SPACES WHOSE FINEST UNIFORMITY IS METRIC

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Those metrizable spaces for which the finest uniform structure is induced by a metric have attracted a certain amount of attention, and M. Atsuji [1] has collected and extended a list of characterizations of them, regarded as uniform spaces. J. Nagata [6] and B. Levshenko [4] have given topological characterizations of these spaces. This note extends Atsuji's list and gives an analogous list of topological characterizations.

I am indebted to the referee for assistance with the references and improvements in the proofs.

Recall that a metric space (or a subset of a metric space) is said to be *uniformly discrete* if for some $\varepsilon > 0$, the distance between two different points is always at least ε.

THEOREM 1. *For a metric uniform space S, either of the following properties implies that the metric uniformity is the finest compatible with the topology; thus they are equivalent to the properties* (1)-(8) *of* [1, Theorem 1].

(9) *All bounded continuous real-valued functions are uniformly continuous.*

(10) *Every closed discrete subspace of S is uniformly discrete.*

THEOREM 2. *For a metrizable topological space S, the following properties are mutually equivalent:*

(a) *The finest uniformity on S is a metric uniformity.*

(b) *The set of all non-isolated points of S is compact.*

(c) *Every subset of S has a compact boundary.*

(d) *Every closed set has a compact boundary.*

(e) *Every closed continuous image of S is metrizable.*

(f) *Every Hausdorff quotient space of S is metrizable.*

(g) *Every Hausdorff quotient space satisfies the first axiom of countability.*

(h) *Every closed set in S has a countable basis of neighborhoods.*

The equivalence of (a) and (b) in Theorem 2 is due to Nagata [6]. Levshenko has given three conditions equivalent to (b) [4]. One is that S is a regular space having a countable family of locally finite converings such that every locally finite covering has a refinement in this family; the other two are obtained by replacing "locally finite" in both

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places by "point-finite", and then by "star-finite". The mutual equivalence of (d), (e), and (h) is also essentially known; it follows at once from results obtained independently by A. H. Stone [7] and by K. Morita and S. Hanai [5].

Proof of Theorem 1. First, (9) implies (10). Suppose on the contrary that T is a closed discrete subspace of S which is not uniformly discrete. Then for each n we can find two points x_n, y_n, in T, at distance less than $1/n$ from each other; moreover, we can assure that x_n and y_n are distinct from the $2n-2$ preceding points $x_1, \cdots, y_{n-1}$. Then the bounded real-valued function f on T which is 0 on all x_n and 1 on all other points of T is continuous, but not uniformly continuous. By Tietze's theorem, f has a bounded continuous extension over S; this contradicts (9).

Next, (10) implies (4) of Atsuji's Theorem 1 [1], which says that any two disjoint closed sets have disjoint ε neighborhoods for some $\varepsilon > 0$. Suppose on the contrary that A and B are disjoint closed sets and $\{x_n\}$ a sequence of points in S, each x_n common to the $1/n$ neighborhoods of A and B. No subsequence of $\{x_n\}$ has a limit point, for such a point would have to be in $A \cap B$; thus $\{x_n\}$ forms a closed discrete subspace of S. By (10), it is uniformly discrete; and since no subsequence has a limit point, it is infinite. Then there is a sequence $\{y_m\}$ of distinct points x_n at distance at least ε from each other, such that for each y_m there are points p_m in A and q_m in B both within $1/m$ of y_m. Again the p_m and q_m have no limit point; but they form a closed discrete subspace which is not uniformly discrete, a contradiction.

Finally, Atsuji shows [1] that (4) implies (8): "All continuous mappings of S into an arbitrary uniform space S' are uniformly continuous". Taking S' to be the topological space S with its finest uniformity, we conclude that the given uniformity is the finest. Obviously this implies (9), and the proof is complete.

Proof of Theorem 2. To begin with, (a) is equivalent to (a′): The diagonal in $S \times S$ has a countable basis of neighborhoods. For since S is metric, the finest uniformity consists of all neighborhoods of the diagonal; and a uniformity is metric if and only if it has a countable basis [3, Chapter 6]. Now if (b) the set N of all non-isolated points is compact, then the corresponding subset of the diagonal (being a compact subset of a metric space) has a countable basis of neighborhoods V_i. If we define U_i as V_i together with all the isolated points (x, x) not in V_i, we have a countable basis $\{U_i\}$ about the diagonal. Thus (b) implies (a′). Conversely, since N is closed, if it is not compact it contains an infinite closed set of points x_n whose distances from each other are bounded below by $\varepsilon > 0$. Given any countable collection $\{U_n\}$ of neighborhoods of the diagonal, we can find a neighborhood of the diagonal V which does not contain as large a neighborhood of (x_n, x_n) as U_n does (since x_n is not isolated); thus no U_n is contained in V, and $\{U_n\}$ is not a basis.

Thus (a) and (b) are equivalent. Now obviously (b) $\Rightarrow$ (c) $\Rightarrow$ (d). For (d) $\Rightarrow$ (e), Stone has shown [7] that a particular closed continuous image of a metric space is metric if and only if all inverse images of points have compact boundaries. For (e) $\Rightarrow$ (h), if S had a closed subset H having no countable basis of neighborhoods, the quotient space obtained by collapsing H to a point would be the image under a closed mapping but not a metric space. Now we can deduce (b) from (h) in the same manner as from (a′), using the supposed points x_n in S instead of (x_n, x_n) in $S \times S$.

Trivially (f) $\Rightarrow$ (g) $\Rightarrow$ (h). It remains to prove (b) $\Rightarrow$ (f). Let Q be a Hausdorff space which is a quotient of S, with quotient mapping $f\colon S \rightarrow Q$. Then the image of N is a compact metric subspace P of Q. Every point of $Q - P$ is isolated, since its inverse image consists of isolated points and thus is open. We shall verify that Q is a regular space with a σ-discrete base, and thus a metric space [3]. For regularity at a point p of P, consider any closed set, H, not containing p. Since $H \cap P$ is a compact set in a Hausdorff space, p has a closed neighborhood F disjoint from $H \cap P$. Then $F - H$ is a still a neighborhood of p, since H is closed; and $F - H$ is closed, since $H - P$ is open. Now for a base, let $\{U_i\}$ be a countable basis of neighborhood of N in S. Each $f(U_i)$ is a neighborhood of P, because $f^{-1}(f(U_i))$ is a neighborhood of $f^{-1}(P)$; and for every neighborhood V of P, $f^{-1}(V)$ contains some U_i, so that V contains $f(U_i)$. Let $\{B_n\}$ be a countable base for the space P. Each B_n is $P \cap C_n$ for some open set C_n in Q; let $D_{ni} = C_n \cap f(U_i)$. To check that $\{D_{ni}\}$ constitutes a basis for each point p of P, we must check that any closed set H not containing p is disjoint from some D_{ni} containing p. There is a B_n containing p such that $\overline{B}_n$ does not meet $P \cap H$; then $\overline{C_n \cap H}$ and P are disjoint, so there exists some $f(U_i)$ disjoint from $C_n \cap H$. But then $p \in D_{ni}$ and D_{ni} does not meet H. Finally we adjoin the discrete collections E_i of all single points in $Q - f(U_i)$; and the proof is complete.

REMARK. The concluding portion of the proof established the equivalence of (b′): *S is a Hausdorff space in which the set of all non-isolated points is compact metric and has a countable basis of neighborhoods.* This suggests the question whether the metrizability assumptions can be weakened or removed from any of the other conditions. It is easy to check that any compact perfectly normal space satisfies (g) and (h); and the standard example [2, p. 31, Problem 19] also has the property that every closed subset has a compact metric boundary. Note, however, Levshenko's conditions [4].

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