ON STRICTLY SEMI-SIMPLE BANACH ALGEBRAS

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I. Introduction. Define the *strict radical* of an algebra to be the intersection of just those of its two-sided ideals which are regular maximal right ideals. Call the algebra *strictly semi-simple* (sss) if its strict radical is the zero ideal. This note proves that the strict radical of a real Banach algebra B contains the set of topologically nilpotent elements of B. Also, it gives a condition which is both necessary and sufficient for B to be sss.

II. Preliminaries. For any ring or algebra A let $T(A)$ denote the set of all those two-sided ideals in A which are regular maximal right ideals. The intersection of the elements of $T(A)$ is the *strict radical* of A. A is *strictly semi-simple* (sss) if its strict radical is the zero ideal.

LEMMA 1. *Let I be a two-sided ideal in the algebra (ring) A. Then the following are equivalent:*

(a) *$I \in T(A)$, that is, I is a regular maximal right ideal.*

(b) *I is a regular maximal left ideal.*

(c) *A/I is a division algebra (division ring).*

Proof. Use is made of the theorem [4, Theorem 24.6.1] that a division algebra has no proper right or left ideals and that an algebra with no proper right ideals either is trivial or is a division algebra.

If (a) holds, then A/I has no proper right ideals. Now A/I is not trivial since if j is a left unit element of A modulo I, $j' \cdot j' = j' \neq 0$ (where x' denotes the image of $x \in A$ under the canonical homomorphism of A onto A/I). The cited theorem shows A/I is a division algebra. Thus (a) implies (c) and, similarly, (b) implies (c). Moreover, if (a) holds, then j' is a left identity for A/I and hence an identity for it, so that I is regular with j as its associated unit element. If $I \subset L$, L a left ideal in A, then L/I is a left ideal in A/I, and an improper ideal by the cited theorem, so that $L = I$ or A and I is a regular maximal left ideal. Thus (a) implies (b).

Suppose (c) holds and e' is a unit of A/I. Then I is regular with

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e as its associated unit element. If $I \subset J$, J a right ideal in A, J/I is a right improper ideal in A/I so that $J = I$ or A and I is a regular maximal right ideal. Thus (c) implies (a).

Theorem 1 relates the strict radical of A to the Jacobson radical [5] and to the Segal [9] or Brown-McCoy radical [2], which is the intersection of the regular maximal two-sided ideals in A. A is called strongly semi-simple (semi-simple) if the Segal (Jacobson) radical is the zero ideal.

A satisfies *Property M* if each of its regular maximal right ideals is a two-sided ideal.

THEOREM 1. *The strict radical contains the Segal and Jacobson radicals so that if A is* sss *then it is necessarily strongly semi-simple and semi-simple. These radicals coincide if Property M is satisfied.*

Proof. Let W be the set of all regular maximal two-sided ideals and W_r the set of all regular maximal right ideals in A. If $I \in T(A)$ then A/I is a division algebra by Lemma 1 so that $I \in W$. Therefore the strict radical contains the Segal radical which contains the Jacobson radical. Now, if Property M holds, $I \in W_r$ implies $I \in T(A)$ which shows the Jacobson radical contains the strict radical and hence all these radicals coincide.

EXAMPLES. 1. An example of an algebra which is semi-simple and strongly semi-simple but not sss is furnished by the algebra of all 2 by 2 matrices, which is radical in the sense of the strict radical.

2. Arens' BQ^*-algebra [1] are examples of Banach algebras which are sss and satisfy Property M. Indeed, Arens establishes that such algebras are semi-simple and have the property that every closed ideal—and, *a fortiori*, every regular maximal right ideal—is two-sided.

3. Let $C(X, D)$ be the ring of all continuous functions on X with values in D, where X is a compact T_0-space and D a division ring that admits a continuous function $f(x)$ such that $xf(x) + yf(y) = 0$ implies that $x = y = 0$. Kaplansky [6, p. 179] notes that such a function $f(x)$ cannot exist in a ring of characteristic 2 (take $x = y$) but exists in every ring of characteristic different from 2 that he has examined. The maximal right (or left) ideals in $C(X, D)$ are two-sided [6, p. 180], so that $C(X, D)$ satisfies Property M and, since it is semi-simple, it is necessarily sss.

4. If a ring A is strongly regular (that is, if for every $a \in A$ there exists $x \in A$ such that $a^2x = a$) then A is semi-simple and every ideal in it is two-sided [2, pp. 462–4]. Hence a strongly regular ring satisfies Property M and is sss.

III. Necessary and sufficient condition for a Banach algebra to be sss. Henceforth the algebras considered are over the real field and the homomorphisms considered are algebraic (real-linear). Let Q denote the quaternions, $H(A, Q)$ the set of nonzero homomorphisms of the algebra A into Q, $|q|$ the absolute value of the quaternion q, and $C(X, Q)$ the algebra of quaternion-valued functions, continuous on and vanishing at the infinite point of a locally compact Hausdorff space X.

LEMMA 2. *An algebra A is mapped onto the reals, onto a field isomorphic to the complexes, or onto the quaternions by any $h \in H(A, Q)$ and the kernel of h belongs to $T(A)$. If A is a Banach algebra each member of $T(A)$ is the kernel of some member of $H(A, Q)$.*

Proof. Let $h(A)$ denote the image of A under h. For any $u,v \in h(A)$

$$|u \cdot v| = |u| \cdot |v| \,. \tag{1}$$

Under the norm $|u|$, $h(A)$ is a normed algebra. A normed algebra in which the norm satisfies property (1) is isomorphic to either the reals, complexes, or quaternions [7, Theorem II]. Hence $A/h^{-1}(0)$ is a division algebra and $h^{-1}(0) \in T(A)$ by Lemma 1.

Let A be a Banach algebra and $I \in T(A)$. Then A/I is a division algebra by Lemma 1 and a Banach algebra since I is closed. A normed division algebra is isomorphic to the reals, complexes, or quaternions [7, Theorem I]. Hence I is the kernel of some $h \in H(A, Q)$.

THEOREM 2. *Any subalgebra A of $C(X, Q)$ is* sss.

Proof. Let $f \in A$, $f \neq 0$. Then there is an $x \in X$ such that $f(x) \neq 0$. Let $I = \{g \in A\colon g(x) = 0\}$. Then A/I is naturally isomorphic to a subalgebra of Q. Hence $I \in T(A)$ by Lemma 2. But $f \notin I$. Therefore A is sss.

THEOREM 3. *If a Banach algebra B is* sss, *then B is isomorphic with a subalgebra of some $C(X, Q)$.*

Proof. Let $X = H(B, Q)$. There is a natural homomorphism of B into $C(X, Q)$: $f \to \varphi$ where $\varphi(x) = x(f)$. It remains only to show that the homomorphism is $1-1$. Let $f \in B$, $f \neq 0$. Since B is sss there is an $I \in T(B)$ such that $f \notin I$. By Lemma 2, $I = x^{-1}(0)$ for some $x \in X$. Hence $\varphi(x) \neq 0$.

COROLLARY 1. *An algebra isomorphic to a subalgebra of a* sss *Banach algebra is itself* sss. *Hence any subalgebra, whether closed or not, of a* sss *Banach algebra is itself* sss.

IV. The strict radical of a Banach algebra contains the set of topologically nilpotent elements. An element x of a normed algebra is called topologically nilpotent if $r(x) = 0$ where $r(x) = \lim_{n\to\infty} ||x^n||^{1/n} = \sup |\beta|$: $\beta \in$ spectrum of x [8, pp. 617–618].

THEOREM 4. *Let N be the set of topologically nilpotent elements of a Banach algebra B and S the strict radical of B. Let J' be the Jacobson radical of any subalgebra of B. Then $J' \subset N \subset S$.*

Proof. That $J' \subset N$ is known [8, Lemma 1.2]. If it is shown that every $h \in H(B, Q)$ maps $x \in N$ into the zero element, then it follows from Lemma 2 that x belongs to every member of $T(B)$ and therefore to S. The spectrum of $h(x)$ contains the spectrum of x; hence $r[h(x)] = 0$ since $r(x) = 0$. Since a topologically nilpotent element is singular [4, p. 121], $h(x) = 0$. Hence $N \subset S$.

COROLLARY 2. *If a Banach algebra is* sss *then zero is its only topologically nilpotent element.*

COROLLARY 3. *Let N and S be defined as in Theorem* 4 *and let J be the Jacobson radical of B. Then $J = S$ if and only if $N = S$. If B satisfies Property M, then $J = N = S$.*

Proof. Theorem 4 yields Corollary 2 as an immediate consequence and also shows that if $J = S$, then $J = N = S$. If $N = S$, N is an ideal composed of topologically nilpotent elements and therefore $N \subset J$ since J is the union of such ideals [8, p. 617]; hence $J = N$. If Property M is satisfied then $J = S$ by Theorem 1 so that $J = N = S$.

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