CONJUGATE SERIES IN SEVERAL VARIABLES

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1. Introduction. For any trigonometric series

$$\sum_{-\infty}^{\infty} a_n e^{inx}$$

the conjugate series is defined to be

$$\sum_{-\infty}^{\infty} -ia_n \, sg\, n \, e^{inx} ,$$

with the convention that $sg\,0 = 0$. A series in two variables

$$\sum_{m,n=-\infty}^{\infty} a_{mn} e^{i(mx+ny)} \tag{1}$$

can be conjugated with respect to x, with respect to y, or with respect to both variables at once. The last possibility gives the series

$$\sum_{m,n=-\infty}^{\infty} -a_{mn}\, sg(mn)\, e^{i(mx+ny)} , \tag{2}$$

and this has been called [3, 5] the conjugate of (1).

For series in one variable, there are several theorems which state that a trigonometric series belonging to some function class always has conjugate in the same, or perhaps in a different function class. These theorems lead to similar results for series in several variables conjugated with respect to *one* of the variables. For example, one proves very easily that

$$\int |\tilde{f}^x| d\sigma \leqq A \int |f| \log^+ |f| d\sigma + B , \tag{3}$$

where $\tilde{f}^x$ is the function conjugate to f with respect to x, $d\sigma = d\sigma(x, y)$ is invariant measure on the torus, and A, B are absolute constants.

In conjugation with respect to x, the coefficients a_{mn} of (1) are multiplied by $-i$ in the right half-plane and by i in the left half-plane. Conjugation in y involves the upper and lower half-planes in the same way. Any half-plane bounded by a line of rational slope can be transformed by a linear change of variables into, say, the upper half-plane, and so there is a whole family of notions of conjugacy with corresponding theorems. If, however, we divide the plane by a line of irrational slope, it is not so clear how to prove the same theorems, although the definition of conjugacy with respect to any line is at hand. It is even

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possible that there are new difficulties, because a line of irrational slope has lattice points close to it on opposite sides.

Sections 2 and 3 are devoted to the extension of several classical theorems about conjugate functions (all contained in [6, Chapter 7]) to several or infinitely many variables, where conjugacy is defined with respect to any given half-space. Like Bochner, who gave the first such extension [1], we use the method of analytic functions. In another paper [2], we used this method to obtain one result of this kind, but it seems worthwhile to present other extensions in detail as well.

In (3), one can replace $\tilde{f}^x$ by $\tilde{f}^{xy} = \tilde{f}$, the full conjugate of f defined by (2), if the right side is increased a little [3]:

$$\int |\tilde{f}| d\sigma \leqq A \int |f| (\log^+ |f|)^2 d\sigma + B . \tag{4}$$

This is proved by applying one-dimensional theorems twice, and obviously this method must introduce a logarithmic factor twice. It has not been settled whether the logarithm in (4) really needs to be squared. Section 4 contains a proof that $(\log^+)^2$ *cannot be replaced in* (4) *by* $(\log^+)^{2-\varepsilon}$ *for any* $\varepsilon > 0$.

Throughout the paper we use the properties of Orlicz spaces, as presented in [4, Chapter 5] or in [6] with different notation.

2. Half-spaces and measures. It is natural to consider these problems on a discrete abelian group G, whose elements are $M, N, \cdots$, and the compact dual group K with elements $X, Y, \cdots$. The value of a character X at a point N will be written $e^{iN \cdot X}$.

DEFINITION. *A subset S of G is called a* half-space *if*

(a) *$M + N$ belongs to S with M and N*

(b) *For any N in G, exactly one of these alternatives holds: $N \in S$, or $-N \in S$, or $N = 0$.*

If S is a half-space in G, we can define a linear order in G compatible with the group operation by defining positive elements to be those belonging to S. Conversely, the set of positive elements in an ordered group is a half-space. Not all discrete groups contain non-trivial half-spaces, but they exist in profusion in the group of integral lattice points of a Euclidean space, and this is the case which interests us most. For example, if $\alpha_1, \cdots, \alpha_k$ are linearly independent real numbers, then the set of points $N = (n_1, \cdots, n_k)$ satisfying

$$n_1\alpha_1 + \cdots + n_k\alpha_k > 0$$

is a half-space in the lattice group of k dimensions.

Let S be a fixed half-space in G. The continuous functions φ on K having Fourier series of the form

$$\varphi(X) \sim a + \sum_{S} a_N e^{iN\cdot X} \tag{5}$$

constitute a Banach algebra C_S in the uniform topology. Among the complex homomorphisms of C_S is a distinguished one defined by[1]

$$\hat{\varphi}(M_0) = \int \varphi d\sigma = a \; . \tag{6}$$

Besides, every point X of K determines a homomorphism

$$\hat{\varphi}(M_X) = \varphi(X) \; . \tag{7}$$

If F is a function analytic on a domain of the complex plane which contains the spectrum of φ, there is an element ψ of C_S such that

$$F(\hat{\varphi}) \equiv \hat{\psi} \; . \tag{8}$$

We can write simply $F(\varphi) = \psi$, because ψ is uniquely determined by its values $\hat{\psi}(M)$. If we apply (8) to the particular homomorphisms given by (6) and (7) we obtain respectively

$$\int F(\varphi) d\sigma = F\left[\int \varphi d\sigma\right] \tag{9}$$

$$F(\varphi(X)) \equiv \psi(X) \qquad \text{(all } X \in K) \; . \tag{10}$$

Determine a measure μ on the complex plane, depending on φ, in the following way. For every Borel set E set

$$\mu(E) = \sigma(\varphi^{-1}(E)) \; . \tag{11}$$

Then μ is obviously a measure with the properties

$$\begin{cases} \mu(E) \geqq 0 & \text{(all Borel sets } E) \\ \int d\mu = 1 & \\ \int F(z) d\mu(z) = F(a) & (F \text{ analytic on the spectrum of } \varphi) \; . \end{cases} \tag{12}$$

(The last property is the same as (9) above, by virtue of (11). To be analytic on the spectrum of φ, it is enough that F be analytic on a simply connected domain containing the support of μ.) A measure μ satisfying (12) will be said to *represent the point* a.)

A measure representing a given point cannot have its mass distributed over the plane in an arbitrary way. In the next section we derive several inequalities which μ must satisfy. These inequalities will be translated into statements about the relative size of the real and imaginary parts of φ, which is to say into theorems about conjugacy.

[1] $d\sigma = d\sigma(X)$ is normalized Haar measure on K.

Not all theorems about conjugate functions are related to the properties of measures in this way, but those which are can be generalized easily to the class of groups we are considering.

3. The main theorems.

THEOREM 1. *Suppose μ represents the point* 0 *and has support in the strip*

$$-\frac{\pi}{2}+\varepsilon \leqq x \leqq \frac{\pi}{2}-\varepsilon \qquad (\varepsilon > 0)\,. \tag{13}$$

Then

$$\int e^{|y|}d\mu(z) \leqq \frac{2}{\cos\left(\frac{\pi}{2}-\varepsilon\right)}\,. \tag{14}$$

Proof. From last formula of (12) with $F(z)=e^{iz}$ we have

$$\begin{aligned} 1 &= \int e^{iz}d\mu(z) = \int e^{-y}(\cos x + i\sin x)\,d\mu(z) \\ &= \int e^{-y}\cos x\,d\mu(x) \geqq \cos\left(\frac{\pi}{2}-\varepsilon\right)\int e^{-y}d\mu(z)\,. \end{aligned}$$

The same calculations, with $F(z)=e^{-iz}$, give

$$1 \geqq \cos\left(\frac{\pi}{2}-\varepsilon\right)\int e^{y}d\mu(z)\,.$$

Adding these inequalities gives the desired result.

COROLLARY. *Let φ belong to C_S and have mean value zero. Write $\varphi = u + iv$, where u and v are real functions. If $|u| \leqq \frac{\pi}{2}-\varepsilon$ for some positive ε, then*

$$\int e^{|v|}d\sigma \leqq \frac{2}{\cos\left(\frac{\pi}{2}-\varepsilon\right)}\,. \tag{15}$$

Indeed, from the definition of μ we have

$$\int e^{|v(X)|}d\sigma(X) = \int e^{|y|}d\mu(z)\,.$$

The hypothesis on u implies that the support of μ lies in the strip (13), and (15) follows then from (14).

Actually the corollary is true for *bounded* functions φ with Fourier series (5) and mean value zero; for φ can be approximated boundedly by trigonometric polynomials satisfying all the hypotheses of the corollary.

If u is a real function, we shall denote by $v = Tu$ that real function having mean value zero such that $u + iv$ has Fourier series (5). Tu is obviously defined if u is a trigonometric polynomial, but in other cases its existence has to be shown.

COROLLARY. *Let u be a real function such that $u \log^+ |u|$ is summable on K. Then $v = Tu$ exists as a summable function and*

$$\int |v| d\sigma \leqq A + B \int |u| \log^+ |u| d\sigma \tag{16}$$

where A and B are absolute constants.

This corollary is the dual, in an appropriate pair of Orlicz spaces, of the first corollary. For $t \geqq 0$ define the complementary functions

$$\Phi(t) = e^t - t - 1$$
$$\Psi(t) = (t + 1) \log (t + 1) - t \, .$$

Denote by L_Φ and L_Ψ the corresponding Orlicz spaces of *real* functions on K. These spaces are paired by the functional

$$(f, g) = \int f g d\sigma \qquad (f \in L_\Phi, g \in L_\Psi) \, ,$$

which exhibits each as a subset of the dual space of the other. It can be proved that L_Φ is exactly the dual of L_Ψ; and L_Ψ is a closed subspace of the dual of L_Φ.

Let us consider T as an operator carrying real trigonometric polynomials in the uniform norm into L_Φ. Suppose that u has mean value zero, and $\|u\|_\infty \leqq \pi/4$. Using the first corollary and an elementary property of the norm in L_Φ, we have for $v = Tu$

$$\|v\|_\Phi \leqq \int e^{|v|} d\sigma + 1 \leqq 2 \sec \frac{\pi}{4} + 1 \, .$$

Therefore T is a bounded operator, on this subset of the continuous functions. The restriction on the mean value of u is immaterial, and so T can be extended to a bounded operator mapping real continuous functions in the uniform norm into L_Φ.

The operator T^* adjoint to T carries linear functionals on L_Φ to signed measures on K. Let u_1 and u_2 be real trigonometric polynomials with mean values a and b respectively. Set $v_1 = Tu_1$, $v_2 = Tu_2$. Then

$$ab = \int (u_1 + iv_1)(u_2 + iv_2)d\sigma = \int (u_1u_2 - v_1v_2)d\sigma$$
$$+ i\int (u_2v_1 + u_1v_2)d\sigma .$$

Since a and b are real, the last term vanishes and

$$\int (Tu_1)u_2 d\sigma = \int u_1(-Tu_2)d\sigma .$$

It follows, by the definition of the adjoint operation, that

$$T^*u_2 = -Tu_2 d\sigma .$$

The statement that T^* is bounded as an operator from L_Ψ to the space of measures is thus

$$\int |Tu| d\sigma \leqq K \| u \|_\Psi .$$

From the general inequality

$$\| u \|_\Psi \leqq \int \Psi(|u|)d\sigma + 1$$

(16) follows at once, at least for trigonometric polynomials. The full statement of the corollary is derived by a conventional limiting process.

It is curious that the duality argument required that S be a half-space, and not merely a cone, which would have sufficed in proving Theorem 1. However, (16) holds *a fortiori* for the real and imaginary parts of a double power series, for example. This result follows also from the corresponding theorem in one variable.

The second corollary is a limiting case of the theorem of Bochner already referred to. The inequality

$$\left(\int |v|^p d\sigma\right)^{1/p} \leqq A_p \int |u| d\sigma \qquad (0 < p < 1) ,$$

which is another limiting case, was established in [2].

THEOREM 2. *Let u be a non-negative trigonometric polynomial with mean value a, and set $v = Tu$. Then*

$$\int u \log^+ u d\sigma \leqq (a+1)\log(a+1) + \frac{\pi}{2}\int |v| d\sigma . \tag{17}$$

Proof. Let $\varphi = u + iv$, so that φ belongs to C_S. It was proved in [2] that the spectrum of φ lies in the right half-plane. Therefore $(z+1)\log(z+1)$ is analytic on the spectrum of φ, and from (9) we have

$$\int (u + 1 + iv) \log (u + 1 + iv) d\sigma = (a + 1) \log (a + 1) .$$

The right side is real. Taking the real part of the left side gives

$$\int (u + 1) \log |u + 1 + iv| - v \arctan v/(u + 1) d\sigma$$
$$= (a + 1) \log (a + 1) .$$

Using the inequality

$$0 \leqq y \arctan y/x \leqq \frac{\pi}{2} |y| \qquad (x > 0)$$

we find (17) directly.

Of course (17) leads by a limiting argument to the fact that if u is a real non-negative function which is summable together with Tu, then (17) continues to hold.

Finally we remark that Theorem 2 could, like Theorem 1, be expressed in terms of a measure representing a given point.[2]

4. THEOREM 3. *Given* $\varepsilon > 0$, *there is a function* f *with Fourier series* (1) *and satisfying*

$$\int |f| (\log^+ |f|)^{2-\varepsilon} d\sigma < \infty \tag{18}$$

such that (2) *is not a Fourier series.*

The proof depends on a number of fairly independent observations.

(a) If f is a function of one variable with Fourier series

$$f(x) \sim \sum_0^\infty a_n e^{inx} ,$$

then according to a well-known theorem of Hardy and Littlewood

$$\sum_0^\infty \frac{|a_n|}{n + 1} \leqq K \int |f(x)| dx . \tag{19}$$

Bochner has remarked that (19) leads to a similar inequality for functions to two variables:

$$f(x, y) \sim \sum_{m,n=0}^\infty a_{mn} e^{i(mx+ny)}$$

implies

[2] Theorem 2 ceases to be true if S is the subset of the two-dimensional lattice group consisting of all (m, n) with $m \geqq 0$. The reason is that the functional assigning to φ in C_S its mean value is not multiplicative. For the corollaries of Theorem 1 the proofs indeed require that S be a half-space, but the results are still essentially true if S is the set just defined.

$$\sum_{m,n=0}^{\infty} |a_{mn}|/(m+1)(n+1) \leqq K\int |f(x,y)|d\sigma(x,y)\ .$$

(b) Suppose, contrary to the statement of the theorem, that (2) is a Fourier series whenever (18) holds, for a certain positive ε which is fixed from now on. We may take $\varepsilon < 1$. Then the functions, $f, \tilde{f}^x, \tilde{f}^y$, and $\tilde{f}^{xy}$ are all summable, and the same must be true for the linear combination

$$\frac{1}{4}\{f + i\tilde{f}^x + i\tilde{f}^y - \tilde{f}^{xy}\} \sum_{m,n=1}^{\infty} a_{mn}e^{i(mx+ny)} + \frac{1}{4}\sum_{m,n=0} a_{mn}e^{i(mx+ny)}\ .$$

The last sum is a Fourier series, merely because f is summable, and we conclude easily that

$$\sum_{m,n=0}^{\infty} a_{mn}e^{i(mx+ny)}$$

is a Fourier series. *It follows from* (a) *that*

$$\sum_{m,n=0}^{\infty} |a_{mn}|/(m+1)(n+1) < \infty \tag{20}$$

whenever (1) *is the Fourier series of a function f satisfying* (18).

(c) For non-negative t define

$$\Psi_1(t) = (t+1)\log^{2-\varepsilon}(t+1)\ .$$

Since $\varepsilon < 1$, Ψ_1 is a convex function vanishing together with its derivative at the origin. Denote by Φ_1 its complementary function. A simple computation shows that for large t

$$\Phi_1(t) > e^{kt^\beta} \tag{21}$$

where k is an appropriately chosen positive number, and $\beta = 1/(2-\varepsilon)$.

(d) The relation (20), valid for every function of the space L_{Ψ_1}, implies that

$$\sum_{m,n=0}^{\infty} \lambda(m)\lambda(n)e^{i\tau_{mn}}e^{i(mx+ny)} \tag{22}$$

(where the τ_{mn} are arbitrary real numbers, and $\lambda(n) = 1/(n+1)$ for $n \geqq 0$) is the Fourier series of a function belonging to the dual space of L_{Ψ_1}. Since $\Psi_1(2t) \leqq M\Psi_1(t)$ for a fixed number M, the dual of L_{Ψ_1} is exactly L_{Φ_1} [4, p. 138], so that (22) belongs to L_{Φ_1} for every choice of the τ_{mn}. Choose constants τ_{mn} in the particular form $\tau_m + \tau_n$, where $\{\tau_n\}_0^\infty$ is a simple sequence of real numbers. Then (22) is formally the product of

$$F(x) \sim \sum_0^\infty \lambda(n) e^{i\gamma_n} e^{inx} \tag{23}$$

and the same series in y. Since (22) represents a function of two variables which is square-summable, the coefficients in (23) are square-summable and F is in L^2 as a function of one variable. Moreover the function represented by (22) really is $F(x)F(y)$, as one can see in an elementary way.

(e). LEMMA. *Let α be a given positive number, and g a non-negative function on some finite measure space with measure $d\mu$, normalized to have unit total mass. A necessary and sufficient condition to have*

$$\int e^{sg^\alpha} d\mu < \infty \text{ for some } s > 0$$

is that g belong to L^q for every finite q and satisfy

$$\|g\|_q = O(q^{1/\alpha}) .$$

In particular, g belongs to L_Φ (where $\Phi(t) = e^t - t - 1$ as heretofore) just if $\|g\|_q = O(q)$.

To prove the lemma, let s be positive and consider

$$\int e^{sg^\alpha} d\mu = 1 + \sum_1^\infty \frac{s^n}{n!} \int g^{n\alpha} d\mu = 1 + \sum_1^\infty \frac{s^n}{n!} \|g\|_{n\alpha}^{n\alpha} . \tag{24}$$

The series converges for some $s > 0$ if and only if

$$\limsup \left[\frac{1}{n!} \|g\|_{n\alpha}^{n\alpha} \right]^{1/n} = \limsup \frac{1}{n} \|g\|_{n\alpha}^{\alpha} < \infty .$$

Because $\|g\|_q$ is monotone in q, this means the same as

$$\limsup q^{-1/\alpha} \|_q g\| < \infty$$

which was to be proved.

Now we can finish the proof of the theorem. Assuming the theorem was false for a certain positive ε, we deduced in (d) that $G(x, y) = F(x)F(y)$ belongs to L_{Φ_1}, where Φ_1 satisfies (21) and F is defined by (23). The lemma in (e) implies then

$$\|G\|_q = O(q^{1/\beta}) .$$

But $\|G\|_q = [\|F\|_q]^2$, and therefore

$$\|F\|_q = O(q^{1/2\beta}) . \tag{25}$$

This is the crucial point of the proof. Since $\beta > 1/2$, $\|F\|_q$ grows less rapidly than q, and we can show that this is false.

Indeed, from (25) and the lemma it follows that

$$\int e^{s|F|^{2\beta}} dx/2\pi < \infty$$

for some positive s. Set $\Phi_2(t) = e^{t^{2\beta}} - 1$ for non-negative t, and let Ψ_2 be the complementary function. For large t we have

$$\Psi_2(t) < t \log^{1/2\beta} t \ .$$

We are interested now in the spaces L_{Φ_2} and L_{Ψ_2}, formed with functions defined on the circle and measure $dx/2\pi$. If f and g belong to L_{Φ_2} and L_{Ψ_2} respectively, and have Fourier coefficients $\{a_n\}$ and $\{b_n\}$, then $\sum_{-\infty}^{\infty} a_n b_n$ is $(C, 1)$-summable at least [6, p. 88]. In particular, taking F for f,

$$\sum_0^\infty \frac{e^{i\tau_n}}{n+1} b_n$$

is $(C, 1)$-summable to a finite value, no matter how the τ_k are chosen. Therefore

$$\sum_0^\infty \frac{|b_n|}{n+1} < \infty$$

whenever the b_n are the Fourier coefficients of a function g in L_{Ψ_2}. This inequality can be sharpened by the uniform boundedness principle to

$$\sum_0^\infty \frac{|b_n|}{n+1} < A\|g\|_{\Psi_2} < B + C\int |g| \log^{+1/2\beta} |g| dx \ . \tag{26}$$

Choose for g the function equal to $1/2\delta$ on $(-\delta, \delta)$ and zero elsewhere on $(-\pi, \pi)$. For $n \geqq 1$ we have $b_n = \sin n\delta/2\pi n\delta$. For small values of δ, the left side of (26) exceeds a constant multiple of $\log 1/\delta$; but the right side is only a constant times $\log^{1/2\beta} 1/\delta$. Since $2\beta > 1$ this is impossible, and the contradiction establishes the theorem.

A modification of the same proof will show that no function of order smaller than $t \log^2 t$ and having sufficiently regular growth can serve in the hypothesis of the theorem on conjugate functions.

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