ALMOST LOCALLY PURE ABELIAN GROUPS

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0. **Introduction.** It is the purpose of this paper to introduce and to give a preliminary investigation of almost locally pure Abelian groups [see definition 1]. For primary groups the concept of almost locally pure Abelian group coincides with that of no elements of infinite height [Theorem 9].

1. DEFINITION. A group (= Abelian group), G, is *almost locally pure* (hereafter abbreviated a.1.p.) if for every finite set of elements $g_1, \cdots, g_n$ of G there exists a finitely generated pure subgroup, P, of G which contains $g_1, \cdots, g_n$.

2. EXAMPLES. Direct sums of cyclic groups are clearly a.1.p. The complete direct sum of copies of the integers is a.1.p. since by [1] every finite subset is contained in a completely decomposable direct summand and each such summand is free of finite rank.

3. REMARK. If one defines a group G to be locally pure if every finite subset generates a pure subgroup, then it is easy to see that G is a direct sum of cyclic groups of prime order, for various primes.

4. THEOREM. *A direct sum of* a. 1. p. *groups is* a.1.p.

Proof. Let $G = \sum_\alpha \oplus H_\alpha$, where $\oplus$ denotes the weak direct sum, and where H_α is a.1.p. for all α. Let $g_1, \cdots, g_n$ be in G. Now let H_β be a summand in which some g_i has a non-zero component, and consider the components $g_{\beta_1}, \cdots, g_{\beta n}$ of $g_1, \cdots, g_n$ in H_β. In each such H_β (there are only a finite number) there exists a finitely generated pure subgroup P_β containing $g_{\beta_1}, \cdots, g_{\beta_n}$. Then $\sum_\beta \oplus P_\beta$ is a finitely generated pure subgroup containing $g_1, \cdots, g_n$.

5. THEOREM. *If G is* a.1.p., *if K is a subgroup of G, and if for every finite set of elements $g_1, \cdots, g_n$ of G, there exists a pure subgroup, P, of G such that the group generated by K and $g_1, \cdots, g_n$ is a subgroup of P and P/K is finitely generated, then G/K is* a.1.p.

If G and G/K are a.1.p., *where K is pure in G, then for every finite set of elements $g_1, \cdots, g_n$ of G, there exists a pure subgroup, P, of G such that the group generated by K and $g_1, \cdots, g_n$ is a subgroup of P, and P/K is finitely generated.*

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Proof. For the proof of the first statement, assume there exists such a P in G for each finite set of elements of G. Then if $g_1 + K, \cdots, g_n + K$ are elements of G/K, there exists a pure subgroup, P, of G such that the group generated by K and $g_1, \cdots, g_n$ is a subgroup of P and P/K is finitely generated. Now P/K is pure in G/K and G/K is a.1.p.

If G/K is a.1.p. and $g_1, \cdots, g_n$ are elements of G, then there exists a finitely generated pure subgroup, P/K, of G/K which contains $g_1 + K, \cdots, g_n + K$. The inverse image, P, of P/K has the desired properties.

6. COROLLARY. *If G is* a.1.p. *and H is a finitely generated subgroup, then G/H is* a.1.p.

7. COROLLARY. *If T is the torsion subgroup of an* a.1.p. *group, then G/T is* a.1.p.

Proof. For $g_1, \cdots, g_n$ in G, let H be a finitely generated pure subgroup containing $g_1, \cdots, g_n$. Then by [2], P, the subgroup generated by H and T is pure. Clearly the subgroup generated by T and $g_1, \cdots, g_n$ is a subgroup of P and P/T is finitely generated. Hence by the theorem G/T is a.1.p.

8. EXAMPLE. A strong direct sum of a.1.p. groups is not necessarily a.1.p. Consider $G = \sum_n \odot C(p^n)$, where $\odot$ denotes the strong direct sum and $C(p^n)$ is cyclic of order p^n. Then if G where a.1.p., G/T would be a.1.p., where T is the torsion subgroup. But a torsion-free a.1.p. group, F, is only finitely divisible (i.e. for each $x \neq 0$ in F there exists a maximum positive integer, m_x, such that $m_x y = x$ has a solution in F), whereas the element $(0, 1/p, 0, 0, 1/p^2, 0, 0, 0, 1/p^3, 0, 0, 0, 0, 1/p^4, \cdots) + T$ is not zero and is divisible by all powers of p.

9. THEOREM. *A torsion group is* a.1.p. *if and only if its p-components have no elements of infinite height.*

Proof. This follows from the footnote on page 79 of [1] and from 4.

10. LEMMA. *Every subgroup of a torsion-free* a.1.p. *group is* a.1.p.

Proof. Let H be a subgroup of the torsion-free group, G, and let $h_1, \cdots, h_n$ be elements of H. Then there exists a finitely generated pure subgroup, P, of G which contains $h_1, \cdots, h_n$. Since $P \cap H$ is a finitely generated pure subgroup of H, H is a.1.p.

11. LEMMA. *The torsion subgroup, T, of an* a.1.p. *group, G, is* a.1.p.

Proof. The proof is similar to the proof of Lemma 10.

12. THEOREM. *Every subgroup of an* a.l.p. *group is* a.l.p.

Proof. Let G be a.l.p., let S be an arbitrary subgroup of G and let T be the torsion subgroup of G. By 7 G/T is a.l.p. and by 10 $(S \cup T)/T$ is a.l.p. Thus $S/(S \cap T)$ is a.l.p. Now let $s_1, \cdots, s_n$ be elements of S. Since $S/(S \cap T)$ is a.l.p. there exists a finitely generated pure subgroup, $P/(S \cap T)$, of $S/(S \cap T)$ such that $s_1 + (S \cap T), \cdots, s_n + (S \cap T)$ are elements of $P/(S \cap T)$. Since $P/(S \cap T)$ is finitely generated and torsion-free, $P = (S \cap T) \oplus K$, where K is finitely generated and torsion-free. Since K is finitely generated, it is clearly a.l.p., and it follows from 11 and 9 that $S \cap T$ is a.l.p. Hence by 4, P is a.l.p. and $s_1, \cdots, s_n$ are elements of P. Thus there exists a finitely generated pure subgroup, P_1, of P containing $s_1, \cdots, s_n$. Since $S \cap T$ is a pure subgroup of S, P is a pure subgroup of S. Hence P_1 is pure in S.

13. LEMMA. *A countable torsion-free* a.l.p. *group, G, is free.*

Proof. Let $H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots$ be an ascending chain of subgroups of G, each having finite rank r. Let $h_1, \cdots, h_r$ be a maximal linearly independent subset of H_1, and hence of all the H_i's. Since G is a.l.p., there exists a finitely generated pure subgroup P containing $h_1, \cdots, h_r$. Hence each H_i is contained in P. Since P is free of finite rank, it satisfies the ascending chain condition, so that by Theorem E, page 168, of [3], G is free.

14. THEOREM. *If the torsion subgroup T, of an* a.l.p. *group, G, has countable index, then T is a direct summand* (and the complementary summand is free).

Proof. By 7 G/T is a.l.p., countable and torsion-free. Thus by 13 G/T is free. Hence $G = T \oplus K$.

15. LEMMA. *A countable* a.l.p. *p-group, G, is a direct of sum cyclic groups.*

Proof. By Theorem 9 this is a restatement of a theorem of Prüfer [4].

Now we prove a generalization of Prüfer's theorem.

16. THEOREM. *A countable* a.l.p. *group, G, is a direct sum of cyclic groups.*

Proof. Let T be the torsion subgroup of G. Then by 14, $G = T \oplus K = T_{p_1} \oplus \cdots \oplus K$, where T_{p_i} is the p_i-component of T. Since G is countable it follows from 4 and 15 that G is a direct sum of cyclic groups.

17. REMARKS. From 12 and 16 it follows that every countable subgroup of an a.1.p. group is a direct sum of cyclic groups.

If one represents the group of rational numbers as a quotient group of a free group, one obtains a pure subgroup (the kernel of the mapping) of an a.1.p. group which is not a direct summand.

From 16 it follows that if H is a pure subgroup of G and G/H is both a.1.p. and countable, then H is a direct summand of G and the complementary summand is a direct sum of cyclic groups.

It follows from Corollary 6 that if T is the torsion subgroup of an a.1.p. group, G, and if H/T is finitely generated then $G/H \cong (G/T)/(H/T)$ is also a.1.p.

18. THEOREM. *If H is pure in G and if H and G/H are* a.1.p., *the G is* a.1.p.

Proof. Let $g_1, \cdots, g_n$ be elements of G. Since G/H is a.1.p. there exists a finitely generated pure subgroup, L/H, of G/H which contains $g_1 + H, \cdots, g_n + H$. Since H is pure and L/H is finitely generated, $L = H \oplus K$, K finitely generated. Since g_i is in L for $i = 1, \cdots, n$, let $g_i = h_i + k_i$. Since H is a.1.p. let P be a finitely generated pure subgroup of H which contains the h_i. Now g_i is in $P \oplus K$ and $P \oplus K$ is pure in L, which is pure in G. Hence $P \oplus K$ is a finitely generated pure subgroup of G which contains the g_i. Hence G is a.1.p.

19. THEOREM. *Every group, G, has a maximal pure* a.1.p. *subgroup, M,* (which may be 0) and 0 is the only pure a. 1. p. subgroup *of G/M.*

Proof. The existence of M is easily proved by applying Zorn's lemma. If P/M were a non-zero pure a.1.p. subgroup of G/M, then P would be a pure subgroup of G and by Theorem 18 P would by a.1.p., contradicting the maximality of M.

20. COROLLARY. *If G is a p-group and M is a maximal pure* a.1.p. *subgroup of G, then G/M is divisible.*

Proof. Otherwise $G/M = D \oplus R$, with D divisible and R reduced and R has a finite cyclic direct summand, P, which is a pure a.1.p. subgroup of G/M.

21. COROLLARY. *If G is a p-group and M is a countable maximal pure* a.l.p. *subgroup, then M is a basic subgroup of G.*

Proof. By Theorem 16 M is a direct sum of cyclic groups and by 20 G/M is divisible. Hence M is a basic subgroup of G.

REFERENCES

1. R. Baer, *Abelian groups without elements of finite order*, Duke Math. J., **3** (1937), 68-122.

2. J. De Groot, *An isomorphism criterion for completely decomposable Abelian groups*, Math. Ann. **132** (1956), 328-332.

3. L. Pontrjagin, *Topological Groups*, Princeton University Press, 1946.

4. H. Prüfer, *Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen, Gruppen*, Math. Zeit. **17** (1923), 35-61.

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