ON A THEOREM DUE TO SZ.-NAGY

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B. Sz.-Nagy [4] has proved the following theorem:

THEOREM A. *Let* $[T_t;\ t \geqq 0]$ *be a strongly continuous semi-group of contraction operators on a Hilbert space* H. *Then there exists a group of unitary operators* $[\mathbf{U}_t,\ -\infty < t < \infty]$ *on a larger Hilbert space* $\mathbf{H}$ *such that*

$$(1) \qquad T_t y = \mathbf{P}\mathbf{U}_t y, \qquad y \in H,\ t \geqq 0\ ;$$

here $\mathbf{P}$ *is the projection operator with range* H. *Then space* $\mathbf{H}$ *can be chosen in a minimal fashion so that* $[\mathbf{U}_t H;\ -\infty < t < \infty]$ *spans* $\mathbf{H}$. *In this case* $[\mathbf{U}_t]$ *is strongly continuous and the structure* $\{\mathbf{H}, \mathbf{U}_t, H\}$ *is determined to within an isomorphism.*[1]

The infinitesimal generator L of the semi-group $[T_t]$ is defined by

$$(2) \qquad \lim_{\delta \to 0+} \delta^{-1}[T_\delta y - y] = Ly$$

for all $y \in H$ for which this limit exists. The operator L is linear and closed with dense domain, $\mathfrak{D}(L)$ (see [1]). It is shown in [2] that L is maximal dissipative in the sense that

$$(3) \qquad (y, Ly) + (Ly, y) \leqq 0, \qquad y \in \mathfrak{D}(L)\ ,$$

and L being maximal with respect to this property. Since $[\mathbf{U}_t]$ is a semi-group as well as a group of operators, the infinitesimal generator $\mathbf{L}$ of $[\mathbf{U}_t]$ also shares these properties; however in the case of a group of unitary operators $i\mathbf{L}$ is in addition self-adjoint.

The purpose of this note is to study the relation between L and $\mathbf{L}$. It turns out that L is a restriction of $\mathbf{L}$ only when L is maximal symmetric. In general L is neither a restriction nor a projection of $\mathbf{L}$; in fact $\mathfrak{D}(\mathbf{L}) \cap H$ may contain only the zero element. Nevertheless we shall obtain $\mathbf{H}$, $\mathbf{L}$, and $[\mathbf{U}_t]$ directly from L, our principal tool being the discrete analogue of the above theorem, which is also due to Sz.-Nagy [4], namely

THEOREM B. *Let* J *be a contraction operator on a Hilbert space* H. *Then there exists a unitary operator* $\mathbf{J}$ *on a larger Hilbert space* $\mathbf{H}$ *such that*

$$(4) \qquad J^n y = \mathbf{P}\mathbf{J}^n y, \qquad y \in H,\ n \geqq 0;$$

here $\mathbf{P}$ *is the projection operator with range* H. *The space* $\mathbf{H}$ *can be*

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[1] Two structures $\{\mathbf{H}, \mathbf{U}_t, H\}$ and $\{\mathbf{H}', \mathbf{U}'_t, H\}$ are isomorphic if there is a unitary map $\mathbf{V}$ of $\mathbf{H}$ onto $\mathbf{H}'$ which is the identity on H and is such that $\mathbf{V}\mathbf{U}_t y = \mathbf{U}'_t\mathbf{V}y$ for all $y \in \mathbf{H}$.

chosen in a minimal fashion in the sense that $[\mathbf{J}^n H; -\infty < n < \infty]$ *spans* $\mathbf{H}$. *In this case the structure* $\{\mathbf{H}, J, H\}$ *is determined to within an isomorphism.*

For a maximal dissipative operator L with dense domain, it is shown in [2, §1.1] that $(I-L)$ is one-to-one with range $\mathfrak{R}(I-L) = H$ and that

$$(5) \qquad J = (I+L)(I-L)^{-1}$$

is a contraction operator with $\mathfrak{D}(J) = H$ and such that $(I+J)$ is one-to-one. Applying Theorem B we obtain the unitary operator $\mathbf{J}$ on the enlarged space $\mathbf{H}$ spanned by $[\mathbf{J}^n H; -\infty < n < \infty]$ with $\mathbf{J}$ satisfying the property (4).

LEMMA 1. *The operator* $(\mathbf{I} + \mathbf{J})$ *is one-to-one.*

Proof. Let S be a contraction operator, set $\mathfrak{Z}(S) = [y; Sy + y = \theta]$, and denote the projection operator with range $\mathfrak{Z}(S)$ by P_S. Then the ergodic theorem (see [3, pp. 400-406]) asserts that

$$\operatorname{st.}\lim_{n\to\infty} (n+1)^{-1}\sum_{n=0}^{n} (-S)^k = P_S$$

and that $SP_S = P_S S = -P_S$. We apply this result first to J and then to $\mathbf{J}$. Making use of (4) we see that

$$\mathbf{P}\mathbf{P}_{\mathbf{J}} y = P_J y, \qquad y \in H .$$

As noted above $P_J = \Theta$, so that $\mathbf{P}\mathbf{P}_{\mathbf{J}}\mathbf{P} = \Theta$. Actually $\mathbf{P}_{\mathbf{J}}\mathbf{P} = \Theta$; for otherwise there would exist a $y \in H$ with $\mathbf{P}_{\mathbf{J}} y \neq \theta$ so that

$$(\mathbf{P}\mathbf{P}_{\mathbf{J}}\mathbf{P} y, y) = (\mathbf{P}_{\mathbf{J}} y, y) = \|\mathbf{P}_{\mathbf{J}} y\|^2 > 0 ,$$

which is impossible. Thus $\mathbf{P}_{\mathbf{J}}\mathbf{P} = \Theta$ and hence $\mathfrak{Z}(\mathbf{J})$ is orthogonal to H. But this means that

$$\mathbf{P}_{\mathbf{J}}\mathbf{J}^n H = \mathbf{J}^n \mathbf{P}_{\mathbf{J}} H = \theta ,$$

and we infer that $\mathbf{J}^n H$ is orthogonal to $\mathfrak{Z}(\mathbf{J})$ for all n. The minimal property of $\mathbf{H}$ therefore requires that $\mathfrak{Z}(\mathbf{J}) = \theta$.

REMARK. Associated with $\mathbf{J}$ is the resolution of the identity $[\mathbf{E}(\sigma); -\pi < \sigma \leqq \pi]$ and the integral representation

$$\mathbf{J}^n = \int_{-\pi}^{\pi} \exp(in\sigma) d\mathbf{E}(\sigma) .$$

Setting the restriction of $\mathbf{PE}(\sigma)$ to H equal to $F(\sigma)$ we see by (4) that

$$J^n = \int_{-\pi}^{\pi} \exp(in\sigma) dF(\sigma) .$$

The argument used in Lemma 1 applied to $S = \exp(i\mu)J$ shows that if

J has no eigenvalues of absolute value one, then neither does $\mathbf{J}$ and hence that both $\mathbf{E}(\sigma)$ and $F(\sigma)$ are strongly continuous in σ. Conversely, $F(\sigma)$ is strongly continuous then as is readily verified

$$(n+1)^{-1}\sum_{k=0}^{n}[\exp(i\mu)J]^k y = \int_{-\pi}^{\pi} K_n(\sigma+\mu)dF(\sigma)y \to \theta\ , \qquad y \in H;$$

here

$$K_n(\sigma) = (n+1)^{-1}\exp(in\sigma/2)\sin\left[\frac{n+1}{2}\sigma\right]\left[\sin\frac{\sigma}{2}\right]^{-1}.$$

It then follows from the ergodic theorem that $\mathfrak{Z}\{-\exp(i\mu)J\} = \theta$ and hence that J has no eigenvalues of absolute value one.

THEOREM. *Set*

$$\mathbf{L} = (\mathbf{J}-\mathbf{I})(\mathbf{J}+\mathbf{I})^{-1}\ . \tag{6}$$

Then $\mathbf{L}$ *generates a strongly continuous group of unitary operators* $[\mathbf{U}_t;\ -\infty < t < \infty]$ *such that*

$$T_t y = \mathbf{P}\mathbf{U}_t y, \qquad y \in H,\ t \geqq 0 \tag{7}$$

and $[\mathbf{U}_t H;\ -\infty < t < \infty]$ *spans* $\mathbf{H}$.

Proof. It follows from the above lemma that $(\mathbf{I}+\mathbf{J})$ is one-to-one and hence that $\mathbf{L}$ is well-defined. Morever $\mathfrak{D}(\mathbf{L}) = \mathfrak{R}(\mathbf{I}+\mathbf{J})$ is necessarily dense in $\mathbf{H}$ since otherwise $(\mathbf{I}+\mathbf{J}^*)$ would nullify some non-zero vector and since $\mathbf{J}^{-1} = \mathbf{J}^*$ the same would be true of $(\mathbf{I}+\mathbf{J})$. Further it is clear that $i\mathbf{L}$ is the Cayley tranform of $i\mathbf{J}$ and hence $\mathbf{L}$ generates a strongly continuous group of unitary operators which we shall denote by $[\mathbf{U}_t]$. In order to verify (7) we proceed to represent the resolvent $R(\lambda, L) = (\lambda I - L)^{-1}$ in terms of J for $\lambda > 0$. We see from (5) that

$$y = 2^{-1}(Ju+u) \text{ and } Ly = 2^{-1}(Ju-u), \qquad u \in H\ . \tag{8}$$

Suppose next that $\lambda y - Ly = f$. Replacing y by u as in (8) we obtain

$$2^{-1}\lambda(Ju+u) - 2^{-1}(Ju-u) = f$$

so that

$$u = 2(1+\lambda)^{-1}\sum_{n=0}^{\infty}[(1-\lambda)(1+\lambda)^{-1}]^n J^n f, \qquad \lambda > 0\ .$$

Again making use of (8) we get

$$y = 2^{-1}(Ju+u) = \sum_{n=0}^{n} a_n(\lambda)J^n f$$

where

$$a_0(\lambda) = (1 + \lambda)^{-1} \text{ and } a_n(\lambda) = 2(1-\lambda)^{n-1}(1 + \lambda)^{-n-1} \text{ for } n > 0 .$$

Thus $R(\lambda, L)$ can be represented by an absolutely convergent series in powers of J for $\lambda > 0$. Taking powers of $R(\lambda, L)$ we see that

$$[R(y, L)]^k = \sum_{n=0}^{\infty} a_n^{(k)}(\lambda) J^n ,$$

where again the series is absolutely convergent. Similarly

$$\mathbf{R}(\lambda, \mathbf{L})^k = \sum_{n=0}^{\infty} a_n^{(k)}(\lambda) \mathbf{J}^n ,$$

and it follows from (4) that

$$(9) \qquad [R(\lambda, L)]^k y = \mathbf{P}[\mathbf{R}(\lambda, \mathbf{L})]^k y, \quad y \in H,\ k \geqq 0,\ \lambda > 0 .$$

According to Yosdia's proof of the Hille-Yosida theorem (see [1]),

$$(10) \qquad T_t = \operatorname*{st.lim}_{\lambda \to \infty} \exp(tB_\lambda) \text{ and } \mathbf{U}_t = \operatorname*{st.lim}_{\lambda \to \infty} \exp(t\mathbf{B}_\lambda), \qquad t \geqq 0 ,$$

where

$$B_\lambda = \lambda^2 R(\lambda, L) - \lambda I \text{ and } \mathbf{B}_\lambda = \lambda^2 R(\lambda, \mathbf{L}) - \lambda \mathbf{I} .$$

Thus for $y \in H$ the relation (9) implies

$$\exp(tB_\lambda) y = \mathbf{P} \exp(t\mathbf{B}_\lambda) y, \qquad y \in H,\ \lambda > 0 ,$$

and this together with (10) gives (7).

It remains to prove that $\mathbf{H}$ is the same as

$$\mathbf{H}_0 = \text{closed linear extension of } [\mathbf{U}_t H; -\infty < t < \infty] .$$

Let $\mathbf{P}_0$ be the projection of $\mathbf{H}$ onto $\mathbf{H}_0$. Then clearly $\mathbf{U}_t\mathbf{H}_0 \subset \mathbf{H}_0$ for all real t, and since $\mathbf{U}_t^* = \mathbf{U}_{-t}$ the same is true of the orthogonal complement to $\mathbf{H}_0$. As a consequence $\mathbf{P}_0\mathbf{U}_t = \mathbf{U}_t\mathbf{P}_0$ for all real t. Hence for $y \in \mathfrak{D}(\mathbf{L})$

$$\mathbf{P}_0\mathbf{L}y = \lim_{\delta \to 0+} \delta^{-1}(\mathbf{P}_0\mathbf{U}_\delta y - \mathbf{P}_0 y) = \lim_{\delta \to 0+} \delta^{-1}(\mathbf{U}_\delta \mathbf{P}_0 y - \mathbf{P}_0 y) = \mathbf{L}\mathbf{P}_0 y .$$

Thus $\mathbf{P}_0$ commutes with $\mathbf{L}$ and hence with $\mathbf{J}$. But since H is obviously contained in $\mathbf{H}_0$ we have

$$\mathbf{J}^n H = \mathbf{J}^n \mathbf{P}_0 H = \mathbf{P}_0 \mathbf{J}^n H \subset \mathbf{H}_0 .$$

The minimal property of $\mathbf{H}$ asserted in Theorem B therefore implies that $\mathbf{H} = \mathbf{H}_0$. This concludes the proof of the theorem.

It should be noted that since $i\mathbf{L}$ is self-adjoint, the largest restriction to H of $i\mathbf{L}$ will be symmetric. On the other hand if iL is symmetric then it is easily verified that J is an isometry and hence that $\mathbf{J}$ is an extension of J; in this case then $\mathbf{L}$ will be an extension of L. However in general if $u \in H$ and $y = \mathbf{J}u + u$, then $z = \mathbf{P}y = Ju + u \in \mathfrak{D}(L)$

and $L\mathbf{P}y = \mathbf{P}Ly$; each $z \in \mathfrak{D}(L)$ can be so represented. A simple example shows that $\mathfrak{D}(L) \cap H$ may contain only the zero element.[2]

References

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[2] Suppose H is one-dimensional and $T_t = \exp(-t)$. The Sz.-Nagy construction for $\mathbf{H}$ in Theorem B then results in $\mathbf{H} = l_2$, the space of complex-valued sequences $y = \{\eta_n; -\infty < n < \infty\}$ with

$$(y, z) = \sum_{n=-\infty}^{\infty} \eta_n \bar{\zeta}_n ,$$

$\mathbf{J}\{\eta_n\} = \{\eta_{n-1}\}$, and $\mathbf{P}\{\eta_n\} = \{\eta'_n\}$ ($\eta'_0 = \eta_0$; $\eta'_n = 0$ for $n \neq 0$). Then relation (8) as applied to $\mathbf{J}$ and $\mathbf{L}$ asserts that for each $\{\eta_n\} \in \mathfrak{D}(\mathbf{L})$ there is a $\{\mu_n\} \in \mathbf{H}$ such that

$$2\eta_n = \mu_{n-1} + \mu_n, \quad 2[\mathbf{L}\{\eta_n\}]_n = \mu_{n-1} - \mu_n .$$

If we also require that $\{\eta_n\} \in H$, then $\mu_{n-1} + \mu_n = 0$ for all $n \neq 0$ and this together with the condition $\sum |\mu_n|^2 < \infty$ implies that $\mu_n = 0$ for all n. It follows that $\mathfrak{D}(L) \cap H = \theta$.