ON ONE-TO-ONE HARMONIC MAPPINGS

ERHARD HEINZ

In this paper we shall prove the following:

THEOREM. *Let* $z = z(w)$ $(z = x + iy,\ w = u + iv)$ *be a one-to-one harmonic mapping of the disc* $|w| < 1$ *onto the disc* $|z| < 1$ *such that* $z(0) = 0$. *Then we have for* $|w| < 1$ *the estimate*

$$(1) \qquad |z_u|^2 + |z_v|^2 \geqq \frac{2}{\pi^2} \,.$$

As an improvement of an earlier result established in [1] J. C. C. Nitsche [4] showed that under the above conditions the inequality

$$(2) \qquad (|z_u|^2 + |z_v|^2)_{w=0} \geqq \frac{1}{2}$$

is satisfied[1]. In contrast to (2) the estimate (1) holds throughout the unit disc $|w| < 1$, but the constant involved is smaller than that of Nitsche.

In order to establish (1) we shall make use of a known result on harmonic functions (the analogue of the Schwarz Lemma)[2]. For the sake of completeness the proof of it will be given here.

LEMMA. *Let* $z = z(w) = x(w) + iy(w)$ *be a complex-valued harmonic function in the disc* $|w| < 1$. *Furthermore, let* $z(0) = 0$ *and* $|z(w)| < 1$ *for* $|w| < 1$. *Then we have the inequality*

$$(3) \qquad |z(w)| \leqq \frac{4}{\pi} \operatorname{arc\,tan} |w| \qquad |w| < 1.$$

Proof. Let θ be an arbitrary real number, and $f(w)$ be the function, which is regular-analytic in the disc $|w| < 1$ and satisfies the relations $f(0) = 0$ and

$$(4) \qquad \Re f(w) = x(w) \cos \theta + y(w) \sin \theta \,.$$

On account of our hypotheses we have

$$(5) \qquad |\Re f(w)| < 1 \qquad |w| < 1,$$

hence,

[1] For further references see [**2**].

[2] See Polya-Szegö [**5**], p. 140.

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$$\Re\left(\exp\left[\frac{i\pi}{2}f(w)\right]\right) > 0 \qquad |w| < 1. \tag{6}$$

Consequently the function

$$g(w) = \frac{\exp\left[\frac{i\pi}{2}f(w)\right] - 1}{\exp\left[\frac{i\pi}{2}f(w)\right] + 1} \tag{7}$$

satisfies the inequality

$$|g(w)| < 1 \qquad |w| < 1, \tag{8}$$

and we have $g(0) = 0$. Applying now the Schwarz Lemma and the elementary inequality

$$\left|\frac{e^{i\zeta} - 1}{e^{i\zeta} + 1}\right| \geqq \tan\frac{1}{2}|\Re\zeta| \qquad |\Re\zeta| \leqq \frac{\pi}{2} \tag{9}$$

we obtain the estimate

$$\tan\frac{\pi}{4}|\Re f(w)| \leqq |g(w)| \leqq |w|\,, \tag{10}$$

hence, by (4)

$$|x(w)\cos\theta + y(w)\sin\theta| \leqq \frac{4}{\pi}\operatorname{arc\,tan}|w| \tag{11}$$

for $|w| < 1$.

Since this holds for every real value of θ the inequality (3) follows, which proves the lemma.

Proof of the theorem. (I) We first prove (1) under the additional hypothesis that the function $z(w)$ and its first derivatives are continuous in the closed disc $|w| \leqq 1$. Since the mapping $w \to z(w)$ is one-to-one and harmonic, its Jacobian $|z_w|^2 - |z_{\bar{w}}|^2$ [3] cannot vanish, in virtue of a theorem of H. Lewy [3]. Furthermore, since hypothesis and conclusion of our theorem remain unchanged, if $z(w)$ is replaced by $\overline{z(w)}$, we may assume without loss of generality that

$$|z_w|^2 - |z_{\bar{w}}|^2 > 0 \qquad |w| < 1. \tag{12}$$

Consequently, the function z_w does not vanish in the disc $|w| < 1$. Furthermore, because of $z_{w\bar{w}} = 0$, it is regular-analytic. From these facts it follows that for $|w| \leqq 1$ the inequality

[3] Here and in the following considerations $\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$ and $\frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$ are the complex derivatives.

(13) $$|z_w| \geqq \min_{|w|=1} |z_w|$$

holds.

We shall now estimate the right-hand side of (13) from below by using our lemma. Let φ and r be two real numbers and $0 < r < 1$. Since by hypothesis the equation $|z(w)| = 1$ holds for $|w| = 1$ we have

(14) $$\left| \frac{z(e^{i\varphi}) - z(re^{i\varphi})}{1-r} \right| \geqq \frac{1 - |z(re^{i\varphi})|}{1-r} \geqq \frac{1 - 4/\pi \operatorname{arc} \tan r}{1-r}$$

If we let r tend to 1, we obtain

(15) $$\left(\left| \frac{\partial z(re^{i\varphi})}{\partial r} \right| \right)_{r=1} \geqq \frac{2}{\pi} \qquad 0 \leqq \varphi < 2\pi.$$

Furthermore, on account of (12) we have

(16) $$\left| \frac{\partial z(re^{i\varphi})}{\partial r} \right| = |z_w(re^{i\varphi})e^{i\varphi} + z_{\bar{w}}(re^{i\varphi})e^{-i\varphi}| \leqq |z_w| + |z_{\bar{w}}| \leqq 2|z_w|$$

for $0 < r \leqq 1$. Combining this with (15) we infer that for $|w| = 1$ the estimate

(17) $$|z_w| \geqq \frac{1}{\pi}$$

holds.

Hence, by (13) we obtain for $|w| \leqq 1$ the inequality

(18) $$\frac{1}{\pi} \leqq |z_w| = \frac{1}{2} |z_u - iz_v| \leqq 2^{-1/2}(|z_u|^2 + |z_v|^2)^{1/2} ,$$

which yields (1).

(II) Now let the mapping $z = z(w)$ merely satisfy the hypotheses of our theorem. Obviously there exists a sequence of numbers $\{R_n\}$ $(n \geqq 2)$ such that the following conditions are satisfied:

(i) We have $0 < R_n < 1$ for all $n \geqq 2$, and

(19) $$\lim_{n\to\infty} R_n = 1 .$$

(ii) The disc $|z| < R_n$ is mapped by the inverse transformation $z \to w$ onto a simply-connected domain D_n such that

(20) $$\left\{ |w| \leqq 1 - \frac{1}{n} \right\} \subset D_n \subset \{|w| < 1\} .$$

Since the mapping $z \to w$ is analytic in x and y, it follows that D_n is bounded by an analytic Jordan curve. By the Riemann mapping theorem there exists a uniquely determined function $w = \Phi_n(\zeta)$, which maps the disc $|\zeta| < 1$ $(\zeta = \xi + i\eta)$ conformally onto D_n such that $\Phi_n(0) = 0$ and $\Phi_n'(0) > 0$. Furthermore, $\Phi_n(\zeta)$ is analytic for $|\zeta| \leqq 1$. Consequently, the function

(21) $$Z(\zeta) = \frac{z(\Phi_n(\zeta))}{R_n}$$

is harmonic for $|\zeta| < 1 + \delta$, where δ is a positive number, and satisfies all the hypotheses of the above theorem. From the facts established in (I) we conclude

(22) $$\frac{|\Phi_n'(\zeta)|^2}{R_n^2}(|z_u|^2 + |z_v|^2) = |Z_\xi|^2 + |Z_\eta|^2 \geqq \frac{2}{\pi^2}\,.$$

Hence we have for $w = \Phi_n(\zeta)$ ($|\zeta| < 1$) the inequality

(23) $$|z_u|^2 + |z_v|^2 \geqq \frac{R_n^2}{|\Phi_n'(\zeta)|^2} \cdot \frac{2}{\pi^2}\,.$$

Furthermore, on account of (20) the estimates

(24) $$\left(1 - \frac{1}{n}\right)|\zeta| \leqq |\Phi_n(\zeta)| \leqq |\zeta|$$

hold for $n \geqq 2$ and $|\zeta| < 1$. Applying the Schwarz Lemma it follows from (24) that there exists a sequence of integers $\{n_k\}$ such that the relations

(25) $$\Phi_{n_k}'(\zeta) \to 1 \qquad (k \to \infty)$$

hold uniformly in every closed disc $|\zeta| \leqq \rho < 1$.

Now let w^* be a fixed complex number with $|w^*| < 1$ and let us determine two positive numbers k_0 and ρ such that the inequalities

(26) $$\frac{|w^*|}{1 - \frac{1}{n_k}} \leqq \rho < 1$$

are satisfied for $k \geqq k_0$. On account of (20) the point w^* belongs to D_{n_k} for $k \geqq k_0$. Hence there exists a sequence of complex numbers $\{\zeta_k\}$ with $|\zeta_k| < 1$ such that the equations

(27) $$w^* = \Phi_{n_k}(\zeta_k)$$

hold for $k \geqq k_0$. By (24) we have

(28) $$|\zeta_k| \leqq \frac{|w^*|}{1 - \frac{1}{n_k}} \leqq \rho < 1$$

for $k \geqq k_0$. Applying now (23) and (25) we conclude

(29) $$(|z_u|^2 + |z_v|^2)_{w=w^*} \geqq \frac{R_{n_k}^2}{|\Phi_{n_k}'(\zeta_k)|^2} \cdot \frac{2}{\pi^2} \to \frac{2}{\pi^2}$$

for $k \to \infty$. Since w^* is an arbitrary point in the disc $|w| < 1$, our theorem is established.

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STANFORD UNIVERSITY