# CONNECTIVITY OF TOPOLOGICAL LATTICES

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In this note we show that compact connected topological lattices have strong acyclicity properties, both globally and locally. This yields a proof of a conjecture of A. D. Wallace [6] in the finite dimensional case.

A *topological lattice* is a topological space (Hausdorff) upon which is imposed a lattice structure compatible with the topology. More explicitly, the topological space $M$ is a topological lattice if there are maps

$$(1) \qquad \bigwedge : M \times M \to M \text{ and } \bigvee : M \times M \to M$$

which define a lattice structure on $M$. This means that for $x, y, z \in M$

$$(2) \qquad \bigwedge(x, x) = x \text{ and } \bigvee(x, x) = x\ ,$$

$$(3) \qquad \bigwedge(x, y) = \bigwedge(y, x) \text{ and } \bigvee(x, y) = \bigvee(y, x)\ ,$$

$$(4) \qquad \begin{aligned} &\bigwedge(x, \bigwedge(y, z)) = \bigwedge(\bigwedge(x, y), z) \text{ and} \\ &\bigvee(x, \bigvee(y, z)) = \bigvee(\bigvee(x, y), z), \text{ and} \end{aligned}$$

$$(5) \qquad \bigwedge(x, \bigvee(x, y)) = x \text{ and } \bigvee(x, \bigwedge(x, y)) = x\ .$$

It is customary to write $x \wedge y$ in place of $\bigwedge(x, y)$ and $x \vee y$ in place of $\bigvee(x, y)$. Relation (5) implies that $x \wedge y = x$ if and only if $x \vee y = y$. We shall say that $x \leqq y$ if and only if $x \wedge y = x$. It is easily seen that the relation $x \leqq y$ induces a *partial ordering* on $M$. The element $1 \in M$ is a *unit* in $M$ provided $m \leqq 1$ for all $m \in M$. Similarly, an element $0 \in M$ is a *zero* in $M$ if $0 \leqq m$ for all $m \in M$. Clearly, if such elements exist they are unique.

We shall need several elementary lemmas on topological lattices. Lemmas 2 and 4 were proved in [1], however for completeness we prove them here. Lemma 1 was proven by A. D. Wallace [7].

LEMMA 1. *If $M$ is a compact topological lattice, then it has a unit and a zero.*

LEMMA 2. *If $M$ is a topological lattice, then*

(a) *if $U$ is a neighborhood of $x \in M$, there is a neighborhood $V$ of $x$ such that if $y, z \in V$, then $y \vee z \in U$ and $y \wedge z \in U$, and*

(b) *if $y \leqq x$ and $U_x$ is a neighborhood of $x$, there are neighbor-*

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*hoods* $V_x$ *of* $x$ *and* $V_y$ *of* $y$ *such that if* $x' \in V_x$ *and* $y' \in V_y$, *then* $x' \vee y' \in U_x$.

To prove (a) we observe that $\bigwedge^{-1}(U) \cap \bigvee^{-1}(U)$ is a neighborhood of $(x, x) \in M \times M$ and that there there is a neighborhood $V$ of $x$ such that $V \times V \subset \bigwedge^{-1}(U) \cap \bigvee^{-1}(U)$. Then if $y$ and $z$ lie in $V$, $(y, z)$ lies in this intersection, and so $y \vee z \in U, y \wedge z \in U$.

In (b) $\bigvee^{-1}(U_x)$ is a neighborhood of $(x, y) \in M \times M$ and there are neighborhoods $V_x \times V_y \subset \bigvee^{-1}(U_x)$. Clearly these neighborhoods have the asserted properties.

LEMMA 3. *If* $M$ *is a compact topological lattice and* $U$ *is a neighborhood of* $x \in M$, *there is a neighborhood* $W$ *of* $x$ *such that if* $y, z \in W$ *and* $m \in M$, *then* $(m \wedge y) \vee z \in U$.

For $m \in M$ let $V(m)$ and $N(m)$ be neighborhoods of $x$ and $x \wedge m$ as in Lemma 2(b). Let $P(m)$ and $Q(m)$ be neighborhoods of $m$ and $x$ such that $P(m) \times Q(m) \subset \bigwedge^{-1}(N(m))$ and let $R(m) = V(m) \cap Q(m)$. Then if $m' \in P(m)$ and $y, z \in R(m), m' \wedge y \in N(m)$ and $(m' \wedge y) \vee z \in U$. Since $M$ is compact, there is a finite set $[m_i]_1^n$ of points of $M$ such that $\bigcup_1^n P(m_i) = M$. Let $W = \bigcap_1^n R(m_i)$. Then $W$ is the required neighborhood of $x$.

For $x, y \in M, x \leqq y$, we define

$$C_{x,y} = \{z \in M \mid x \leqq z \leqq y\} . \tag{6}$$

Such sets $C_{x,y}$ will be called *convex sets*. It is clear that if $M$ is a compact topological lattice, then its convex subsets are also compact topological lattices in their natural lattice and topological structures.

LEMMA 4. *If* $M$ *is a compact topological lattice and* $U$ *is a neighborhood of* $x$, *then there is a nighborhood* $V$ *of* $x$ *such that if* $y \in V$, *then* $C_{y,y\vee x} \cup C_{x,y\vee x} \subset U$.

Let $W$ be a neighborhood of $x$ as in Lemma 3 and $V$ be a neighborhood of $x$ for $W$ as in Lemma 2(a). Then if $y \in V, y \vee x \in W$. If $y \leqq z \leqq y \vee x$, then $z = (z \wedge \{y \vee x\}) \vee y \in U$ and if $x \leqq z \leqq y \vee x$, then $z = (z \wedge \{y \vee x\}) \vee x \in U$.

A space $X$ is *acyclic* if $H^*(X) = 0$, where $H^*(\ )$ denotes the reduced cohomology ring; $X$ is *clc* if for each $x \in X$ and closed neighborhood $U$ of $x$, there is a closed neighborhood $V$ of $x, V \subset U$, such that the homomorphism of reduced groups $H^*(U) \to H^*(V)$ induced by inclusion is trivial.

Before proceding to the proof of Theorem 1 we recall a well-known generalization of the fact that homotopic maps induce the same homomorphisms of cohomology.

THEOREM. *If $X$ and $Y$ are compact, $N$ is compact and connected, $n_1, n_2$ are two points of $N$, $f, g: X \to Y$, and $F: X \times N \to Y$ are given such that $F|X \times n_1 = f$ and $F|X \times n_2 = g$, then $f^* = g^*: H^*(Y) \to H^*(X)$.*

THEOREM 1. *If the compact topological lattice $M$ is connected, then it is acyclic and* clc.

The fact that $M$ is acyclic was first proven by A. D. Wallace [8]. We give here a slightly different proof.

Let $\hat{M} = (M \times 1) \cup (1 \times M) \subset M \times M$. Since $M$ is connected and $1 \times 1 \in (M \times 1) \cap (1 \times M)$, $\hat{M}$ is connected. For $x, y \in M$ define

$$f_{x,y}: \hat{M} \to C_{y,y\vee x} \cup C_{x,y\vee x}$$

by

$$f_{x,y}(m, 1) = (m \wedge \{y \vee x\}) \vee y \text{ and } f_{x,y}(1, m) = (m \wedge \{y \vee x\}) \vee x .$$

Note that

$$(1 \wedge \{y \vee x\}) \vee y = y \vee x = (1 \wedge \{y \vee x\}) \vee x .$$

Define $G: M \times \hat{M} \to M$ by $G(m, \hat{m}) = f_{0,m}(\hat{m})$. Then

$$G(m, (0, 1)) = f_{0,m}(0, 1) = (0 \wedge \{m \vee 0\}) \vee m = m$$

and

$$G(m, (1, 0)) = f_{0,m}(1, 0) = (0 \vee \{m \vee 0\}) \wedge 0 = 0 .$$

If $i: M \to M$ is the identity and $j: M \to 0 \xrightarrow{\subset} M$, then $i = G|M \times (0, 1)$ and $j = G|M \times (1, 0)$. Hence $i^* = j^*$. But $i^*$ is the identity isomorphism of $H^*(M)$, and $j^*$ is trivial. Hence, $M$ is acyclic.

For a closed neighborhood $U$ of $x \in M$, let $V$ be a closed neighborhood of $x$ as in Lemma 4. Define $F: V \times \hat{M} \to U$ by

$$F(v, \hat{m}) = f_{x,v}(\hat{m}) \subset C_{v,v\vee x} \cup C_{x,v\vee x} \subset U .$$

Note that $F| V \times (0, 1)$ is the inclusion map of $V$ into $U$ and that $F| V \times (0, 1)$ is the trivial map of $V$ onto $x$. It follows as before that the inclusion map induces the trivial cohomology homomorphism, and hence, that $M$ is clc.

In this connection we remark that Lee Anderson [1] has shown that a locally compact connected lattice is locally connected.

An immediate consequence of Theorem 1 and results of E. G. Begle [2] is the following.

COROLLARY 1. *If $M$ is a finite dimensional, compact connected topological lattice, then $M$ has the fixed point property.*

A slightly stronger statement is also true; namely, if $M$ is such a lattice and $f$ is an upper semi-continuous mapping of $M$ into the set of its convex subsets, then some element of $M$ lies in its image.

THEOREM 2. *If the compact metric topological lattice $M$ is connected, then it is contractible and locally contractible.*

Since $M$ is clc°, it is locally connected. Thus, $\hat{M}$ is a compact, connected, locally connected metric space. It follows that there is a mapping $h: I \rightarrow \hat{M}$ such that $h(0) = (0, 1) \in \hat{M}$ and $h(1) = (1, 0) \in \hat{M}$. Here $I$ denotes the unit interval.

Define $H: M \times I \rightarrow M$ by $H(m, t) = G(m, h)$. Then $H$ is the contracting homotopy sought. For $V \subset U$ as in the proof of Theorem 1, define $J: V \times I \rightarrow U$ by $J(v, t) = F(v, h(t))$. Then $J$ is a contraction of $V$ to $x$ within $U$.

A consequence of this theorem and standard results on absolute neighborhood retracts (see, for example, [5] Propositions 12.2b, 16.4, 19.2) is the following.

COROLLARY 2. *If $M$ is a finite dimensional, compact metric, connected topological lattice, then it is an absolute retract.*

Any convex subset of a compact connected topological lattice has these same properties itself, and is thus acyclic and clc. Furthermore, the intersection of finitely many convex subsets is a convex subset. We shall show that if $M$ satisfies certain additional conditions, it has a neighborhood basis of convex subsets.

A lattice $M$ is said to be *distributive* if for $x, y, z \in M$

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) . \tag{7}$$

A lattice $M$ is said to be of *breadth* $b$ if for each finite set $[x_i]$ of more than $b$ elements of $M$, there is a subset $[y_i]$ of $b$ elements such that $x_1 \wedge x_2 \wedge \cdots = y_1 \wedge y_2 \wedge \cdots \wedge y_b$, and $b$ is the least number for which this holds. Similarly, one can define the breadth, $b_1$, using joins instead of meets. It is then a simple fact that $b = b_1$ (see [3] p. 20).

THEOREM 3. *If $M$ is a compact distributive topological lattice of finite breadth and $U$ is a neighborhood of a point $x \in M$, then there is a convex set $C_{y,z}$ that is a neighborhood of $x$ and lies in $U$.*

Let $b$ denote the breadth of $M$, let $W_1$ denote a closed neighborhood of $x$ as in Lemma 3 for the neighborhood $U$, and let $W_2, \cdots, W_{2b}$ denote neighborhoods of $x$ such that for $2 \leqq i \leqq 2b$, if $y, z \in W_i$ then $y \vee z \in W_{i-1}$ and $y \wedge z \in W_{i-1}$. Let $R$ denote the union of the ranges of all lattice polynomials over the domain $W_{2b}$. By Theorem 12, p. 145 of [3], any such polynomial can be written in the form

$$\bigwedge_{h=1}^{r}\left[\bigvee_{k=1}^{n(h)} x_{i(h,k)}\right].$$

Since each $\bigvee x_{i(h,k)}$ is the join of not more than $b$ elements of $W_{2b}$, every such element lies in $W_b$. Hence, any element in the range of a lattice polynomial over $W_{2b}$ is the meet of not more than $b$ elements of $W_b$ and so lies in $W_1$. Thus, $R \subset W_1$. Since $R$ is a sublattice of $M$, the closure $\overline{R}$ of $R$ is a sublattice of $M$, and $\overline{R} \subset W_1$. $\overline{R}$ is a compact topological lattice and by Lemma 1 has a unit $a$ and a zero $b$. By Lemma 3, $C_{b,a} \subset U$. $C_{b,a}$ is a neighborhood of $x$ since $W_{2b} \subset R \subset \overline{R} \subset C_{b,a}$.

In closing we would like to note the following conjectures.

Suppose $M$ is a compact, metric, connected, distributive topological lattice. Then

(i) $M$ admits sufficiently many lattice homomorphisms onto the unit interval to separate points;

(ii) $M$ is an absolute retract;

(iii) $\dim M =$ breadth of $M$; and

(iv) if $\dim M = n$, $M$ is homeomorphic to a subset of an $n$-cell.

D. E. Edmondson has announced [4] an example of a compact, metric connected two dimensional lattice that is modular but not distributive, and that cannot be imbedded in the plane. Lee Anderson has a proof (unpublished) that breadth $M \leqq \dim M$. Therefore in Theorem 3 the hypothesis that $M$ has finite breadth may be replaced by the hypothesis that $M$ is finite dimensional.

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