HOMOGENEOUS STOCHASTIC PROCESSES*

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Summary. The form of a stationary translation-invariant Markov process on the real line has been known for some time, and these processes have been variously characterized as infinitely divisible or infinitely decomposable. The purpose of this paper is to study a natural generalization of these processes on a homogeneous space (X, G). Aside from the lack of structure inherent in the very generality of the spaces (X, G), the basic obstacles to be surmounted stem from the presence of non trivial compact subgroups in G and the non commutativity of G, which precludes the use of an extended Fourier analysis of characteristic functions, a tool which played a dominant role in the classical studies. Even in the general situation there is a striking similarity between homogeneous processes and their counterparts on the real line.

A homogeneous process is a process in the terminology of Feller [3] on a locally compact Hausdorff space X, whose transition probabilities $P(t, x, dy)$ are invariant under the action of elements $g \in G$ of a transitive group of homeomorphisms of X, in the sense that $P(t, g[x], g[dy]) = P(t, x, dy)$. It is shown that if every compact subset of X is separable or G is commutative the family of measures $t^{-1}P(t, x, \cdot)$ converges to a not necessarily bounded Borel measure $Q_x(\cdot)$ on X-$\{x\}$ as $t \to 0$, meaning that for every bounded continuous, complex valued function f on X which vanishes in a neighborhood of x and is constant at infinity $t^{-1}P(t, x, f) \to Q_x(f)$.

In 3 we show that the paths of a separable homogeneous process are bounded on every bounded t-interval and have right and left limits at every t with probability one. If the action of G on X is used to translate the origin of each jump to x, it is shown for suitably regular compact sets D that the probability of a jump into D while $t \in [0, T]$ is given by 1–exp $\{-TQ_x(D)\}$. The maps $f \to P(t, \cdot, f) = (T_t f)(\cdot)$ map the Banach space, $C(X)$, of continuous functions generated by the constants and functions with compact support into itself, and by a suitable normalization can be assumed strongly continuous for $t \geq 0$. Indeed, T_t is a strongly continuous semi-group. The domain $D(A)$ of the infinitesimal generator A of T_t admits a smoothing operation whose precise

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nature is described in Corollary 1 of Theorem 2.2. Roughly speaking, if $g \in D(A)$, $\varepsilon > 0$, and $f \in C(X)$ we can find an $h \in D(A)$ such that $\|h - f\|_\infty < \varepsilon$ and $f = h$ on any preassigned compact subset of interior $(\{x \mid f(x) = g(x)\})$.

A family of measures $P(x, A)$ which is a probability measure on the Borel subsets of X for fixed $x \in X$, measurable in x for fixed A, and invariant under G(i.e. $P(g[x], g[A]) = P(x, A)$), is called a homogeneous transition probability of norm one. Such a family generates a continuous endomorphism $f \to P(\cdot, f) = (Pf)(\cdot)$ of $C(X)$, and a homogeneous process $T_t = \exp\{tr(P-1)\}$. This latter process is called a compound Poisson process. In 4 we study strong convergence for compound Poisson processes (Definition 4.1) and prove among other facts that every homogeneous process is a strong limit of compound Poisson processes $\exp\{tr_i(P_i - 1)\}$, and if $r_i \leq M < +\infty$ the limit process is necessarily compound Poisson. If X is given the discrete topology every homogeneous process on X is compound Poisson. In case the Q_x associated with $P(t, x, dy)$ vanishes identically, or equivalently $P(t, x, dy)$ has continuous paths, we show in 5 that $P(t, x, dy)$ is the strong limit of compound Poisson processes whose $P_i(x, dz)$ have support arbitrarily closed to x.

In 7 we study subordination of homogeneous processes as defined by Bochner [1]. By phrasing the definition in terms of a probabilistically run clock it is shown that many processes are maximal in the partial order induced by subordination. If we follow the notation in (7.2) where $\exp\{tS(b, A, F, z)\}$ denotes the family of characteristic functions of a homogeneous process $X(t, \omega)$ on Euclidean n-space, we obtain the following type of result. When support (F) is compact, $X(t, \omega)$ is not subordinate to any process but itself unless support (F) is also contained in the half line R^+b and $A = 0$. In this latter case $X(t, \omega)$ is subordinate to the Bernoulli process $Z(t, \omega) = tb$. Actually somewhat stronger statements can be made but they are proven only for the real line.

In our notation we have not distinguished between the application of a measure μ to a function f and the measure of a measurable set E, denoting these respectively by $\mu(f)$ and $\mu(E)$. The set X–E is denoted by E^c and the usual convention is adopted in letting C, R, R^+, Z, and Z^+ represent respectively the complex numbers, the reals, the non-negative reals, the integers, and the non-negative integers.

I would like to take this opportunity to express my gratitude to Professor Bochner who has patiently encouraged this work, and whose own ideas are at the base of § 7.

1. Introduction. Let G be a Hausdorff, locally compact topological group and H one of its compact subgroups. We consider G as a group of homeomorphisms of the left cosets of G modulo H, which we denote by G/H, the left coset xH being mapped by $a \in G$ into the left coset axH. Moreover, if we are primarily interested in the space G/H and the action of G on this space, there is no reason why the subgroup H should play a dominant role, for the homeomorphism $x \to xb$ of G carries the left coset xH of H into the left coset $xbb^{-1}Hb$ of $b^{-1}Hb$. As far as left multiplication by elements of G is concerned this is an operator homeomorphism and G/H is equivalent to $G/b^{-1}Hb$. For this reason we use the neutral letter X for the space G/H and denote the operation of $a \in G$ on $x \in X$ by $a[x]$. We call the system (X, G) a homogeneous space and note that X is naturally homeomorphic to any coset space of the form G/G_z where $G_z = \{a \in G \mid a[z] = z\}$.

For the sake of exposition let $N(X)$ be the Banach space of regular bounded complex Borel measures on X; $C_c(X)$ the linear space of continuous complex valued functions with compact support; $C_\infty(X)$ the closure of $C_c(X)$ in the uniform norm; $C(X)$ the Banach space of functions generated by $C_\infty(X)$ and the constant functions with the uniform norm. We use the current notation of W. Feller and denote linear transformations of $N(X)$ by postmultiplication. In this notation a linear transformation T of $C_\infty(X)$ is denoted by the same letter as its adjoint transformation on $N(X)$, viz. $\mu T(f) = \mu(Tf)$. By the expression $\mu \geq 0$, $\mu \in N(X)$, we mean μ is a real valued non-negative measure, and by the transformation L_a we refer to the isometries of $C_c(X)$, $C_\infty(X)$, $C(X)$ and $N(X)$ generated by translation of X by $a \in G$, viz. $(L_a f)(x) = f(a^{-1}[x])$, $\mu L_a(E) = \mu(a[E])$. When $x \in X$ we denote by δ^x the measure placing a unit mass at x, so that $\delta^x(E) = 1$ if $x \in E$ and 0 otherwise. For example, we shall often use the relationship $\delta^x L_{a^{-1}} = \delta^{a[x]}$. Finally, when we say that a directed sequence $(\mu_q)_{q \in Q}$ of measures-commonly called a net in $N(X)$ - converges weakly to $\mu \in N(X)$, in symbols $\mu_q \to \mu$, we mean for every $f \in C_c(X)$, $\mu_q(f) \to \mu(f)$ as complex numbers.

DEFINITION 1.1. A *homogeneous transition probability* is a continuous endomorphism $P: N(X) \to N(X)$ satisfying:

(i) $\mu \geq 0$ implies $\mu P \geq 0$;
(ii) $\mu_q \to \mu$ implies $\mu_q P \to \mu P$;
(iii) $PL_a = L_a P$.

An endomorphism, P, with properties (i) and (ii) is usually called a transition probability and (iii) makes the transition probability homogeneous.

When $f \in C_\infty(X)$ it follows from (ii) that $\delta^x P(f)$ is continuous in x. In addition $\delta^x P(f) \in C_\infty(X)$. To show this let $a_i[z]$ be any directed sequence in X tending to infinity, then by virtue of the assumed compactness of G_z, $a_i[D] \to$ infinity for any compact set $D \subset X$ and

$L_{a_i^{-1}}f(y) = f(a_i[y]) \to 0$ boundedly and uniformly on every compact set. It follows immediately that

$$\delta^{a_i[z]}P(f) = \delta^z L_{a_i^{-1}}P(f) = \delta^z P(L_{a_i^{-1}}f) \to 0 \ ,$$

proving $\delta^x P(f) \in C_\infty(X)$. Now for any given $\mu \in N(X)$ let $\mu_q = \sum_i m_{qi}\delta^{x_{qi}}$ be a bounded, directed sequence of purely atomic measures in $N(X)$ approaching μ weakly, $\mu_q \to \mu$. Then separate calculations show that on the one hand $\mu_q P(f) \to \mu P(f)$, while on the other

$$\mu_q P(f) = \sum_i m_{qi}\delta^{x_{qi}}P(f) = \mu_q(\delta^x P[f]) \to \mu(\delta^x P[f]) \ .$$

Accordingly, for $f \in C_\infty(X)$ and $\mu \in N(X)$

$$\mu(\delta^x P[f]) = \mu P(f) \ ; \tag{1.1}$$

the adjoint of P transforms $C_\infty(X)$ into itself; and when $f \in C_\infty(X)$

$$(Pf)(x) = \delta^x P(f) \ . \tag{1.2}$$

By letting $f \uparrow 1$ in (1.1) we see that (1.1) and (1.2) hold for $f \in C(X)$ as well. If P and Q are homogeneous transition probabilities

$$||P|| = \delta^x P(1), \tag{1.3}$$

$$||PQ|| = ||P|| \ ||Q|| \ . \tag{1.4}$$

We obtain (1.3) by noting first from (iii) that $\delta^x P(1)$ does not depend on $x \in X$, and then it follows from (i) that $\mu \geq 0$ implies $||\mu P|| = \mu P(1) = ||\mu||\delta^x P(1)$; so $||P|| \geq \delta^x P(1)$. The opposite inequality is obtained by using the preceeding remarks on a Jordan decomposition of $\mu \in N(X)$, while (1.4) follows easily from (1.3).

We remark at this point that if $z \in X$ and m is the normalized Haar measure of G_z, then the map $P \to \tilde{P} \in N(G)$, given for $g \in C_\infty(G)$ by $\tilde{P}(g) = \int \delta^z P(dy) \int m(dw)g(yw)$, maps the Banach algebra generated by the homogeneous transition probabilities isometrically onto the subalgebra $m * N(G) * m$ of $N(G)$.

DEFINITION 1.2. A *homogeneous process* is a one-parameter semigroup, $(T_t)_{t>0}$, of homogeneous transition probabilities which is temporally continuous in the sense

(iv) $\mu \in N(X)$ and $f \in C(X)$ imply $\mu T_t(f)$ is continuous in t.

Using (1.4), (1.3) and (iv) we see that $||T_t|| = e^{pt}$. Therefore, we replace T_t by the equivalent process $e^{-pt}T_t$ and assume in the rest of this paper that (v) below is satisfied unless explicitely stated otherwise.

(v) $||T_t|| = 1$,

The requirement of temporal continuity for a homogeneous process is equivalent to the weak continuity of the restricted adjoint process $T_t: C(X) \to C(X)$, and by well known results, [5], implies the strong continuity of this last process for $t > 0$. The theory of semi-groups shows that T_t is strongly continuous at $t = 0$ on the closure of $\bigcup_{t>0} T_t C(X)$. This may be a proper subspace of $C(X)$. For example, let (G, G) be the homogeneous space of a group acting on itself by left translations with $\mu T_t = \mu * m$, where m is the normalized Haar measure of a non trivial compact subgroup. G. Hunt [7, pp. 291–293] has shown that by enlarging the subgroups G_z if necessary we can always assume a homogeneous process possesses the following quality.

Property I. For any fixed $x \in X$ and any Borel measurable neighborhood N of x, $\delta^x T_t(N) \to 1$ or equivalently $\delta^x T_t(N^c) \to 0$ as $t \to 0$.

It is a routine calculation to show that Property I is equivalent to the strong continuity of $T_t: C(X) \to C(X)$ at the origin, so we henceforth assume our processes satisfy Property I, and by Hunt's result we can do this without loss of generality.

In view of the above statements we can apply the Hille-Yosida theory of strongly continuous semi-groups to the semi-group $T_t: C(X) \to C(X)$. An elementary application of this theory shows that there exists a dense linear subspace of $C(X)$ which we denote by $D(A)$, and a closed linear operator $A: D(A) \to C(X)$ with the property that for $f \in D(A)$ $\limsup_{t\to 0} \| (Tf - f)t^{-1} - Af \|_\infty = 0$.

2. Properties of $D(A)$. In this section we investigate the domain of the infinitesimal generator for homogeneous processes which satisfy Property II below. Later we show that Property II is automatically satisfied if either X is separable or G is commutative.

Property II. There is a regular Borel measure Q_z on $X - \{z\}$, such that $t^{-1}(T_t f)(z) \to Q_z(f)$ as $t \to 0$ for all $f \in C(X)$ which vanish on any neighborhood of z.

Q_z is positive and $Q_z L_{a^{-1}} = Q_{a[z]}$. In general, of course, Q_z will be unbounded, although its values on any set E lying in the complement of a fixed neighborhood N of z must be bounded, or equivalently $\delta^z T_t(E) = O(|t|)$ as $t \to 0$. This is easily checked if one notes that $D(A)$ includes the constant functions and is invariant under G. For then we can choose an $f \in D(A)$ which is everywhere positive on X, vanishes at z, and is greater than 1 on N^c. Clearly

$$Q_z(E) \leq (Af)(z) = \lim_{t\to 0} t^{-1}(T_t f)(z)$$

is bounded independently of $E \subset N^c$.

The homogeneity of the process T_t entails a uniformity in this convergence to Q_z which may be stated as follows.

THEOREM 2.1. *If J is an open subset of X, $f \in C(X)$, and $f(J) = 0$; then $t^{-1}T_t f$ converges to its limit as $t \to 0$ uniformly on every compact subset of J. If in addition J^c is compact, this convergence is even uniform on every closed subset of J.*

Proof. To prove the first assertion it suffices to show that the approach is uniform on a neighborhood of $z \in J$ of the form $N[z]$ where $N^2[z] \subset J$ and $N = N^{-1}$ is compact. If $a, b \in N$, and K is chosen so that $t^{-1}\delta^z T_t(N[z]^c) \leq K$, then

$$|t^{-1}\delta^{a[z]}T_t(f) - t^{-1}\delta^{b[z]}T_t(f)| \leq K\|L_{ba^{-1}}(f) - f\|_\infty .$$

The family $\{h_t(x) | t > 0, h_t(x) = t^{-1}\delta^x T_t(f)\}$ is accordingly, equicontinuous on $N[z]$, and $h_t(x) \to Q_x(f)$ as $t \to 0$. It follows that this approach is uniform on $N[z]$. The second assertion is a consequence of the fact that for each $\varepsilon > 0$ there is a compact set D_ε and a $t_\varepsilon > 0$, such that $x \notin D_\varepsilon$ and $t \leq t_\varepsilon$ imply $|t^{-1}(T_t f)(x)| \leq \varepsilon$. Assume on the contrary that there is a sequence $a_i[z] \to$ infinity, together with a sequence $t_i \to 0$, such that $|t_i^{-1}(T_{t_i} f)(a_i[z])| > \varepsilon$. Now choose a bounded sequence of functions g_j from $C(X)$ which converges to zero monotonely while their supports approach infinity, and which satisfy the crucial inequalities $g_j \geq \sup_{i \geq j} |L_{a_i^{-1}}(f)|$. The inequalities

$$\varepsilon \leq \liminf_i |t_i^{-1}(T_{t_i} f)(a_i[z])| \leq \lim_j Q_z(g_j) = 0$$

yield the contradiction which proves the second assertion.

Suppose $h \in D(A)$ and on some open subset J of X we have $f = h$, where $f \in C(X)$. Suppose further that either $\bar{J}$ or J^c is compact. Let D be a closed subset of J, containing a neighborhood of infinity if J^c is compact; and let $m \in C(X)$ be constructed so that $0 \leq m \leq 1$, $m(D) = 0$, and $m(J^c) = 1$. The map $S: X \times X \times (0, \infty) \to R$ defined by $S(x, y, t) = (T_{m(y)t} f)(x)$ is, because of the strong continuity of T_t, defined and continuous in (x, y, t). Consequently its restriction to the diagonal in the first two components is continuous on $X \times (0, \infty)$. When $x \in J^c$, $(T_{m(x)t} f)(x) = (T_t f)(x)$, while if J^c is compact $(T_{m(x)t} f)(x) = (T_0 f)(x) = f(x)$ close to infinity. The maps $W_t: C(X) \to C(X)$ defined by $f(x) \to T_{m(x)t} f(x)$ form a strongly continuous family of bounded linear transformations, so that we may form the Rieman integral $g(x) = r^{-1}\int_0^r (W_t f)(x) dt$. If $x \in D$, $m(x) = 0$ and $g(x) = f(x)$; while for any x,

$$\begin{aligned} |g(x) - f(x)| &= r^{-1}\left|\int_0^r \{W_t f(x) - f(x)\} dt\right| \\ &\leq \sup_{t<r} \|W_t f - f\|_\infty \to 0 \text{ as } r \to 0 . \end{aligned}$$

Thus for any given $\varepsilon > 0$ we may guarantee that $\|g - f\|_\infty < \varepsilon$ by a sufficiently small choice of r.

THEOREM 2.2. *Let D be a closed subset of the open set J in X, where either $\bar{J}$ or J^c is compact. Let $h \in D(A)$, $f \in C(X)$, and $f = h$ on J. Then for each $\varepsilon > 0$ there exists a $g_\varepsilon \in D(A)$, such that* (i) *$h = f = g_\varepsilon$ on D, and* (ii) *$\|g_\varepsilon - f\|_\infty < \varepsilon$.*

Proof. We only need to prove that the g defined above lies in $D(A)$. To do this we show that

$$s^{-1}\{T_s g - g\} = r^{-1}\int_0^r s^{-1}W_t\{T_s f - f\}dt$$

converges to its limit uniformly on X. The computations are divided into two cases.

Case I. $m(x) \leq 1/2$. On this closed subset of J

$$s^{-1}\{T_s f - f\} = s^{-1}\{T_s h - h\} + s^{-1}\{T_s(f - h) - (f - h)\} .$$

Now $s^{-1}\{T_s h - h\} \to Ah$ in the uniform norm as $s \to 0$, and by Theorem 2.1 $s^{-1}\{T_s(f - h) - (f - h)\} \to Q_x(f - h)$ uniformly on $\{x \mid m(x) \leq 1/2\}$. Accordingly,

$$s^{-1}\{T_s g - g\} \to r^{-1}\int_0^r W_t\{Ah + Q_x(f - h)\}dt$$

uniformly on $\{x \mid m(z) \leq 1/2)$.

Case II. $m(x) \geq 1/4$. On this set

$$\begin{aligned} s^{-1}\{T_s g - g\} &= (rs)^{-1}\int_0^r \{T_{s+m(x)t}f - T_{m(x)t}f\}dt \\ &= \{rsm(x)\}^{-1}\Big[{}_{m(x)r}\int^{m(x)r+s}\{T_u f\}du - \int_0^s T_u f du\Big] \\ &\to \{rm(x)\}^{-1}\{T_{m(x)r}f(x) - f(x)\} \end{aligned}$$

as $s \to 0$, uniformly on $\{x \mid m(x) \geq 1/4\}$. These observations show that $g \in D(A)$.

In the above proof we did not really need the fact that $h \in D(A)$. Indeed, had we replaced h by $h + u$, where $u \in C(X)$ and $u(J) = 0$, there would have been no change in the proof. Furthermore, since $g(x) = r^{-1}\int_0^r (T_{m(x)t}f)(x)dt$, if f is real valued we have the relation $\inf_x f(x) \leq \inf_y g(y) \leq \sup_z g(z) \leq \sup_x f(x)$. These remarks allow us to state a corollary.

COROLLARY 1. *Let $J_1, J_2, \cdots, J_n$ be disjoint open sets of X and let D_i be a closed subset of J_i. Suppose that for each i either $\bar{J}_i$ is compact or for at most one i, J_i^c is compact. Let $h_i \in D(A)$ and $f \in C(X)$ satisfy $f = h_i$ on J_i. Then for each $\varepsilon > 0$ there exists a $g_\varepsilon \in D(A)$, such that*

(i) $g_\varepsilon = h_i = f$ on D_i,

(ii) $||g_\varepsilon - f||_\infty < \varepsilon$, and

(iii) *if f is real valued then* $\inf_x f(x) \leq \inf_y g_\varepsilon(y) \leq \sup_z g_\varepsilon(z) \leq \sup_x f(x)$.

Specializing the preceding we get the following.

COROLLARY 2. *If B is compact in X, H is closed, and $H \cap B = \phi$; there exists an $f \in D(A)$, such that $0 \leq f \leq 1, f(H) = 0$,* and $f(B) = 1$.

The above results have been derived under Property II and we now show that we can replace Property II by a condition on $D(A)$.

Property III. For each $\varepsilon > 0$ and $f \in C(X)$ which is zero in a neighborhood J of x, there exists an $f_\varepsilon \in D(A)$, such that $f_\varepsilon = 0$ in a neighborhood U of x which is independent of ε, $||f_\varepsilon - f||_\infty < \varepsilon$, and if f is real valued so is f_ε with $\inf_x f(x) \leq \inf_y f_\varepsilon(y) \leq \sup_z f_\varepsilon(z) \leq \sup_x f(x)$.

THEOREM 2.3. *Property* III *is equivalent to Property* II *for homogeneous processes.*

Proof. We have already seen that Property II implies Property III and will now demonstrate the converse. First, let J be a neighborhood of x, and choose a function $h \in C(X)$, such that $h = 0$ in a neighborhood $U \subset J$ of x, and $h(J^c) = 1$. We also require that $h \geq 0$ everywhere. Now for $0 < \varepsilon < 1$ choose an h_ε in accordance with Property III.

$$t^{-1}\delta^x T_t(J^c) \leq (1-\varepsilon)^{-1}\delta^x T_t(h_\varepsilon)t^{-1} \leq K(U) < +\infty$$

for some constant $K(U)$ depending on U. If $f \in C(X)$ and $f(J) = 0$, Property II is equivalent to the fact that $t^{-1}\delta^x T_t(f) \to a$ limit as $t \to 0$. The inequalities

$$\begin{aligned} &|\lim_{t\to 0} t^{-1}\delta^x T_t(f_{\varepsilon_1}) - \lim_{t\to 0} t^{-1}\delta^x T_t(f_{\varepsilon_2})| \leq \\ &\quad \leq \limsup_{t\to 0} t^{-1}\delta^x T_t(|f_{\varepsilon_1} - f_{\varepsilon_2}|) \\ &\quad \leq (\varepsilon_1 + \varepsilon_2)\limsup_{t\to 0} t^{-1}\delta^x T_t(U^c) \leq (\varepsilon_1 + \varepsilon_2)K(U) \end{aligned}$$

show that $\lim_{\varepsilon\to 0}\lim_{t\to 0} t^{-1}\delta^x T_t(f_\varepsilon)$ exists. If we call this limit b, the inequality

$$|t^{-1}\delta^x T_t(f) - b| \leq |t^{-1}\delta^x T_t(f_\varepsilon) - b| + \varepsilon K(U)$$

shows by first letting $t \to 0$ and then $\varepsilon \to 0$, that $t^{-1}(T_t f)(x) \to b$ as $t \to 0$, completing the proof.

By the use of well-known smoothing techniques which exist on a differentiable manifold we can conclude that if G is a Lie group or if X is a differentiable manifold on which G acts differentiably, and $D(A)$ includes $C^\infty(X) \cap C(X)$, then Property III and Property II are automatically satisfied.

We close this section by remarking that the only place where we have used the homogeneity of T_t was in the proof of Theorem 2.1, so that a strongly continuous semi-group on $C(X)$ satisfying the conclusions of Theorem 2.1 also satisfies the conclusions of Corollary 1 if its adjoint preserves positivity.

3. The paths of a homogeneous process and Property II. Before we discuss the nature of the paths of a homogeneous process it is necessary to take certain precautions which will assure us that the properties we want to discuss can be handled by the theory of probability. Given a consistent set of transition probabilities for a particle moving in a locally compact Hausdorff space it is possible, using a theorem of Kolmogoroff's, to construct an infinite product space in which these transition probabilities determine the finite dimensional distributions. From this we can construct, in the usual manner, a set of paths and a probability measure on this path space. Alternatively, we can consider a process as a family $X(t, \omega)$ of measurable transformations from some abstract sample space Ω to the range space X. Since there is some freedom in the definition of $X(t, \omega)$ given only the finite dimensional distributions, it is important to notice that the concept of separability as used on the real line is available in this case.

DEFINITION 3.1. A process $\{X(t, \omega), 0 \le t\}$ will be called *separable relative to the class A* of closed subsets of the locally compact Hausdorff space X if, and only if, (1) there exists a denumerable subset $\{t_j\} \subset [0, \infty)$, and (2) an event $K \subset \Omega$ with $P\{K\} = 0$, such that for every open interval $I \subset [0, \infty)$, and every set $F \in A$, the event

$$\{\omega \mid X(t_j, \omega) \in F, t_j \in I \cap \{t_j\}\} - \{\omega \mid X(t, \omega) \in F, t \in I\} \subset K .$$

The importance of the concept of separability rests on the validity of the following theorem.

THEOREM 3.1. *Let X be a separable locally compact Hausdorff space, and let $\{X(t, \omega), t \le 0\}$ be an X-valued process. Then there exists an X^*-valued stochastic process $\{X(t, \omega), t \ge 0\}$ such that:* (1) $X(t, \omega)$ *is* *defined on the same* ω*-space,* Ω*, as* $X(t, \omega)$ *and takes values in the one point compactification,* X^**, of* X*;* (2) $X(t, \omega)$ *is separable relative to the class of closed sets of* X^**;* (3) *for every* $t \geq 0$ $\{P\omega \mid X(t, \omega) = X(t, \omega)\} = 1$.

Proof. This theorem may be proved in a manner entirely analogous to the comparable theorem on the real line, and its proof is given, for example, in Doob [2 p. 57]. We remark that the separability of the space is necessary for this proof.

In the following we discuss the displacements of the path $X(t, \omega)$ during a closed interval of time. The success of our technique depends directly on the possibility of comparing two such displacements with different origins. If our homogeneous space (X, G) is that of a group acting on itself, we can translate all displacements origins to the identity and their endpoints are uniquely determined. On a general homogeneous space, however, the endpoint of a translated displacement is not determined by its new origin. We introduce several concepts from the calculus of relations and a space of displacements to handle this ambiguity.

By a relation (U) on X we mean a subset of $X \times X$ which contains the diagonal. The notations $(U)^{-1} = \{(y, x) \mid (x, y) \in (U)\}$, $(W) \circ (U) = \{(x, y) \mid$ for some $z, (x, z) \in (U)$ and $(z, y) \in (W)\}$, and $(U)[A] = \{x \mid (y, x) \in (U)$ for some $y \in A\}$ are standard. It is sometimes convenient to substitute U_x for $(U)[\{x\}]$ and in this notation $(W) \circ (U)_x = \bigcup_{y \in U_x} W_y$. A relation (U) is called homogeneous whenever $(x, y) \in (U)$ implies $(a[x], a[y]) \in (U)$ for all $a \in G$, and it is convenient to describe a relation by giving a property possessed by all the sets U_x. For example, we call a relation (U) compact, open, closed, or a neighborhood relation if each U_x is compact, open, closed or a neighborhood of x respectively. Using a double coset representation it is easily shown that the class of homogeneous neighborhood relations forms a base for the natural uniformity of the homogeneous space (X, G). We say the displacement from x to y belongs to the homogeneous relation (U) if $(x, y) \in (U)$. The reason for this mention of relations is that they appear to be exactly what is needed to generalize the statement and proof of a theorem from Kinney [9], p. 292–293.

THEOREM 3.2. *Let* (X, G) *be a homogeneous space satisfying the first axiom of countability, and let* X^* *be the space* X *compactified by adding a single point at infinity. Let* $\{X(t, \omega), t \geq 0\}$ *be an* X^**-valued homogeneous stochastic process governed by the transition probabilities* $\delta^x T_t$*. and separable with respect to the closed subsets of* X^**. Then if* $T > 0$*, there is an* ω*-set* E_T *with* $P\{E_T\} = 0$*, such that* $\omega \notin E_T$ *implies the statements below.*

(1) $X(t, \omega)$ *is bounded on* $t \in [0, T)$*. By which we mean*

$$Cl[\{X(t, \omega) \mid t \in [0, T)\}]$$

is a compact subset of X.

(2) *$X(t, \omega)$ has finite right and left hand limits at every $t \in [0, T)$.*

(3) *The number of jumps in $[0, T)$ whose displacements lie outside a homogeneous neighborhood relation (U) is finite. Furthermore, for any homogeneous neighborhood relation, (U), the maximum number of disjoint subintervals (t, s) of $[0, T)$ for which $(X(t^-, \omega), X(s^-, \omega)) \notin (U)$ is finite, where $X(t^-, \omega) = \lim_{h \uparrow 0} X(t + h, \omega)$.*

In particular the use of the one point compactification of X was only necessary to cover the processes constructed in Theorem 3.1 and may be eliminated as soon as (1) is proved.

The displacement or jump of a particle from x to y can be considered as a point in the space $X \times X$. It is natural to consider classes of similar displacements, and this involves the introduction of an equivalence relation on $X \times X$, two points, (x, y) and (x', y'), being considered equivalent if there is an $a \in G$ such that $(a[x], a[y]) = (x', y')$. This is a closed equivalence relation and the quotient space is homeomorphic to the space of double cosets $\{G_x a G_x\} = Y$. Let $p: X \times X \to Y$ be the canonical projection. Y is a locally compact Hausdorff space known as the space of displacements, and p is a continuous open mapping. If we fix the first component at x, so we only consider jumps origination at x, we get a map $p': X \to Y$ given by $z \to p(x, z)$. Using this map the commutativity of T_t with L_a shows that we can very properly place the measures $\delta^x T_t$ and Q_x on the space Y without losing a thing. If $f \in C_c(X)$, and m is the normalized Harr measure of G_x, the equation $\delta^x T_t(f) = \int f(a[z]) m(da) \delta^x T_t(dz)$ indicates a means of returning $\delta^x T_t$ and Q_x to X from Y. The following theorem is the key to the results of this section. It and Theorem 3.4 are generalizations of similar results for homogeneous processes on the real line which may be found, for example, in Doob [2, p. 422–424].

THEOREM 3.3. *Let (X, G) be a homogeneous space satisfying the first axiom of countability, and $X(t, \omega)$ a separable homogeneous process on X governed by the transition probabilities $\delta^X T_t(\cdot)$. Suppose that for some sequence of t-values, $t_j \to 0$, there is a not necessarily bounded regular Borel measure Q_X on $X - \{x\}$; for which $f \in C_c(X)$, $x \notin$ support (f), imply $t_j^{-1}(T_{t_j} f)(x) \to Q_x(f)$ as $t_j \to 0$. Let Y be the space of displacements of (X, G), and denote the measures $\delta^x T_t$ and Q_x after transference to Y by the same symbols. Suppose $X(0, \omega) = x$ a.s,, $C(D) = \{\omega \mid \lim_{n \to \infty} p(X(t - n^{-1}, \omega), X(t + n^{-1}, \omega)) \in D$ for some $t \in [0, T)\}$, and let P_* and P^* be the inner and outer measures induced by P on Ω.*

(1) *Then for any compact subset D of $Y - \{x\}$:*

$$1 - \exp\{T \limsup_j t_j^{-1}\delta^x T_t(D)\} \leq P_*\{C(D)\}\,; P^*\{C(D)\} \leq 1 - e^{-TQ_x(D)}\ .$$

(2) *If D is a compact subset of $Y - \{x\}$ satisfying, either*

(i) *there is a U open $\subset D$ for which $\delta^x T_t(U - D) = o(t)$ as $t \to 0$, or*

(ii) *there exists a sequence C_k compact $\subset$ interior (D) for which $Q_x(C_k) \to Q_x(D)$ as $k \to \infty$, then*

$$P_*\{C(D)\} = P^*\{C(D)\} = 1 - e^{-TQ_x(D)}$$

Proof. Using the monotone sequence $t_j \to 0$ we define a sequence of partitions of $[0, T)$. The jth partition being given by

$$\{[0, t_j), [t_j, 2t_j), \cdots, [(k_j - 1)t_j, k_j t_j), [k_j t_j, T)\}\ ,$$

where k_j is the largest integer $< T/t_j$. Define Y-valued random variables

$$H(j, n) = p(X(\{n-1\}t_j^-, \omega), X(nt_j^-, \omega)) \qquad 1 \leq n \leq k_j$$
$$H(j, k_j + 1) = p(X(k_j t_j^-, \omega)X(T^-, \omega))\ ,$$

where as usual $X(t^-, \omega) = \lim_{s \uparrow t} X(s, \omega)$. For any measurable subset $F \subset Y$ we put $F(j, n) = \{\omega \mid H(j, n) \in F\}$ and $F(j) = \bigcup_{n=1}^{k_j+1} F(j, n)$. Since $\{H(j, n), 1 \leq n \leq k_j\}$ are independent and identically distributed random variables, it follows that

$$P\{F(j)\} = 1 - \{1 - \delta^x T_{T-k_j t_j}(F)\}\{1 - \delta^x T_{t_j}(F)\}^{k_j}\ ,$$

where $1 \geq \varepsilon_j \to 0$, $0 \leq T - k_j t_j < t_j \downarrow 0$, and $k_j = Tt_j^{-1} - \varepsilon_j$. Let D be a compact subset of Y, and U an open neighborhood of D whose closure does not contain x. Our knowledge that the paths have right and left hand limits at every point shows that $C(D) \subset \liminf_j U(j)$. If $\omega \in \limsup_j D(j)$, choose a sequence of semiclosed intervals $[n_j t_j, (n_j + 1)t_j)$ which converge to a point, and across whose length the path ω has a displacement $H(j, n_j) \in D$. By passing to further subsequences if necessary we can assume that the sequences of endpoints are monotone. There are four cases;

(a) $n_j t_j \uparrow, (n_j + 1)t_j \downarrow$ leads to $\omega \in C(D)$;

(b) $n_j t_j \downarrow, (n_j + 1)t_j \uparrow$ is impossible; while

(c) $n_j t_j \uparrow, (n_j + 1)t_j \uparrow$ and

(d) $n_j t_j \downarrow, (n_j + 1)t_j \downarrow$ both lead to ω's having infinitely many displacements close to D in $[0, T)$, and, accordingly, occuring with probability zero by condition (3) of Theorem 3.2. Therefore, $\limsup_j D(j) \subset C(D) \subset \liminf_j U(j)$ which in turn implies

$$1 - \exp\{- T \limsup_j t_j^{-1}\delta^x T_{t_j}(D)\} \leq P_*\{C(D)\}\ ,$$

and

$$P^*\{C(D)\} \leq 1 - \exp\{- T \liminf_j t_j^{-1}\delta^x T_{t_j}(U)\}\ .$$

It can be shown by a standard arguement that if V open $\supset U$ and $x \notin \overline{V}$, then $Q_x(D) \leq \liminf_j t_j^{-1}\delta^x T_{t_j}(U) \leq Q_x(V)$, and by letting V and U shrink to D it follows that

$$P^*\{C(D)\} \leq 1 - e^{-TQ_x(D)} .$$

If in addition D satisfies either (i) or (ii) in (2), it is easy to see that

$$Q_x(D) \leq \limsup_j t_j^{-1}\delta^x T_{t_j}(D) ,$$

which implies the conclusion of (2).

We now show that the conditions (i) and (ii) placed on the compact set D in (2) of Theorem 3.3 are sufficiently unrestrictive for us to prove Property II for homogeneous processes.

THEOREM 3.4. *Let (X, G) be a separable locally compact homogeneous space, and let T_t be a homogenous process on (X, G). Then there exists a unique not necessarily bounded Borel measure Q_x on $Y - \{x\}$, the space of displacements of (X, G), such that $f \in C(X)$ and $x \notin$ support (f) imply*

$$t^{-1}(T_t f)(x) \to Q_x(f) \qquad \text{as } t \to 0 .$$

Proof. The use of the Hille-Yosida theory of strongly continuous semi-groups shows that when restricted to the complement of any neighborhood U of $x \in Y$, the family of measures $t^{-1}\delta^x T_t$ is bounded. We compactify Y by adding a point at infinity and denote the compactified space by Y^*. Using the compactness of bounded sets in the weak star topology, and the first axiom of countability for Y, we can find a sequence $t_j \to 0$ and a not necessarily bounded Borel measure Q_x on $Y^* - \{x\}$, such that for any $f \in C(Y) = C(Y^*)$, $x \notin$ support (f) implies $t_j^{-1}\delta^x T_{t_j}(f)$ as $t_j \to 0$. There remain two problems.

(a) to prove Q_x is unique, and

(b) to show that $Q_x(\{\infty\}) = 0$. Using the separability of (X, G) construct a representation, $X(t, \omega)$, of the paths of T_t satisfying all the conditions in Theorem 3.2.

Suppose that $s_j^{-1}\delta^x T_{s_j} \to Q_x'$ were another limit and let C be any compact subset of $Y - \{x\}$. For an arbitrary $\varepsilon > 0$ let U_ε be chosen so that $C \subset U_\varepsilon$ open $\subset \overline{U}_\varepsilon$ compact $\subset Y - \{x\}$, $Q_x(U_\varepsilon) < Q_x(C) + \varepsilon$, and $Q_x'(U_\varepsilon) < Q_x'(C) + \varepsilon$. Now construct a function $f \in C_c(Y)$, such that $f(U_\varepsilon^c) = 0, 0 \leq f \leq 1$, and $f(C) = 1$. Select a $b \in (0, 1)$ such that

$$Q_x(\{x \mid f(x) = b\}) = Q_x'(\{x \mid f(x) = b\}) = 0$$

and put $D = \{x \mid f(x) \leq b\}$. Note that

$$C \text{ compact} \subset \text{interior } (D) \subset D \text{ compact} \subset U_\varepsilon,$$

Q_x (interior (D)) $= Q_x(D)$, and Q_x' (interior (D)) $= Q_x'(D)$. Accordingly, D satisfies condition (ii) of Theorem 3.3 (2), and thus for any fixed $T > 0$,

$$P^*\{C(D)\} = 1 - e^{-T}Q_x(D) = 1 - e^{-TQ_x'(D)} ,$$

so that $Q_x(D) = Q_x'(D)$. This shows

$$|Q_x'(C) - Q_x(C)| \leq |Q_x(D) - Q_x(D - C) + Q_x'(D - C) - Q_x'(D)| \leq 2\varepsilon$$

for every $\varepsilon > 0$; so $Q_x = Q_x'$.

To show that Q_x is really a Borel measure on $Y - \{x\}$, and not on $Y^* - \{x\}$, we must show that $Q_x(\{\infty\}) = 0$. In order to do this we make a final appeal to the paths. Except for an event of probability zero we know that $\mathrm{Cl}\,[\{X(t, \omega) | t \in [0, T)\}]$ is a compact subset of X, and, consequently, its projection on Y will also be compact. Using the method above, choose a sequence of compact sets $D_n \downarrow \{\infty\}$ in Y^* and satisfying condition (ii) of Theorem 3.3 (2). Then if $Q_x(\{\infty\}) \neq 0$,

$$P\{C(D)\} = 1 - e^{-T}Q_x(D_n) \geq r > 0 ,$$

and $P\{\bigcap_{n=1}^{\infty} C(D_n)\} > 0$. Any path with a jump in every D_n during the time interval $[0, T)$ certainly contains ∞ as a limit point. Thus our process violates the condition that the paths are a.s. bounded. Hence $Q_x(\{\infty\}) = 0$.

The temporal continuity (weak continuity) of T_t enables us to restrict consideration to a sigma-compact subset of X, namely $\bigcup_{t>0}$ support $(\delta^x T_t)$. As a consequence of the preceeding theorem this remark proves the following corollary.

COROLLARY. *Let (X, G) be a homogeneous space where every compact subset of X is separable. Then every homogeneous process on (X, G) possesses Property* II.

The following theorem gives an accurate description of the important set support (Q_x).

THEOREM 3.5. *Let (X, G), Y, $X(t, \omega)$, $\delta^x T_t$, and Q_x be as in Theorem* 3.4. *For those ω's which have one sided limits let*

$$F(\omega) = \{\lim_{n\to\infty} p(X(t - n^{-1}, \omega), X(t + n^{-1}, \omega)) | t \geq 0\} .$$

Then $\overline{F(\omega)}$ = support $(Q_x) \cup \{x\}$ a.s..

Proof. By definition $C(\bigcup_\alpha F_\alpha) = \bigcup_\alpha C(F_\alpha)$ for F_α measurable $\subset Y$. Now choose a sequence of compact sets D_i, such that $\bigcup_i D_i = \{\text{support } (Q_x) \cup \{x\}\}^c$. $P^*\{C(D)\} \leq 1 - e^{-T\cdot 0} = 0$, so

$$P^*\{C(\bigcup_i D_i)\} = P^*\{\bigcup_i C(D_i)\} \leq \textstyle\sum_i P^*\{C(D_i)\} = 0 ,$$

which shows $F(\omega) \subset \text{support}\,(Q_x) \cup \{x\}$. Let U open $\subset \overline{U}$ compact $\subset Y - \{x\}$, and $U \cap \text{support}\,(Q_x) \neq \phi$. Choose $f \in C_c(Y)$, such that $0 \leq f \leq 1$, support $(f) \subset U$, and $Q_x(f) > 0$. Let

$$r(U) = \limsup_{t \to 0} t^{-1} \delta_x T_t(\overline{U} \geq Q_x(f) > 0 .$$

If $\{U_i, i = 1, 2, \cdots, n\}$ is a finite class of such U's, let

$$P_*\{\omega | \omega \text{ has a jump in every } \overline{U}_i \text{ when } t < T\} = A(T) ;$$

$$P_*\{\omega | \omega \text{ has a jump in } \overline{U}_k \text{ when } T(k-1)n^{-1} \leq t < Tkn^{-1}\} = B(k, T) .$$

We have the following inequalities relating the above numbers

$$A(T) \geq B(1, T)B(2, T) \cdots B(n, T)$$
$$\geq (1 - \exp\{-n^{-1}Tr(\overline{U}_1)\}) \cdots (1 - \exp\{-n^{-1}Tr(\overline{U}_n)\}) .$$

Letting T approach infinity one sees that

$$P_*\{\omega | \omega \text{ has a jump in each } \overline{U}_i \text{ for some } t > 0\} \geq 1.$$

If we now choose a countable sequence of finite classes of the type $\{U_i\}$ above, and let their sets become arbitrarily fine while their unions swell out and eventually cover support (Q_x),

$$P_*\{\omega | \omega \text{ has a jump in each } \overline{U}_i \text{ of the } k\text{th covering}\} = 1 .$$

Hence the inner measure of their intersection is one. Now the paths in their intersection have jumps in the closure of each of the finer and finer covering sets, and consequently for these paths $\text{Cl}\,[\overline{F(\omega)}] \supset$ support (Q_x). The existence of left hand limits for $X(t, \omega)$ implies $x \in [F(\omega)]$.

4. Compound Poisson processes. The poisson process with rate parameter $r \geq 0$ on the real line is a homogeneous process with transition probabilities $\delta^0 T_t(E) = \exp\{tr(\delta^1 - \delta^0)\}(E)$. It can be generalized to a compound Poisson process by replacing δ^1 by any positive regular Borel measure μ of norm one. Probabilistically one thinks of a compound Poisson process in the following manner. A simple Poisson process is run at a rate r, and when a jump occurs in this simple process, the particle ruled by the compound Poisson process jumps from its position x into the set $E + x$ with probability $\mu(E)$.

Suppose we observe two Poisson processes, $\exp\{tr_i(\delta^1 - \delta^0)\}$ $i = 1, 2$, running simultaneously. We can then define a new process as follows. The state of our process will be described by a finite sequence of x_1's and x_2's called a word, and we change states when a jump occurs in either of our two simple Poisson processes. If a jump occurs in the ith process we lengthen our state by placing the symbol x_i to the right of the current word. One can calculate that the probability of starting with the empty state at time zero, and being at a fixed state with $n_1 x_1$'s and $n_2 x_2$'s at time T, is independent of their order, and given by $\exp\{-(r_1+r_2)T\}(r_1T)^{n_1}(r_2T)^{n_2}/(n_1+n_2)!$. To give an alternative description of this process, let H be the free group generated by the symbols $\{x_1, x_2\}$. Topologize H with the discrete topology, and consider the compound Poisson process on H given by

$$\delta^e T_t = \exp\{t(r_1+r_2)(p\delta^{x_1}+q\delta^{x_2}-\delta^e)\ ,$$

where $p = r_1(r_1+r_2)^{-1}$ and $q = r_2(r_1+r_2)^{-1}$. An elementary expansion shows this process is identical with the word generating process defined above. It would seem natural to define the superposition of the two Poisson processes $\exp\{tr_i(\delta^{x_i}-\delta^e)\}$ to be the process $\delta^e T_t$. This symbol generating process can also be interpreted by running a simple Poisson process at the rate r_1+r_2, and each time a jump occurs, multiplying on the right by x_1 with probability p and by x_2 with probability q.

The analogue of the compound Poisson processes for a homogeneous space (X, G) is the class of processes of the form $T_t = \exp\{tr(P-1\}$, where P is a homogeneous transition probability of norm one. We shall, accordingly, call these processes compound Poisson processes. An easy computation shows that the infinitesimal generator of such a process is $A = r(P-1)$ and $D(A) = C(X)$. Thus the superposition of the two processes in the preceeding paragraph corresponds to the addition of their infinitesimal generators. We use this last remark to define the superposition of an arbitrary homogeneous process and a compound Poisson process.

DEFINITION 4.1. The sequence $T_t^{(n)}$ of semi-groups on $C(X)$ is said to *converge in the sense of Bernoulli* {*strongly*} *to the semi-group* T_t if, and only if, whenever $f \in C(X)$, $(T_t^{(n)}f)(x) \to (T_tf)(x)$ for each fixed x and t as $n \to \infty$ {if, and only if, the following condition is satisfied. For each $\delta > 0$ and each $f \in D(A)$ where A is the infinitesimal generator of T_t, there exists an integer $N_{\delta,f}$, such that $n \geq N_{\delta,f}$ implies $\|(T_t^{(n)}f - T_tf\|_\infty \leq \delta t$ for all $t > 0$}.

It is an elementary consequence of this definition that $T_t^{(n)} \to T_t$ strongly implies $T_t^{(n)} \to T_t$ in the sense of Bernoulli. We now recall a fact from the theory of semi-groups which we need in the proof of the next theorem. Put $A_\varepsilon = \varepsilon^{-1}(T_\varepsilon - 1)$, then $\|e^{tA_\varepsilon}f - T_tf\|_\infty \to 0$ as $\varepsilon \to 0$ for any $f \in C(X)$, and uniformly for $t \in (0, M)$, $M < \infty$. More precisely,

$$\|(e^{tA_\varepsilon} - T_t)f\|_\infty \leq t \limsup_{s\to 0} \|(A_\varepsilon - A_s)f\|_\infty\ , \tag{4.1}$$

which for $f \in D(A)$ becomes $\|(e^{tA_\varepsilon} - T_t)f\|_\infty \leq t\|(A_\varepsilon - A)f\|_\infty$.

THEOREM 4.1. *Every homogeneous process is a strong limit of a sequence of compound Poisson processes.*

Proof. Let T_t be a homogeneous process, $f \in D(A)$, and A_{t_0} be as above. Then $\|(\exp\{tA_{t_0}\} - T_t)f\|_\infty \leq t\delta_{t_0}$, where $\delta_{t_0} \to 0$ as $t_0 \to 0$.

We now study the concept of strong convergence in more detail for compound Poisson processes. Our main results are stated in Theorems 4.2 – 4.5, but before proceeding to these theorems we establish the following useful lemma.

LEMMA 4.1. *Let $\pi_{n,t}$ $n = 1, 2, \cdots, \infty$ be positive Borel measures on (X, G) while t ranges over the compact separable set F. Suppose for each $f \in C(X)$, $\pi_{n,t}(f)$ is continuous in t, and $\pi_{n,t}(f) \to \pi_{\infty,t}(f)$ as $n \to \infty$ uniformly for $t \in F$. Under these conditions*

$$\sup\{|\pi_{n,t}(L_a f) - \pi_{\infty,t}(L_a f)| : a \in G\} \to 0$$

uniformly on F as $n \to \infty$.

Proof. We show first that given $\varepsilon > 0$ there exists a compact set B, for which $\pi_{n,t}(B^c) < \varepsilon$ whatever n and t. Choose $\{t_i\}$ as a countable dense set in F, and consider the union, E, of the supports of all π_{n,t_i}. $\overline{E}$ is sigma-compact as the closure of a sigma-compact set in a uniformly locally compact space, and includes the supports of all the measures $\pi_{n,t}$; because by the continuity of $\pi_{n,t}(f)$ if a point does not lie in $\overline{E}$ it cannot lie in the support of any $\pi_{n,t}$. Select a sequence of functions $f_n \in C(X)$, such that $f_n \downarrow 0$ on $\overline{E}$, $f_n(0_n^c) = 1$, and $f_n(V_n) = 0$, where V_n is a compact set contained in the open set $0_n \subset V_{n+1}$. For fixed k the sequence of continuous $t \to \pi_{k,t}(f_n)$ converges monotonically to zero on the compact set F as $n \to \infty$. This convergence is then uniform. Accordingly, we can find an M_k, such that $n \geq M_k$ implies $0 \leq \pi_{k,t}(f_n) < \varepsilon/2$ all $t \in F$. Using the hypotheses of the lemma, we can find a K_n, such that $k \geq K_n$ implies $|\pi_{k,t}(f_n) - \pi_{\infty,t}(f_n)| < \varepsilon/2$ for all t. Put $n' = M_\infty$. Then for any $K_{n'} \leq k \leq \infty$, and every $t \in F$,

$$\begin{aligned}\pi_{k,t}(0_{n'}^c) &\leq \pi_{k,t}(f_{n'}) \\ &\leq \pi_{\infty,t}(f_{n'}) + |\pi_{k,t}(f_{n'}) - \pi_{\infty,t}(f_{n'})| \leq \varepsilon .\end{aligned}$$

Whereas for $k \leq K_{n'}$, and every $t \in F$,

$$\pi_{k,t}(0_{M_k}^c) \leq \pi_{k,t}(f_{M_k}) < \varepsilon/2 .$$

If we put W_ε equal to the union of $0_{n'}$ and $\bigcup_{j=0}^{K_{n'}} 0_{M_j}$, $B_\varepsilon = \overline{W}_\varepsilon$ satisfies the desired condition.

Now put $f = 1$ and observe that from $\pi_{n,t}(f) \to \pi_{\infty,t}(f)$ uniformly on F, we can conclude there is an M, such that $0 \leq \|\pi_{n,t}\| \leq M$, and $\|\pi_{n,t}\| \to \|\pi_{\infty,t}\|$ uniformly on F. This shows we need only to prove the lemma for $f \in C_\infty(X)$. Choose a $\delta > 0$, an $f \in C_\infty(X)$, and put $D_\delta = \{x : |f(x)| \geq \delta\}$. D_δ is compact. If $H_\delta = \{a \in G : a[D_\delta] \cap B_\varepsilon = \phi\}$, H_δ is open, and $a \in H_\delta$ implies.

$$|\pi_{n,t}(L_{a^{-1}}f) - \pi_{\infty,t}(L_{a^{-1}}f)| \leq 2\varepsilon \|f\|_\infty + 2\delta M .$$

In this paragraph we show that

$$\limsup_n \sup \{|\pi_{n,t}(L_{a^{-1}}f) - \pi_{\infty,t}(L_{a^{-1}}f)| : t \in F, a \in H_\delta^c\} = 0 .$$

Regard $a \to |\pi_{n,t}(L_{a^{-1}}f) - \pi_{\infty,t}(L_{a^{-1}}f)| = h_{a,n}(t)$ as a map from the compact set H_δ^c to the space of continuous real valued functions on F. If b is sufficiently close to a, $\|L_{a^{-1}}f - L_{b^{-1}}f\|_\infty < \varepsilon'$, from which $|h_{a,n}(t) - h_{b,n}(t)| \leq 2M\varepsilon'$, so that the maps $h_{a,n}(\cdot)$ are equicontinuous in a. Now $\lim_{n\to\infty} h_{a,n}(t) = 0$ uniformly in t by hypotheses, so that we have a sequence of equicontinuous functions, $a \to h_{a,n}(\cdot)$, defined on a compact set, H_δ^c, with values in a normed vector space and converging pointwise to zero. As a trivial consequence, they converge uniformly to zero and

$$\limsup_n \sup \{h_{a,n}(t) : t \in F, a \in H_\delta^c\} = 0 .$$

Collecting results we have shown

$$\limsup_n \sup \{h_{a,n}(t) : t \in F, a \in G\} \leq 2\varepsilon \|f\|_\infty + 2\delta M .$$

Since ε and δ are arbitrary, this gives the conclusion of Lemma 4.1.

THEOREM 4.2. *If $r_n \to r$ and $P^{(n)}$, P are homogeneous transition probabilities, such that for each $f \in C(X)$, $P^{(n)}f \to Pf$ pointwise as $n \to \infty$. Then.*

$$\exp\{tr_n(P^{(n)} - 1)\} \to \exp\{tr(P - 1)\}$$

strongly as $n \to \infty$.

Proof. Given $f \in C(X)$ define $W(n, t)$ by

$$tW(n, t) = \|(\exp\{tr_n(P^{(n)} - 1)\} - \exp\{tr(P - 1)\})f\|_\infty .$$

We must show that $\limsup_n \sup \{W(n, t) : t > 0\} = 0$. An expansion gives

$$\begin{aligned} W(n, t) \leq & |t^{-1}(e^{-tr_n} - e^{-tr})| \, \|f\|_\infty \\ & + \sup\{|r_n e^{-tr_n}\delta^x P^{(n)}(L_a f) - re^{-tr}\delta^x P(L_a f)| : a \in G\} \\ & + \sup\{|t^{-1}Q^x_{n,t}(L_a f)| : a \in G\} , \end{aligned}$$

where $Q^x_{n,t}$ is a difference of two positive measures and satisfies: $\|Q^x_{n,t}\| \leq 2$; $\|t^{-1}Q^x_{n,t}\| \to 0$ uniformly in n as $t \to 0$; $t^{-1}Q^x_{n,t}(g) \to 0$ for each $g \in C(X)$ as $n \to \infty$ uniformly in every compact t subinterval of $(0, \infty)$. We handle these three terms separately. Clearly

$$\limsup_n \sup \{|t^{-1}(e^{-tr_n} - e^{-tr})| : t > 0\} = 0 .$$

Lemma 4.1 shows that the second term is also zero in the limit. For the last terms we let t range over a bounded interval $[t_0, T]$ where $0 < t_0$ and again use Lemma 4.1. If t_0 is small enough and T large enough the extra pieces are arbitrarily small, so that for sufficiently large n we can make this last term small too. This completes the proof of Theorem 4.2.

For later use we weaken the hypothesis of Theorem 4.2 by generalizing the concept of a homogeneous process to allow an escape of mass to infinity. Let $X^* = \{X, x^*\}$ be the cannonical one point compactification of X with x^* denoting the point at infinity. Extend the operations of G to X^* as proper maps, so that $G[x^*] = x^*$. In order to use our earlier notation we consider the spaces $C_\infty(X)$, $C_c(X)$ as imbedded in $C(X^*) = C(X)$. A homogeneous transition probability on (X^*, G) is a continuous endomorphism $P: N(X^*) \to N(X^*)$ satisfying (i), (ii) and (iii) of Definition 1.1. It follows as before that P satisfies equations (1.1), (1.2), (1.3) and (1.4) of § 1 with $f \in C(X)$ and $\mu \in N(X^*)$. On the subspace $N(X)$ of $N(X^*)$, P can be expressed as

$$P = P' + kS , \tag{4.1}$$

where P' is a homogeneous transition probability on (X, G) and $\mu S = \mu(X)\delta^{x^*}$.

A homogeneous process on (X^*, G) is a weakly continuous one-parameter semi-group of homogeneous transition probabilities of norm one acting on $C(X)$, or as adjoints on $N(X^*)$. If X is not compact it is easy to show that whenever $x \neq x^*$ there is an $r \geq 0$, such that

$$\delta^x T_t(E) = \delta^x T_t(X \cap E) + \{1 - e^{-rt}\}\delta^{x^*}(E) , \tag{4.2}$$

and

$$\delta^{x^*} T_t(E) = \delta^{x^*}(E) . \tag{4.3}$$

THEOREM 4.3. *If $r_n \to r$ and $P^{(n)}$, P are homogeneous transition probabilities on (X^*, G), such that for each $f \in C(X)$, $P^{(n)}f \to Pf$ pointwise as $n \to \infty$. Then*

$$\exp\{tr_n(P^{(n)} - 1)\} \to \exp\{tr(P - 1)\}$$

strongly as $n \to \infty$.

Proof. An expansion shows immediately that one only need consider those $f \in C_\infty(X)$ in the criterion for strong convergence. Since for these f, $Pf = P'f$, the conclusion of Theorem 4.3 follows by the methods used in the proof of Theorem 4.2.

THEOREM 4.4. *Let* $\exp\{tr_n(P^{(n)} - 1)\}$ *be a sequence of compound Poisson processes on* (X, G). *Suppose* $r_n \leq K < \infty$ *and* $\exp\{tr_n(P^{(n)} - 1)\}$ *converges in the sense of Bernoulli to the homogeneous process* T_t *on* (X, G), *where* $\| T_t \| = 1$. *Then* $T_t = \exp\{tr(P - 1)\}$ *is a compound Poisson process on* (X, G), *and for some subsequence* $\{n_i\}$ *of* $\{n\}$, $r_{n_i} \to r$, *and* $P^{(n_i)}f \to Pf$ *pointwise whenever* $f \in C(X)$

Proof. Choose a subsequence $\{n_i\}$ so that r_{n_i} approaches some r and $\delta^x P^{(n_i)} \to \delta^x P' + k\delta^{x^*}$ in the weak topology generated by $C(X)$. By Theorem 4.2 it suffices to show $k = 0$. If $k \neq 0$ it follows from Theorem 4.3. that

$$T_t = \exp\{tr(P' - 1)\} + (1 - e^{-ktr})\delta^{x^*} ,$$

violating the condition that $\| T_t \| = 1$ on (X, G).

The following theorem is known for homogeneous processes on a commutative group. A proof based on an analysis of characteristic functions is given in Bochner [1, p. 76].

THEOREM 4.5. *Let* X *be a discrete space so that every one point subset of* X *is an open set, and let* T_t *be a homogeneous process on* (X, G). *Then* T_t *is a compound Poisson process.*

Proof. Select a sequence of compound Poisson processes

$$\exp\{tr_n(P^{(n)} - 1)\}$$

converging strongly to T_t and normalized by $\delta^x P^{(n)}(\{x\}) = 0$. Using the discretness of X the characteristic function of the set $X - \{x_0\}$, $I_{X-\{x_0\}}$, is in $C(X)$. For any $\varepsilon > 0$ we can choose an $f \in D(A)$, such that $\| I_{X-\{x_0\}} - f \|_\infty < \varepsilon$. By addition of the constant $-f(x_0)$ to f we obtain an $h = f - f(x_0) \in D(A)$ which vanishes at x_0 and is greater than $1 - 2\varepsilon$ elsewhere. Using this h in the definition of strong convergence we find that on the one hand

$$\| t^{-1}(\exp\{tr_n(P^{(n)} - 1)\} - T_t)(h) \|_\infty < \delta_n \to 0$$

as $n \to \infty$, while on the other this first expression approaches

$$\| r_n(P^{(n)} - 1)(h) - Ah \|_\infty$$

as $t \to 0$. From this it follows that

$$(1-2\varepsilon)r_n \leq r_n\delta^{x_0}P^{(n)}(h) \leq \delta_n + Ah(x_0) ,$$

which implies

$$r_n \leq \{\delta_n + Ah(x_0)\}(1-2\varepsilon)^{-1} \leq K < \infty .$$

Then T_t is a compound Poisson process by Theorem 4.4.

5. Processes with continuous paths. We define in this section a class of processes having continuous paths, and we give conditions for a process to belong to this class.

DEFINITION 5.1. We say that a homogeneous process T_t on (X, G) has the property G_w $\{G_s\}$, or belongs to the class of processes designated by the symbol G_w $\{G_s\}$, if, and only if, for any fixed $x_0 \in X$ and any neighborhood N_{x_0} of x_0, there is a sequence of compound Poisson processes $\exp\{tr_n(P^{(n)}-1)\}$ converging in the sense of Bernoulli {strongly} to T_t, with support $(\delta^{a[x_0]}P^{(n)}) \subset a[N_{x_0}]$ for all $a \in G$

THEOREM 5.1. *If $T_t \in G_s$ and satisfies Property* II *of* § 2, *it follows that $\delta^x T_t(N_x^c) = o(t)$ as $t \to 0$ for every measurable neighborhood N_x of x.*

Proof. Choose a compact neighborhood, V_x, of x contained in the interior of N_x, and in accordance with Corollary 2 of Theorem 2.2 let $f \in D(A)$, $0 \leq f \leq 1$, $f(V_x) = 0$, and $f(N_x^c) = 1$. Now select a sequence of compound Poisson processes $\exp\{tr_n(P^{(n)}-1)\}$ converging strongly to T_t and satisfying support $(\delta^x P^{(n)}) \subset V_x$. It suffices to show $(T_t f)(x) = o(t)$ as $t \to 0$, or that for every $t_n \to 0$, $\limsup_n t_n^{-1}(T_{t_n} f)(x) = 0$. It is no restriction to assume $t_n r_n$ and $t_n r_n^2$ are both less than n^{-1}. The condition of strong convergence applied to f, then shows that

$$\begin{aligned}
\limsup_n t_n^{-1}(T_{t_n} f)(x) &= \limsup_n t_n^{-1}[\exp\{t_n r_n(P^{(n)}-1)\}(f)](x) \\
&\leq \limsup_n t_n^{-1} e^{-t_n r_n}(e^{t_n r_n} - 1 - r_n t_n) \\
&\leq \limsup_n t_n^{-1} e^{-t_n r_n}(t_n r_n)^2 \\
&\leq \limsup_n n^{-1} e^{-t_n r_n} = 0
\end{aligned}$$

as we desired to prove.

In the proof of a partial converse to Theorem 5.1 we will need the following lemma.

LEMMA 5.1. *If* $t, \varepsilon, \delta > 0$ and $k \geq (te^2+1)\varepsilon^{-1} + 1 - \log\delta$, then

$$\sum_{n=k}^{+\infty} \exp\{-t\varepsilon^{-1}\}(t\varepsilon^{-1})^n/n! \leq \varepsilon\delta .$$

Proof. Rather than prove this lemma in detail we will indicate a method of proof. If the sum is overestimated by an integral and the $n!$ in that integral is underestimated using $(2\pi)^{1/2}n^{n+(1/2)}e^{-n} \leq n!$, one obtains the statement that $k \geq \varepsilon^{-1}te^2$ implies

$$\sum_{n=k+1}^{+\infty} \exp\{-t\varepsilon^{-1}\}(t\varepsilon^{-1})/n! \leq (2\pi k)^{-1/2} \exp\{-(k+t\varepsilon^{-1})\} .$$

The desired conclusion follows easily from this estimate.

Lemma 5.1 gives more than we need to prove a weak converse to Theorem 5.1, but not enough to prove a converse. The best we are able to achieve in this direction using the above estimate is stated below.

THEOREM 5.2. (i) *If* $\delta^x T_t(N_x^c) = o(t)$ *as* $t \to 0$ *for every neighborhood* N_x *of* $x \in X$, $T_t \in G_w$.

(ii) *If* $\delta^x T_t(N_x^c) = o(t^2)$ *as* $t \to 0$ *fore very neighborhood* N_x *of* $x \in X$, $T_t \in G_s$.

Proof. These results are stated together because their proofs parallel one another. Let T_t be (at first) any homogeneous process, and suppose N_{x_0} is a compact neighborhood of x_0. Put $W_{x_0} = \bigcap_{a\in G_{x_0}} a[N_{x_0}]$ and $W_y = b[W_{x_0}]$ where $b[x_0] = y$. This choice of W_{x_0} insures that W_y is well-defined, $W_{a[x_0]} \subset a[N_{x_0}]$, and W_x is a compact neighborhood of x. Now define a homogeneous transition probability, P_ε, by $\delta^y T_\varepsilon(F) = \delta^y T_\varepsilon(W_y \cap F)$. Let $s(\varepsilon) = \delta^x T_\varepsilon(W_x^c)$, and $q(\varepsilon) = 1 - s(\varepsilon) = \delta^x T_\varepsilon(W_x)$. We show that the compound Poisson processes $\exp\{t\varepsilon^{-1}([1-s(\varepsilon)]^{-1}P_\varepsilon - 1)\}$ approximate T_t in the desired sense as $\varepsilon \to 0$. Since support $(\delta^x P_\varepsilon) \subset W_x \subset a[N_{x_0}]$ whenever $a[x_0] = x$, this will be sufficient to prove Theorem 5.2. Since it is known that the processes $\exp\{t\varepsilon^{-1}(T_\varepsilon - 1)\}$ approximate T_t in the strong sense, it suffices to show

$$U(t, \varepsilon) = \|\exp\{t\varepsilon^{-1}[q(\varepsilon)^{-1}P_\varepsilon - 1]\} - \exp\{t\varepsilon^{-1}(T_\varepsilon - 1)\}\| \to 0$$

as $\varepsilon \to 0$ for fixed t in conclusion (i), and that $t^{-1}U(t, \varepsilon) \to 0$ uniformly in t as $\varepsilon \to 0$ in conclusion (ii). Calculation shows

$$U(t, \varepsilon) \leq \sum_{n=0}^{+\infty} \exp\{-t\varepsilon^{-1}\}(t\varepsilon^{-1})^n B(n, \varepsilon)/n! ,$$

where

$$B(n, \varepsilon) = \|\{q(\varepsilon)^{-1}P_\varepsilon\}^n - T_\varepsilon^n\| .$$

Since $\delta^x(T_\varepsilon^n - P_\varepsilon^n) \geq 0$, it follows that $\|T_\varepsilon^n - P_\varepsilon^n\| = 1 - q(\varepsilon)^n$; so

$$B(n, \varepsilon) \leq 2q(\varepsilon)^{-n}\{1 - q(\varepsilon)^n\} .$$

For large n there is a better bound because $B(n, \varepsilon) \leq 2$ in any case. We will also use the fact that $1 \geq s \geq 0$ implies $\log(1-s) \geq -s(1-s)^{-1}$.

Proof of (i). Let $s(\varepsilon) = f(\varepsilon)\varepsilon$ where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$, and suppose t is fixed and ε small enough so that $\varepsilon^{-1}K \geq (te^2 + 1)\varepsilon^{-1} + 1$. Putting $\delta = 1$ in Lemma 5.1, and using the substitution

$$E(n, r) = \exp\{r\varepsilon^{-1}\log[1 - \varepsilon^n f(\varepsilon)]\} ,$$

we find

$$\begin{aligned} U(t, \varepsilon) &\leq \sum_{n \geq K\varepsilon^{-1}} \{\text{terms above}\} + \sum_{K\varepsilon^{-1} < n} \{\text{terms above}\} \\ &\leq E(1, -K)\{1 - E(1, K)\} + 2\varepsilon \to 0 \end{aligned}$$

as $\varepsilon \to 0$.

Proof of (ii). For $\delta > 0$. We will show $\limsup_{\varepsilon\to 0} \sup_{t>0} t^{-1}U(t, \varepsilon) \leq \delta$.

Since $U(t, \varepsilon) \leq 2$, we only need to consider those $t \leq 2\delta^{-1}$. For these t choose $K > 0$ and a range of ε's sufficiently close to zero, so that the hypothesis of Lemma 5.1 is satisfied for all ε's and t's which come under consideration. Let $s(\varepsilon) = \varepsilon^2 f(\varepsilon)$ where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Computing as before and noting that

$$\sum_{n>0} \exp\{-t\varepsilon^{-1}\}(t\varepsilon^{-1})^{n-1}\varepsilon^{-1}/n! \leq \varepsilon^{-1} ,$$

we find

$$\begin{aligned} t^{-1}U(t, \varepsilon) &\leq \varepsilon^{-1}\{1 - E(2, K)\}E(2, -K) + \varepsilon K\varepsilon^{-1} \sum_{n > K\varepsilon^{-1} - 1} \{\text{terms above}\} \\ &\leq \varepsilon^{-1}(1 - \exp\{-Kf(\varepsilon)\varepsilon[1 - \varepsilon^2 f(\varepsilon)]^{-1}\}E(2, -K) \\ &\quad + 2\varepsilon\delta K^{-1} \to 0 \end{aligned}$$

as $\varepsilon \to 0$, which completes the proof of Theorem 5.2.

In Euclidean spaces it is easy to see that $\delta^0 T_t(N_0^c) = o(t)$ implies $\delta^0 T_t(N_0^c) = o(t^m)$ for any $m \to 0$ as $t \to 0$. Accordingly, in these spaces the $o(t)$ condition and the G_s condition are equivalent. This may be true in general but we will not explore the question further here.

6. The commutative case. This section contains an independent proof of Property II that is more general than the proof in § 3 in that X need not be separable, and more restrictive because G must be commutative. Let $G=\{x, y, z, \cdots\}$ be a commutative, Hausdorff, locally compact topological group, and let $\hat{G}=\{a, b, c, \cdots\}$ be its character group. Denote the Fourier-Stieltjes transform of the measure $\mu \in N(G)$ by $\check{\mu}(x)=\int (x, a)\mu(da)$, and write $\mu_n \xrightarrow{c} \mu$ if, and only if, for every $g \in C(G)$, $\mu_n(g) \rightarrow \mu(g)$. If f_n, f are uniformly continuous bounded functions on G, we write $f_n \xrightarrow{p} f$ as a substitute for the fact that $f_n \rightarrow f$ pointwise and uniformly on compact sets. It is convenient to denote the Haar measure of a Borel set E by $|E|$, and to let $N_+(\hat{G})$ be an abbreviation for the cone of positive measures in $N(\hat{G})$. We need the following results which we state without proof from harmonic analysis.

(a) If $\mu_n, \mu \in N_+(\hat{G})$, then $\mu_n \xrightarrow{c} \mu$ is equivalent with $\check{\mu}_n \xrightarrow{c} \check{\mu}$.

(b) If $\mu_n \in N_+(\hat{G})$ and $\check{\mu}_n \rightarrow f$ pointwise, f being continuous at $e \in G$, then f is continuous on G and there is a $\mu \in N_+(G)$, such that $f=\check{\mu}$ and $\check{\mu}_n \xrightarrow{c} \check{\mu}$.

(c) If $\mu_n \in N_+(\hat{G})$, $\check{\mu}_n \rightarrow \check{\mu}$ almost everywhere with respect to the Harr measure on G, and $\check{\mu}_n(e) \rightarrow \mu(e)$; then $\check{\mu}_n \xrightarrow{p} \check{\mu}$.

(d) If G is connected, $\mu_n \in N_+(\hat{G}), \|\mu_n\| \leq M<+\infty$, and $\check{\mu}_n \rightarrow M$ on a set, A, of positive Harr measure in G; then $\check{\mu}_n \xrightarrow{c} M$ on G.

(e) If $\|\mu_n\| \leq M<+\infty$, then $\mu_n \rightarrow \mu$ if, and only if,

$$\int_E \check{\mu}_n(x)|dx| \rightarrow \int_E \mu(x)|dx|$$

for every compact $E \subset G$.

Let $P^+(\hat{G})$ be the cone of regular not necessarily bounded Borel measures, Q, on $\hat{G}-\{\hat{e}\}$ for which the integral $Q'(x)=\int\{1-(x, a)\} Q(da)$ exists and is continuous on G. If U is a compact symmetric neighborhood of $e \in G$, we define the function h on G by

$$h(a)=|U|^{-1} \int_U\{1-(y, a)\}|dy|,$$

and observe that this h has the following properties:

(i) h is real valued and continuous;

(ii) $0 \leq h(a)<2$;

(iii) it G is connected, $h(a)=0$ implies $a=\hat{e}$;

(iv) $h(a) \rightarrow 1$ as $a \rightarrow$ infinity, so that $h \in C(\hat{G})$.

By choosing if necessary a new Haar measure we can assume $|U|=1$. We do this in the proofs below, and in addition denote the measure $\mu(F)=\int_F h(a) Q(da)$ by hQ.

LEMMA 6.1. *If* $Q \in P^+(\hat{G}), 0 \leq \int h(a) Q(da)=\|hQ\|<+\infty$.

Proof. $\int_U Q'(x)|dx| = \int_U |dx| \int_{\hat{G}-\{\hat{e}\}} \{1-(x,a)\}Q(da) = \int h(a)Q(da).$

The first term above is clearly bounded, so $\|hQ\|$ is finite.

Let $\{Q_q, q \in L\}$ be a directed set with $Q_q \in P^+(\hat{G})$, and g a continuous complex valued function on G. We say $Q'_q \to g$ boundedly on every compact set if $Q'_q \to g$ pointwise, and for every compact set $D \subset G$ there is a positive K_D and an $N_D \in L$, such that $q > N_D$ and $x \in D$ imply $|Q'_q(x)| \leq K_D$. We denote this by $Q'_q \xrightarrow{b} g$. The following theorem collects the Fourier analysis we need for $P^+(\hat{G})$.

THEOREM 6.1. *For $Q \in P^+(\hat{G})$ define $Q^h(x) = \int (x,a)h(a)Q(da)$, and let Q_q be a directed sequence from $P^+(\hat{G})$. Then*

(1) $Q'_q \xrightarrow{b} Q' \Rightarrow Q^h_q \to Q^h$ pointwise $\Leftrightarrow Q^h_q \xrightarrow{p} Q^h \Leftrightarrow hQ_q \xrightarrow{c} hQ$;

(2) $Q'_q \xrightarrow{b} g$ continuous $\Rightarrow Q^h_q \to$ some continuous $f \Leftrightarrow Q^h_q \xrightarrow{p} f \Leftrightarrow hQ_q \xrightarrow{c}$ some $\mu \in N_+(\hat{G})$.

Proof. Only the first implications need proof in each case. Calculation shows

$$\begin{aligned} Q^h(x) &= \int (x,a)h(a)Q(da) = \int_U |dy| \int Q(da)\{(x,a)[1-(y,a)]\} \\ &= \int_U |dy| \int Q(da)\{[(x,a)-1]+[1-(yx,a)]\} \\ &= \int_U Q'(yx)|dy| - Q'(x) . \end{aligned}$$

The implications follow after an application of the Lebesgue bounded convergence criterion.

THEOREM 6.2. *Let T_t be a homogeneous process on G. Put $\delta^{\hat{e}}T_t = w_t$. Then $\hat{w}_t(x) = e^{tf(x)}$ where f is a uniquely determined continuous complex function on G.*

Proof. This is an immediate consequence of the semi-group property $w_t(x)\check{w}_s(x) = \check{w}_{t+s}(x)$ and Property I, $w_t \xrightarrow{p} 1$ as $t \to 0$. For small enough values of t we can even define f directly at a particular $x \in G$ by putting $f(x) = t^{-1}\log \check{w}_t(x)$ and using the principal branch of the logarithm.

We can now prove a slightly weaker statement than Property II for commutative groups.

THEOREM 6.3. *If T_t is a homogeneous process on the locally compact abelian group $\hat{G}$, there is an $m \in N_+(\hat{G})$, such that $t^{-1}h\delta^{\hat{e}}T_t \xrightarrow{c} m$ as $t \to 0$.*

Proof. $\int (x, a)\delta^{\hat{e}}T_t(da) = e^{-tg(x)}$ from Theorem 6.2. If we put $Q_t = t^{-1}\delta^e T_t$,

$$t^{-1}(1 - \exp\{-tg(x)\}) = Q_t'(x) \xrightarrow{b} g(x)$$

as $t \to 0$, and Theorem 6.1 shows $hQ_t \to$ some $m \in N_+(\hat{G})$.

In general it is not possible to choose a single h in the above manner which vanishes just at $\hat{e}$. For example, if G is the real line with the discrete topology, denoted by R_d, then in order to satisfy the condition $|U| < +\infty$, U must be a finite point set and generate a proper subgroup K of R_d. In this case the associated h will surely vanish on $K^* = R_d/K$ which is certainly not equal to $\{\hat{e}\}$. By K^* we mean, as usual, the set of characters identically equal to one on K. By varying U we can, however, prove Property II. In the following we denote by J the class of h's defined by $h(a) = |U|^{-1}\int_U \{1 - (y, a)\}|dy|$ for some compact symmetric neighborhood U of e.

LEMMA 6.2. *If $h \in J$, then $H_h = \{a : h(a) = 0\}$ is a compact subgroup of $\hat{G}$.*

Proof. H_h is closed and compact because $h \to 1$ as $a \to$ infinity. Since $a \in H_h$ is equivalent to $(x, a) = 1$ for all $x \in U$, H_h is also a subgroup of $\hat{G}$

LEMMA 6.3. *For each $a \in \hat{G} - \{\hat{e}\}$ there is an $h \in J$ with $h(a) > 0$.*

Proof. Choose a $y \in G$ for which $(y, a) \neq 1$, and for any U satisfying the above conditions construct a new $U' = yU \cup U \cup Uy^{-1}$. U' satisfies the required conditions and since $(x, a) \neq 1$ for every $x \in U'$, the h corresponding to U' satisfies $h(a) > 0$.

With this preparation Property II is immediate.

THEOREM 6.4. *Given any open neighborhood $N_{\hat{e}}$ of $\hat{e} \in \hat{G}$ and any homogeneous process T_t on $\hat{G}$, $t^{-1}\delta^e T_t \xrightarrow{c}$ some $\mu \in N_+(N_{\hat{e}}^c)$ when restricted to $N_{\hat{e}}^c$.*

Proof. Compactify $\hat{G}$ by adding a point at infinity. Then taking complements in the compactified space $\{H_h^c \cap N_{\hat{e}}^c : h \in J\}$ forms an open covering of the compact $N^c_{\hat{e}}$. Choose a finite subcovering $\{H^c_{h_i} \cap N^c_{\hat{e}}, 1 \leq i \leq n\}$, then $\sum_{i=1}^n h_i > 0$ on $N^c_{\hat{e}}$. By Theorem 6.3 $h_i t^{-1}\delta^{\hat{e}} T_t \xrightarrow{c} m_{h_i}$ on $\hat{G}$ as $t \to 0$ and, accordingly,

$$\sum_{i=1}^n h_i t^{-1}\delta^{\hat{e}} T_t \xrightarrow{c} \sum_{i=1}^n m_{h_i} ,$$

so that

$$t^{-1}\delta^{\hat{e}} T_t \xrightarrow{c} \left(\sum_{i=1}^n h_i\right)^{-1} \sum_{i=1}^n m_{h_i}$$

on $N^c_{\hat{e}}$.

7. Subordination of stochastic processes. In this section we give a new definition of the concept of subordination introduced by Bochner for homogeneous processes on Euclidean spaces. Using this definition we are able to show that there are a great many processes which are not subordinate to any but themselves. We introduce this topic by discussing subordination from the characteristic function viewpoint.

Bochner calls a map $v: (0, +\infty) \to (0, +\infty)$ a completely monotone mapping if for every completely monotone function $f: R^+ \to R$, the function $f \cdot v: R^+ \to R$ is completely monotone. A mapping, v, is a completely monotone mapping if, and only if, dv/dx is a completely monotone function, or equivalently e^{-tv} is a completely monotone function for every $t > 0$. If v is a completely monotone mapping, it can be extended to a map $v: R^+ + iR \to R^+ + iR$ of the closed right half complex plane into itself, which is analytic on the interior of its domain, and is of the form $v(z) = c_0 + cz + Q_v(z)$ where $c_0, c \geq 0$;

(i) $\Re e\{Q_v(z)\} \geq 0$;

(ii) $\Re e\{Q_v(z)\} = 0, \Re e(z) > 0$ implies $Q_v \equiv 0$;

(iii) $\Re e\{Q_v(z)\} = 0, \Re e\{z\} = 0$ implies $\Im m\{Q_v(z)\} = 0$;

(iv) $\Re e\{Q_v(z)\} = o(|z|)$ as $|z| \to +\infty$ with $\Re e(z) \geq 0$. If v is a completely monotone mapping $v(0+)$ exist, and we call v a subordinator when $v(0+) = 0$. v is a subordinator if, and only if, $e^{-tv(x)} = \int_0^\infty e^{-xs}\pi_t(ds)$, where $\pi_t \geq 0$, $\|\pi_t\| = 1$, $\pi_0(\{0\}) = 1$, $\pi_t * \pi_s = \pi_{s+t}$, and $s \to t$ implies $\pi_s \to \pi_t$. If v is a subordinator and $e^{-th(x)} = \int (x, a)\delta^e T_t(da)$ is the family or Fourier-Stieltjes transforms of a homogeneous process on $\hat{G}$, then $\exp\{-tv[h(x)]\}$ is also the family of Fourier-Stieltjes transforms of a homogeneous process on $\hat{G}$.

DEFINITION 7.1. (Bochner) A process $e^{-th(x)} = \int (x, a)\delta^{\hat{e}} T_t(da)$ is called *subordinate* to a process $e^{-th(x)} = \int (x, a)\delta^{\hat{e}} T_t(da)$ if, and only if, one of the three following equivalent conditions is satisfied.

(1) There is a subordinator v, with $e^{-tv(x)} = \int_0^\infty e^{-sx}\pi_t(ds,)$ such that

$$v\{h(x)\} = h(x).$$

(2) $e^{-th(x)} = \int_0^\infty e^{-sh(x)}\pi_t(ds)$.

(3) $\hat{\delta^e} T_t = \int_0^\infty \hat{\delta^e} T_s \pi_t(ds)$.

If T_t is a homogeneous process with infinitesimal generator A and $A_\varepsilon = \varepsilon^{-1}(T_\varepsilon - 1)$, then the equation $\exp\{-tv(-A_\varepsilon)\} = \int \exp\{sA_\varepsilon\}\pi_t(ds)$ shows that subordination leads from a process with infinitesimal generator A to a process with infinitesimal generator $-v(-A)$ when suitable interpreted. Bochner has proven that an infinitely decomposable process with a Fourier-Stieltjes transform of the form $\exp\{-t(bx^2 + ib'x)\}$ on the real line is not subordinate to any processes but those with ch.f.$(X_t) = \exp\{-tc(bx^2 + ib'x)\}$ and $c > 0$. In this case subordination is simply a linear change of time scale. In general we denote the relationship T_t' is subordinate to T_t' by $T_t' < T_t$. If processes differing only by a linear change of time scale are identified, it is easy to show using Bochner's result that the relation $<$ is a proper partial ordering of homogeneous processes.

An alternative definition of subordination rests on a probabilistic choice of time scale. If $X(t, \omega)$ is a measurable stochastic process and $Y(t, \omega)$ is a non negative stochastic process which is independent of $\{X(s, \omega)\}$, and whose paths are almost surely non decreasing in R^+, we can form the composite process $Z(t, \omega) = X(Y(t, \omega), \omega)$. Under special circumstances this composition corresponds to subordination of the $X(t, \omega)$ process by the $Y(t, \omega)$ process. In general the transformation $X(t, \omega) \to Z(t, \omega)$ will preserve any of those properties of the $X(t, \omega)$ process which depend only on the order relations of the time scale. For example, if $X(t, \omega)$ is Markov, a semi-martingale, a martingale, or spatially homogeneous, so is $Z(t, \omega)$. The stationarity of $X(t, \omega)$, a property depending not only on the order of the time scale but on the magnitude of certain time differences, will in general not be preserved unless $Y(t, \omega)$ is a homogeneous process (i.e. spatially homogeneous and stationary). In this last case if $Y(0, \omega) = 0$ with probability one, we say $Z(t, \omega)$ is subordinate to $X(t, \omega)$ with subordinator $Y(t, \omega)$.

After making the observation that $P\{Y(t, \omega) \in E \mid Y(0, \omega) = 0\}$ should correspond to $\pi_t(E)$ in Bochner's definition, we can show the coincidence of transition probabilities in the two concepts of subordination. If $X(t, \omega)$ is a homogeneous process on the commutative group G with $X(t, \omega) = e$ and

$$P\{X(t+s, \omega) \in E | X(s, \omega)\} = \delta^{X(s,\omega)} T_t(E) ,$$

then subordinating by $Y(t, \omega)$ according to Definition 7.1, we get $\delta^e T'_t(E) = \int \delta^e T_s(E) \pi_t(ds)$. If we subordinate the same process using the definition in terms of paths, we find

$$\delta^e T'_t(E) = P\{Z(t, \omega) \in E | Z(0, \omega) = e\}$$

$$\delta^e T'_t(E) = \int P\{X(u, \omega) \in E | Y(t, \omega) = u\} P\{Y(t, \omega) \in du\}$$

$$= \int P\{X(u, \omega) \in E\} \pi_t(du) = \int \delta^e T_u(E) \pi_t(du) .$$

Consider a homogeneous process on a not necessarily commutative locally compact topological group, G, described by the transition probabilities $\delta^x T_t$. For such a process, define the set $F(T_t) = \bigcap_{t>0}$ support $(\delta^e T_t)$. $F(T_t)$ is a closed possibly empty semi-group of G.

THEOREM 7.1. *Let T'_t be a homogeneous process subordinate to T_t on G, and supporse $t^{-1}\delta^e T'_t \to Q_e$ on $G - \{e\}$ in the sense of Theorem* 3.4 *as $t \to 0$. The either $T'_t = T_{ct}$ for some $c > 0$, or support $(Q_e) \supset F(T_t)$.*

Proof. Since the paths of $Y(t, \omega)$ are non decreasing and accordingly, of finite length in any time interval,

$$E\{\exp\{iuY(t, \omega)\}\} = \exp\{t(icu + \int_0^\infty [e^{ixu} - 1] J(dx))\} , \tag{7.1}$$

where $c \geq 0$ and $\int_0^\varepsilon xJ(dx) < +\infty$ for any $\varepsilon > 0$. If $J = 0$, $Y(t, \omega) = ct$ a.s. and $T'_t = T_{ct}$. If $J \neq 0$, the $Y(t, \omega)$ process will have jumps with probability one, and during any of these jumps there is a positive probability that the T_t process will move from its position to a neighborhood of any specific point in $F(T_t)$. This means the subordinated process may have jumps anywhere in $F(T_t)$, then from Theorem 3.3 $F(T_t) \subset$ support (Q_e).

That support $(Q_e) \neq F(T'_t)$ in general can be seen by subordinating a Bernoulli process on the real line (see Bochner [1] for definitions). It is easy to refine Theorem 7.1.

THEOREM 7.2. *Let T'_t be a homogeneous process subordinate to T_t on G. Let $t^{-1}\delta^e T'_t \to Q'_e$ and $t^{-1}\delta^e T_t \to Q_e$ as $t \to 0$ in the sense of Theorem* 3.4. *Let $Y(t, \omega)$ be the subordinating process as in* (7.1). *Then if $c < 0$,*

$$\text{support } (Q'_e) = \text{Cl}\,[\,\cup\,\{\text{support } (\delta^e T_s) : s \in \text{support } (J)\} \cup \text{support } (Q_e)] ,$$

and if $c = 0$,

$$\text{support}\,(Q'_e) = \text{Cl}\,[\,\cup\,\{\text{support}\,(\delta^e T_s) : s \in \text{support}\,(J)\}]\,.$$

In particular, support $(Q'_e) = \phi$ *if, and only if,* $J = 0$ *and either* Q_e *or* c *equals zero.*

Proof. Let $z \in$ support $(\delta^e T_s)$ and $s \in$ support (J). Then given any neighborhood U_z of z there is a neighborhood V_s of s, such that $t \in V_s$ implies $\delta^e T_t(U_z) > 2^{-1}\delta^e T_s(U_z) > 0$. We can show this by choosing a compact set $D \subset U_z$ and a function $f \in C_c(G)$, such that $f(D) = 1, f(U_z^c) = 0$ and $0 \leq f \leq 1$. If D is chosen so that $\delta^e T_s(f)$ is close to $\delta^e T_s(U_z)$ the assertion follows from the continuity of $\delta^e T_t(f)$. There is a positive probability that $Y(t, \omega)$ will have a jump while $t \in V_s$ and the T'_t process a corresponding jump in U_z. Thus $z \in$ support (Q'_e). If $c > 0$ it is clear from the definition of subordination, and the fact that the paths of a homogeneous process can be chosen to have limits from both the left and the right at all time points, that support $(Q_e) \subset$ support (Q'_e). If, conversely, $y \in$ support (Q'_e), the subordinated paths will have a jump in every neighborhood of y and the only manner in which this can happen is for y to lie in the right hand side of the above expressions.

COROLLARY 1. *If* $F(T_t) =$ support $(\delta^e T_s)$ *for every* $s > 0$, *and* $J \neq 0$, *then* support $(Q'_e) = F(T_t)$. *If* $J = 0$ support $(Q'_e) =$ support (Q_e).

COROLLARY 2. *If all motion in the* T_t *process occurs by jumps, as is the case if* T_t *is compound Poisson, and* $J \neq 0$, *then*

$$\text{support}\,(Q'_e) = F(T_t) = \text{Cl}\,[\,\textstyle\bigcup_{k=1}^{\infty}\{\text{support}\,(Q_e)\}^k]\,.$$

Proof. In this case $F(T_t) =$ support $(\delta^e T_s)$ for every $s > 0$ and the first corollary applies.

If G is commutative we can use a different method of description.

THEOREM 7.3. *Let* T'_t *be a homogeneous process subordinate to* T_t *on the commutative group* G. *Let* $t^{-1}\delta^e T'_t \to Q'_e$ *and* $t^{-1}\delta^e T_t \to Q_e$ *as* $t \to 0$. *Then*

$$\text{support}\,(Q'_e) \supset \text{support}\,(Q'_e) \cdot \text{support}\,(Q_e)\,.$$

Proof. In this case it is clear that

$$\text{support}\,(\delta^e T_t) \cdot \text{support}\,(Q_e) \subset \text{support}\,(\delta^e T\,)$$

and the conclusion follows from Theorem 7.2.

Let us now restrict our attention to Euclidean n-space. In R^n we denote the inner product by $\langle x, y\rangle$ and the norm by $|x|$. The characteristic functions of a homogeneous process $X(t, \omega)$ have the form

$$(7.2) \qquad E[\exp\{i\langle z, X(t, \omega)\rangle\}] = \exp\{tS(b, A, F, z)\}:$$

where $b \in R^n$, A is a positive semi-definite linear transformation of R^n, F is a positive, bounded, regular Borel measure on $R^n - \{0\}$,

$$(7.3) \qquad S(b, A, F, z) = i\langle b, z\rangle - \langle Az, z\rangle + Q(F, z)\,,$$

and

$$(7.4) \qquad Q(F, z) = \int_{|u|>0}\left[\exp\{i\langle u, z\rangle\} - 1 - \frac{i\langle u, z\rangle}{1 + |u|^2}\right]\frac{1 + |u|^2}{|u|^2}\,F(du)\,.$$

Corollary 1 shows that if $X'(t, \omega)$ is a process subordinate to $X(t, \omega)$ in a non trivial manner and

$$(7.5) \qquad E[\exp\{i\langle z, X'(t, \omega)\rangle\}] = \exp\{tS(b', A', F', z)\}\,,$$

then support (F) contains the subspace of R^n orthogonal to $\{x \mid Ax = 0\}$. If, in particular, A is positive definite support $(F') = R^n$. Theorem 7.3 states

$$\text{support }(F') + \text{support }(F) \subset \text{support }(F')\,.$$

This shows if support (F') is compact that $X(t, \omega)$ is not subordinate to any process but itself and possibly a Bernoulli process of the form $X(t, \omega) = tb$. In the latter case $X'(t, \omega) = Y(t, \omega)b$ and all displacement of the $X'(t, \omega)$ process takes place along the ray $\{sb: s \geq 0\} = R^+b$. These observations are summarized below.

THEOREM 7.4. *Let $X'(t, \omega)$ be a homogeneous process on R^n for which support (F') is compact, then $X'(t, \omega)$ is not subordinate to any process but itself unless support $(F') \subset R^+b$, and $A' = 0$. In the latter case $X'(t, \omega)$ is subordinate to the Bernoulli process $Z(t, \omega) = tb$.*

Theorem 7.4 does not exhaust the results which can be obtained by the above technique. In particular we will improve Theorem 7.4 for the real line.

Let $X'(t, \omega)$ be a homogeneous process on the real line subordinate to $X(t, \omega)$ as above. For convenience put

$$L(f) = \int_{x\neq 0}[f(x)]^{-1}F(dx)\,,$$

and

$$B(s) = [b + L(x)]s + \mathrm{Cl}\left[\bigcup_{n=1}^{+\infty}\{\text{support }(F)\}^n\right]$$

when $L(|x|) < +\infty$. Then if $\delta^0 T_s$ denotes the transition probabilities of $X(t, \omega)$, and $X(t, \omega)$ has no Gaussian component, it is clear that support $(\delta^0 T_s) = B(s)$ when $L(|x|) < +\infty$, and support $(\delta^0 T_s) = R$ otherwise. Using Theorem 7.2 this leads to the following.

THEOREM 7.5. *Let the homogeneous process $X'(t, \omega)$ on the real line be subordinated by $Y(t, \omega)$ to the process $X(t, \omega)$, where their Fourier-Stieltjes transforms are given by* (7.1) *and* (7.2).

(1) *If $A \neq 0$ or $L(|x|) = +\infty$*, support $(F') = R$.

(2) *If $A = 0$ and $L(|x|) < +\infty$*, then

$$\text{support}\,(F') = [\,\cup\,\{B(s) : s \in \text{support}\,(J)\}] \cup \text{support}\,(J)$$

if $c > 0$, and

$$\text{support}\,(F') = \cup\,\{B(s) : s \in \text{support}\,(J)\}$$

if $c = 0$.

It should be noted that we are using the notation of a multiplicative group, so that $\bigcup_{n=1}^{+\infty} \{\text{support}\,(F)\}^n$ in $B(s)$ refers to the additive semi-group generated by support (F).

If we use the easily proved fact that a closed additive semi-group of the real line which contains both a positive and a negative number is necessarily a subgroup of R, the following corollary of Theorem 7.5 is immediate.

COROLLARY. *If $X'(t, \omega)$ is non trivially subordinate to a homogeneous process, then support (F') has one of the forms $(H + S) \cup W$ or $H + S$, where W is a closed set which generates the closed additive semi-group S, and H is not empty and is contained in either $[0, +\infty)$ or $(-\infty, 0]$. If S is not a closed subgroup of R it is contained in either $[0, +\infty)$ or $(-\infty, 0]$.*

This rules out, among others, sets like $\{\ldots, -2, -1, 0\} \cup (0, +\infty)$ as the support of the F of a non trivially subordinated process.

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