 = 3s - 4, s > 5$. We consider two subcases.

Subcase i. There exists k such that $s = 6k + 1$ or $s = 6k + 3$. In this case, we let

$$\begin{aligned} A_1 &= \{1, \cdots, s - 2\} \\ A_2 &= \{s - 1, \cdots, 2s - 4\} \\ A_3 &= \{2s - 3, \cdots, 3s - 6\} . \end{aligned}$$

There is a tricover K of (A_1, A_2, A_3) such that $(1, s - 1, 2s - 3) \in K$. For $i = 1, 2, 3$ we define

$$R_i = A_i \cup \{3s-5, 3s-4\}\ .$$

and let S_i be an admissible R_i-copt such that $(1, 3s-5, 3s-4) \in S_1$, $(s-1, 3s-5, 3s-4) \in S_2$ and $(2s-3, 3s-5, 3s-4) \in S_3$. We define $S = K \cup S_1 \cup S_2 \cup S_3$. It is easy to see that S is an n-copt, and that S has

$$(s-2)^2 + \frac{3s(s-1)}{6} = \frac{3s^2-9s+8}{2} = \frac{n^2-n+4}{6}$$

members. Since $(1, 3s-5, 3s-4)$, $(s-1, 3s-5, 3s-4)$, $(2s-3, 3s-5, 3s-4)$, $(1, s-1, 2s-3)$ are members of S, S is admissible.

Subcase ii. There exists k such that $s = 6k+5$. We define

$$A_1 = \{1, \cdots, s-2\}$$
$$A_2 = \{s-1, \cdots, 2s-4\}$$
$$A_3 = \{2s-3, \cdots, 3s-6\}$$

and let $R_i = A_i \cup \{3s-5, 3s-4\}$ for $i = 1, 2, 3$. By the inductive hypothesis, there exists an admissible R_i-copt S_i such that S_i contains the set B_i, where

$$B_1 = \{1, 2, 3), (1, 3s-5, 3s-4), (2, 3s-5, 3s-4), (3, 3s-5, 3s-4)\}\ ,$$
$$B_2 = \{(s-1, s, s+1), (s-1, 3s-5, 3s-4), (s, 3s-5, 3s-4), (s+1, 3s-5, 3s-4)\}\ .$$
$$B_3 = \{(2s-3, 2s-2, 2s-1), (2s-3, 3s-5, 3s-4), (2s-2, 3s-5, 3s-4), (2s-1, 3s-5, 3s-4)\}\ .$$

Let $G = \{1, 2, 3, s-1, s, s+1, 2s-3, 2s-2, 2s-1, 3s-5, 3s-4\}$. G has 11 members, and hence there exists an admissible G-copt M.

We choose a tricover T_1 for $(\{1, 2, 3\}, \{s-1, s, s+1\}, \{2s-3, 2s-2, 2s-1\})$ and extend T_1 to a tricover T for (A_1, A_2, A_3).

We now define

$$S = (S_1 - B_1) \cup (S_2 - B_2) \cup (S_3 - B_3) \cup M \cup (T - T_1)\ .$$

It is a routine matter to verify that S is an n-copt. The number of members of S is

$$3\left[\frac{s^2-s+4}{6} - 4\right] + 19 + \left[(s-2)^2 - 9\right] = \frac{3s^2-9s+8}{2} = \frac{n^2-n+4}{6}\ .$$

Since $S \supset M$ and M is admissible, it follows that S is admissible.

3. Properties of minimal n-copts. Let S be a minimal n-copt. If $n \equiv r \pmod 6$, for $r = 0, 2, 4, 5$, then the covering is not exact and some pairs must be contained in more than one member of S. However, it is possible to state precisely the way in which this sort of "multiple covering" takes place. Our results are contained in the next three theorems.

THEOREM 2. *Let $n = 6k$, and let S be an n-copt for which $C(S)=\varphi(n)$. There exists a partition of $\{1, 2, \cdots, n\}$ into $3k$ pairs $P_1, P_2, \cdots, P_{3k}$, each of which is contained in exactly two members of S. Every other pair (u, v), $1 \leqq u < v \leqq n$, is contained in exactly one member of S.*

Proof. For $1 \leqq j \leqq n$, let $f(j)$ be the number of members of S that contain j. It is clear that $f(j)$ is at least $n/2$, so that $f(j) = n/2 + g(j)$, $g(j) \geqq 0$. We obtain

$$\sum_{j=1}^{n} f(j) = 3\varphi(n) .$$

Thus

$$\sum_{j=1}^{n}\left[\frac{n}{2} + g(j)\right] = 3 \cdot \frac{n^2}{6} , \text{ and}$$

$$\frac{n^2}{2} + \sum_{j=1}^{n} g(j) = \frac{n^2}{2} .$$

We see that $g(j) = 0$ for $j = 1, \cdots, n$ and $f(j) = n/2$. Since for each $k \neq j$ there is at least one member of S which contains (j, k), there must exist $j^* \neq j$ such that (j, j^*) is contained in exactly two members of S, and (j, k) is contained in exactly one member of S for $j \neq k \neq j^*$. Moreover, $j^{**} = j$, and hence the pairs (j, j^*) are the $n/2$ pairs $P_1, P_2, \cdots, P_{3k}$.

THEOREM 3. *Let $n = 6k + 2$ or $n = 6k + 4$, and let S be an n-copt for which $C(S) = \varphi(n)$. There exist $n/2 + 1$ pairs $P_1, \cdots, P_{n/2+1}$ which are contained in exactly two members of S. Every other pair is contained in exactly one member of S. There exists an integer m which is contained in exactly three of the pairs $P_1, \cdots, P_{n/2+1}$. Every other integer is contained in exactly one of the pairs $P_1, \cdots, P_{n/2+1}$.*

Proof. Let $f(j)$ be the number of members of S that contain the integer j. Since $f(j) \geqq n/2$, we can write

$$f(j) = \frac{n}{2} + g(j) , \qquad g(j) \geqq 0 .$$

Then

$$\sum_{j=1}^{n} f(j) = \frac{n^2}{2} + \sum_{j=1}^{n} g(j) = 3 \cdot \varphi(n) = \frac{n^2+2}{2} .$$

Thus $\sum_{j=1}^{n} g(j) = 1$. There exists an integer m such that $g(m) = 1$ and $g(j) = 0$ for $j \neq m$.

Now suppose $j \neq m$. There must exist j^* such that (j, j^*) is contained in exactly two members of S, and (j, h) is contained in exactly one member of S for $j \neq h \neq j^*$.

Since there are $n/2 + 1$ members of S that contain m, and each pair (m, j) is contained in at least one and not more than two members of S, there exist a, b, c, such that $(m, a), (m, b), (m, c)$ are each contained in exactly two members of S, and (m, j) is contained in exactly one member of S if $j \neq a, j \neq b$, and $j \neq c$.

If j is a member of $T = \{1. \cdots, n\} - \{m, a, b, c\}$, then $j^{**} = j$. Hence T is partitioned into pairs $P_1, P_2, \cdots, P_{(n-4)/2}$, each of which is contained in exactly two members of S. These pairs, together with $(m, a), (m, b), (m, c)$ form the set $P_1, \cdots, P_{n/2+1}$.

THEOREM 4. *If* $n = 6k + 5$ *and* S *is a minimal* n*-copt for which* $\varphi(n) = (n^2 - n + 4)/6$, *then one pair is contained in three members of* S *and every other pair is contained in exactly one member of* S.

Proof. For $1 \leqq j \leqq n$, we define $f(j)$ to be the number of members of S that contain j. Clearly $f(j) \geqq (n-1)/2$. We define $g(j) = f(j) - (n-1)/2$. Since $\sum_{j=1}^{n} f(j) = 3\varphi(n) = (n^2 - n + 4)/2$, we obtain

$$\sum_{j=1}^{n} g(j) = 2 .$$

There exists j_1 such that $g(j_1) > 0$. Since there are more than $(n-1)/2$ triples of S that contain j_1, there exists j_2 such that the pair (j_1, j_2) is contained in at least two triples $(j_1, j_2, j_3), (j_1, j_2, j_4)$. The integer j_2 must be in triples with $n - 4$ integers other than j_1, j_3, j_4, and it requires at least $(n-3)/2$ triples to satisfy this condition. Thus $f(j_2) \geqq (n+1)/2$ and $g(j_2) > 0$. We now see that $g(j_1) = g(j_2) = 1$ and $g(j) = 0$ if $j_1 \neq j \neq j_2$.

It now follows that if (u, v) is a pair for which $g(u) = 0$ or $g(v) = 0$, then (u, v) is contained in exactly one member of S. Since $3\varphi(n) = n(n-1)/2 + 2$, the pair (j_1, j_2) must be contained in three members of S.

Our Theorem 1 is of a constructive nature, and indicates how minimal n-copts can be constructed out of minimal m-copts for $m < n$. There are other methods, however, of constructing minimal n-copts out of minimal m-copts for $m <$ n. We give a lemma and theorem due to Reiss [2] which are useful in this connection. Our final theorem is analogous to the Reiss Theorem.

REISS LEMMA. *Let n be a positive integer. Let*

$$P = \{(u, v) \mid 1 \leqq u < v \leqq 2n\} .$$

Then there exists a partition of P into sets $S_1, S_2, \cdots, S_{2n-1}$, each containing n elements, such that for each $i, i = 1, 2, \cdots, 2n-1$, the coordinates of the n pairs in S_i constitute the integers $1, 2, \cdots, 2n$.

Proof. Let j be an integer such that $1 \leqq j \leqq 2n-1$. We define

$$T_j = \{(a, b) \mid 1 \leqq a < b \leqq j+1 \text{ and } a+b = j+2\}$$

and

$$R_j = \{(a, b) \mid j+1 < a < b < 2n \text{ and } a+b = j+2n+1\} .$$

Let $S_{2n-1} = T_{2n-1}$. For j even, $1 \leqq j \leqq 2n-2$, let

$$S_j = T_j \cup R_j \cup \left\{\left(\frac{j+2}{2}, 2n\right)\right\} .$$

For j odd, $1 \leqq j \leqq 2n-3$, let

$$S_j = T_j \cup R_j \cup \left\{\left(\frac{j+1+2n}{2}, 2n\right)\right\} .$$

It may be verified that the sets S_j have the desired properties.

REISS THEOREM. *Let m be odd and let S be an m-copt for which $C(S) = \varphi(m)$. Then there exists a $(2m+1)$-copt T such that $T \supset S$ and $C(T) = \varphi(2m+1)$.*

Proof. Let $P = \{(u, v) \mid m < u < v \leqq 2m+1\}$. We use the Reiss lemma to partition P into sets $S_1, \cdots, S_m$, each containing $(m+1)/2$ elements, such that for each $i, i = 1, 2, \cdots, m$, the coordinates of the $(m+1)/2$ pairs in S_i constitute the integers $m+1, m+2, \cdots, 2m+1$. We now define

$$T = S \cup \{(i, j, k) \mid 1 \leqq i \leqq m \text{ and } (j, k) \in S_i\} .$$

It is easily verified that T is a $(2m+1)$-copt. If $m \equiv 1$ or $m \equiv 3 \pmod 6$, then

$$C(S) = \frac{m(m-1)}{6} + \frac{m(m+1)}{2} = \frac{4m^2+2m}{6} = \frac{(2m+1)(2m)}{6} = \varphi(2m+1) .$$

If $m \equiv 5 \pmod 6$, then

$$C(S) = \frac{m^2-m+1}{6} + \frac{m(m+1)}{2} = \frac{4m^2+2m+4}{6}$$

$$= \frac{(2m+1)^2-(2m+1)+4}{6} = \varphi(2m+1) .$$

THEOREM 5. *Let n be an even integer and let S be an n-copt for which $C(S) = \varphi(n)$. Then there exists a $2n$-copt T such that $C(T) = \varphi(2n)$ and $S \subset T$.*

Proof. According to the Reiss Lemma there exists a partition of the set

$$P = \{(u, v) \mid n + 1 \leqq u < v \leqq 2n\}$$

into $n - 1$ sets $A_1, A_2, \cdots, A_{n-1}$ such that for each i, $i = 1, 2, \cdots, n - 1$, the coordinates of the $n/2$ pairs in A_i constitute the integers $\{n + 1, \cdots, 2n\}$. Let $A_n = A_{n-1}$, and let

$$T = S \cup \{(i, j, k) \mid i = 1, 2, \cdots, n\,;\ (j, k) \in A_i\}\ .$$

It is easy to prove that T satisfies the desired conditions.

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