ON THE BREADTH AND CO-DIMENSION OF A TOPOLOGICAL LATTICE

LEE W. ANDERSON

Consider the following two conjectures:

Conjecture 1. (E. Dyer and A. Shields [7]) If L is a compact, connected, metrizible, distributive topological lattice then $\dim(L) =$ breadth of L.

Conjecture 2. (A. D. Wallace [10]) If L is a compact, connected topological lattice and if $\dim(L) = n$ then the center of L contains at most $2^n - 2$ elements.

The purpose of this note is to prove the following results:

(1) If L is a locally compact distributive topological lattice and if each pair of comparable points is contained in a closed connected chain then the breadth of $L \leqq$ codim (L).

(2) If L is a compact, connected, distributive topological lattice and if codim $(L) \leqq n$ then the center of L contains at most $2^n - 2$ elements.

1. NOTATION.

The terminology and notation used in this paper is the same as in [1] [2] and [3]. If L is a lattice, then the *breadth of* L [4], hereafter denoted by $Br(L)$, is the smallest integer n such that any finite subset, F, of L has a subset F' of at most n elements such that $\inf(F) = \inf(F')$.

If A is a subset of a lattice, let $\wedge A^n$ denote the set of all elements of the form $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ where $x_i \in A$.

2. $Br(L) \leqq cd(L)$.

The proof of the following lemma is quite straight forward and will be omitted.

LEMMA 1. *If L is a lattice then the following are equivalent:*

(i) $Br(L) \leqq n$

(ii) *If A is an $n+1$ — element subset of L then A contains an n-element subset B, such that* $\inf(A) = \inf(B)$.

(iii) *If A is a subset of L and if $m, p \geqq n$ then $\wedge A^m = \wedge A^p$.*

If L is a topological lattice, then L is *chain-wise connected* if for each pair of elements, x and y, in L with $x \leqq y$ there is a closed connected chain from x to y. Clearly a compact connected topological lattice is chainwise connected.

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Problem. Is a locally compact (or locally connected), connected topological lattice chain-wise connected?

THEOREM 1. *If L is a distributive (chain-wise connected) topological lattice then $Br(L) \leqq n$ if, and only if, L does not contain a sublattice topologically isomorphic with a Cartesian product of $n+1$ nondegenerate (closed and connected) chains.*

Proof. If $Br(L) \nleq n$ then L contains an $n+1$ element subset, A, such that if B is any proper subset of A then $\inf(A) \neq \inf(B)$. Let $x_1, \cdots, x_{n+1}$ be an enumeration of A. Let $b_i = \inf(A \backslash x_i)$, $i = 1, 2, \cdots, n+1$ and let $a = \inf(A)$. Then $b_i \neq a$, $i = 1, 2, \cdots, n+1$ and $b_i \neq b_j$ if $i \neq j$. Let C_i, $i = 1, 2, \cdots, n+1$ be a chain from a to b_i. If L is chain-wise connected we can choose C_i closed and connected. Let $C = C_1 \times C_2 \times \cdots \times C_{n+1}$ and define $f\colon C \to L$ by $f(x_1, x_2, \cdots, x_{n+1}) = x_1 \vee x_2 \vee \cdots \vee x_{n+1}$. It is shown in [3] that f is a topological isomorphism, hence the result follows.

If L contains a sublattice, L', isomorphic with a product of $n+1$ nondegenerate chains then $Br(L) \nleq n$ since $Br(L) \geqq Br(L') \geqq n+1$.

COROLLARY 1. *If L is a locally compact, chain-wise connected, distributive topological lattice then $Br(L) \leqq cd(L)$.*

Proof. Suppose $cd(L) \leqq n$ and $Br(L) \nleq n$. Since L is locally compact and connected it follows that L is also locally convex [1]. Since L is locally convex, the chains $C_1, \cdots, C_{n+1}$ chosen in the proof of Theorem 1 can be taken to be compact [2], hence L contains a sublattice topologically isomorphic with a Cartesian product of $n+1$ nondegenerate compact connected chains. It follows from a result of Cohen [6] that the Cartesian product of $n+1$ nondegenerate compact connected chains has codimension $n+1$. Thus it follows that $cd(L) \geqq n+1$ which is a contradiction.

If X is a compact metric space, we denote by 2^X the set of all closed nonvoid subsets of X with the usual Hausdorff metric.

LEMMA 2. *If L is a compact, connected, metrizable topological lattice and if $f\colon 2^L \to L$ defined by $f(A) = \inf(A)$ is continuous then L is an absolute retract.*

Proof. If L is a compact topological lattice and if A is a nonvoid subset of L then $\inf(A)$ exists, hence $f(A)$ is defined. If we embed L in 2^L in the usual way and if f is continuous then L is a retract of 2^L. Since L is compact, connected and metrizable, it follows that L is a

Peano continuum [2]. Therefore 2^L is an absolute retract [9] and so L is also an absolute retract.

COROLLARY 2. (Dyer and Shields [7]) *If L is a compact, metrizable, distributive topological lattice and if $cd(L)$ is finite then L is an absolute retract.*

Proof. If $cd(L) = n$ then $Br(L) \leqq n$ and so $\wedge A^n = \wedge A^{n+1} = \cdots$ for all $A \subset L$. Let $\mathscr{A}$ denote the set of $A \in 2^L$ such that $\inf(A) \in A$. It is known [5] that $f\colon \mathscr{A} \to L$ defined by $f(A) = \inf(A)$ is continuous. Define g: $2^L \to \mathscr{A}$ by $g(A) = \wedge A^n$ then clearly g is continuous and so $F\colon 2^L \to L$ defined by $F(A) = f(g(A)) = \inf(A)$ is continuous. Thus it follows from Lemma 2 that L is an absolute retract.

Problem. Is $A \to \inf(A)$ continuous if L is not distributive and not finite dimensional?

3. *On the set $\mathscr{B}(x)$.* If L is a lattice and $a \in L$, let $\mathscr{M}(a)$ denote the set of all subsets, M, of L that satisfy

(i) $M \wedge M \subset M$
(ii) $a \notin M$
(iii) M is maximal with respect to (i) and (ii).

Let $\mathscr{B}(a)$ denote the set of all complements of elements in $\mathscr{M}(a)$.

LEMMA 3. *If L is a lattice and $a \in L$ then $\cap \{B\colon B \in \mathscr{B}(a)\} = \{a\}$.*

Proof. If $x \in \cap \{B\colon B \in \mathscr{B}(a)\}$ and if $x \neq a$ then by the Hausdorff Maximality Principle, there is a maximal $\wedge$-closed set, M, containing x but not containing a. But then $M \in \mathscr{M}(a)$ and so $x \notin L\backslash M \in \mathscr{B}(a)$. It is clear that $a \in \cap \{B\colon B \in \mathscr{B}(a)\}$, hence the result is established.

LEMMA 4. *If L is a lattice and if $a \in L$, $B \in \mathscr{B}(a)$ then $a \vee L \subset B$ if, and only if, $a = 1$.*

Proof. If $a \vee L \subset B$ then $a \vee L \subset \cap \{B\colon B \in \mathscr{B}(a)\} = \{a\}$ and so $a = 1$. If $a = 1$ then $a \vee L = \{1\} = B$.

LEMMA 5. *If L is a lattice and if $a \in L$, $a \neq L$, $B \in \mathscr{B}(a)$, $M = L\backslash B \in \mathscr{M}(a)$ then $x \in B$ if, and only if, $a \in x \wedge M$.*

Proof. If $a \in x \wedge M$ and if $x \notin B$ then $x \in M$ and so $a \in x \wedge M \subset M$ which is a contradiction. If $x \in B$ and $x = a$ then, since $a \neq 1$, by

Lemma 4 we have $M \cap (a \vee L) \neq \varphi$ and hence $a \in a \wedge M$. If $x \in B$ and if $x \neq a$ and if $a \notin x \wedge M$ then, since

$$(\{x\} \cup (x \wedge M)) \wedge (\{x\} \cup (x \wedge M)) \subset \{x\} \cup (x \wedge M) ,$$

we have $\{x\} \cup (x \wedge M) \subset M$. This, however, is a contradiction since $x \in B = L \backslash M$.

LEMMA 6. *If L is a lattice and if $a \in L$, $b \in B \in \mathscr{B}(a)$ and if $y \geqq a$ then $y \wedge b \in B$.*

Proof. If $b \in B \in B\mathscr{B}(a)$, there is an $x \in M = L \backslash B$ such that $b \wedge x = a$. Now $x \wedge (b \wedge y) = (x \wedge b) \wedge y = a \wedge y = a$ and so, by Lemma 5, $b \wedge y \in B$.

LEMMA 7. *If L is a lattice and if $a \in L$, $b \in B_0$, $b \neq a$ and $b \notin \cup \{B\colon B \in \mathscr{B}(a),\ B \neq B_0\}$ then*

$$\{y \in L\colon y \wedge b = a,\ y \neq a\} \subset \cap \{B\colon B \in \mathscr{B}(a),\ B \neq B_0\} \cap M_0$$

where $M_0 = L \backslash B_0$. Moreover if $y \wedge b = a$ and $y \neq a$ then

$$B_0 = \{x \in L\colon x \wedge y = a\} .$$

Proof. Let $y \in L$ such that $y \wedge b = a$ and let $B \in \mathscr{B}(a)$ be distinct from B_0. Now if $y \notin B$ then $y, b \notin B$ and so $y, b \in L \backslash B \in \mathscr{M}(a)$. But $y \wedge b = a$ which is a contradiction and so $y \in \cap \{B\colon B \in \mathscr{B}(a),\ B \neq B_0\}$. Now $y \neq a$ and $y \wedge y = y$, thus there is an $M \in \mathscr{M}(a)$ with $y \in M$. However $y \in \cap \{B\colon B \in \mathscr{B}(a),\ B \neq B_0\}$ and therefore $M = M_0$. Now if $y \wedge b = a$ and $x \in B_0$ then $y \wedge x \in \cap \{B\colon B \in B(a)\} = \{a\}$ and so $y \wedge x = a$. Also if $y \wedge b = a$ and $y \neq a$ then $y \in M_0$, and so if $y \wedge x = a$ then $x \in B_0$.

LEMMA 8. *If L is a distributive lattice and if $Br(L) = n$ then* $\sup \{\text{card}\,(\mathscr{B}(x))\colon x \in L\} = n$.

Proof. Suppose that for some $a \in L$, $\text{card}\,(\mathscr{B}(a)) \geqq n + 1$. Pick $n + 1$ distinct members of $\mathscr{B}(a)$, say $B_1, \cdots, B_{n+1}$. Since L is distributive, we can pick, for each $i = 1, 2, \cdots, n + 1$, an $x_i \in B_i$ such that $x_i \notin B_i$ if $i \neq j$. Thus it follows that

$$\inf \{x_i\colon i = 1, 2, \cdots n + 1\} \in B_1 \cap B_2 \cap \cdots \cap B_{n+1}$$

but $\inf \{x_i\colon i \neq j \text{ and } i = 1, 2, \cdots, n + 1\} \notin B_j$ and so $Br(L) \geqq n + 1$. Therefore $\text{card}\,(B(x)) \leqq n$ for all $x \in L$.

Now $Br(L) = n$ and so there is an n-element set, say A, such that

$\inf(A) \neq \inf(A')$ for all proper subsets A' of A. Thus for each $a \in A$ we can find a $B \in \mathscr{B}(\inf(A))$ such that $a \in B$ and $A \backslash \{a\} \subset L \backslash B$ and so card $(\mathscr{B}(\inf(A)) \geqq n$.

LEMMA 9. *If L is a distributive topological lattice and if $a \in L$ and if* card $(\mathscr{B}(a))$ *is finite then each $B \in \mathscr{B}(a)$ is a closed sublattice of L.*

Proof. Let $B_1, B_2, \cdots, B_n$ be an enumeration of $\mathscr{B}(a)$. We will show that B_1 is a closed sublattice of L. Since L is distributive, we can pick $b \in B_1$ so that $b \notin B_i$ if $i \neq 1$. Thus there is a $y \in B_2 \cap \cdots \cap B_n$ such that $y \neq a$ and $y \wedge b = a$. By Lemma 7, $B_1 = \{x \in L: x \wedge y = a\}$ and so B_1 is closed. Since L is distributive, B_1 is clearly a sublattice of L.

Problem. If L is a topological lattice and if $a \in L$, and $B \in \mathscr{B}(a)$ is B closed?

THEOREM 2. *If L is a compact, connected, distributive topological lattice and if $cd(L) \leqq n$ and if $a \in L$ and $B \in \mathscr{B}(a)$ then $cd(B) \leqq n-1$.*

Proof. We first prove the theorem for the case $n > 1$. By way of a contradiction let us assume that $cd(L) \leqq n$ and $cd(B) > n - 1$. Then for some closed set $A \subset B$ we have $H^n(B, A) \neq 0$. Since B is a closed sublattice of L we have, letting $b = \sup(B)$, $b \in B$. To simplify our notation, we let $C = \{x \in L: x \wedge b = a\}$, $c = \sup(C)$, $D = c \vee L$, $E = C \vee A$ and $F = B \cup E \cup D$. It follows that $B \cap C = \{a\}$ and $B \cap (E \cup D) = A$, and that C, D, E, and F are closed. We will now show that if $p > 0$, $H^p(E \cup D) = 0$. Define f: $(E \cup D) \times C \to E \cup D$ by $f(x, y) = x \vee y$. Clearly f is defined and continuous. For each $y \in C$ define F_y: $E \cup D \to E \cup D$ by $F_y(x) = f(x, y)$ then, since $E \cup D$ is compact and C is connected, it follows from the Generalized Homotopy lemma that $F_a^* = F_c^*$.

Now F_c retracts $E \cup D$ onto D and, since $H^n(D) = 0$, it follows that $F_c^* = 0$. Also F_a is the identity function and therefore $H^p(E \cup D) = 0$. Now consider the following Mayer-Victoris exact sequence [8]:

$$H^{n-1}(E \cup D) \times H^{n-1}(B) \xrightarrow{I^*} H^{n-1}(A) \xrightarrow{\Delta^*} H^n(F) \xrightarrow{J^*} H^n(E \cup D) \times H^n(B) .$$

Now $H^{n-1}(E \cup D) = H^{n-1}(B) = H^n(E \cup D) = H^n(B) = 0$, and so Δ^* is an isomphorphism onto. It therefore follows that $H^n(F) \neq 0$ which contradicts the fact that $cd(L) \leqq n$ and $H^n(L) = 0$.

In the case $n = 0$, L is a single point and therefore the result is trivial. If $n = 1$ then L is a chain [1] and so B is at most a single point which implies that $cd(B) \leqq 0$.

We recall (see e.g. [3] or [4]) that if L is a lattice with 0 and 1 then the center of L, denoted by $\mathrm{Cen}(L)$, is the set of all $x \in L$ *other then* 0 *and* 1 such that for some $y \in L$, $x \wedge y = 0$ and $x \vee y = 1$. If L is distributive and if $x \in \mathrm{Cen}(L)$ then there is a unique element, denoted by $c(x)$, such that $x \wedge c(x) = 0$ and $x \vee c(x) = 1$.

COROLLARY. *If L is a compact, connected, distributive topological lattice and if $cd(L) \leqq n$ then* $\mathrm{card}\,(\mathrm{Cen}(L)) \leqq 2^n - 2$.

Proof. We proceed by finite induction. If $cd(L) \leqq 1$ then L is a chain and so $\mathrm{card}\,(\mathrm{Cen}(L)) \leqq 0$. Suppose the theorem is true for all $n < k$ and suppose $cd(L) \leqq k$. If $a \in \mathrm{Cen}(L)$, choose $M \in \mathscr{M}(0)$ such that $a \in M$ so that $B = L \backslash M \in \mathscr{B}(0)$. Thus if $\mathscr{B}(0)$ is empty then $\mathrm{Cen}(L)$ is also empty and the result is established. If $\mathscr{B}(0)$ is not empty, let $B \in \mathscr{B}(0)$. It follows from lemma [9] that B is a closed sublattice of L. Letting $b = \sup(B)$ we have that $b \in B$. We will now show that if $a \in \mathrm{Cen}(L)$ then either $b \wedge a = 0$, $b \wedge a = b$, $a \in \mathrm{Cen}(B)$ or $c(a) \in \mathrm{Cen}\,(B)$. If $a \wedge b \neq 0$, b and if $a \notin B$ and if $c(a) \notin B$ then $a, c(a) \in L \backslash B$ and so $a \wedge c(a) \neq 0$ which is a contradiction. Therefore $a \in B$ or $c(a) \in B$. Now if $a \in B$ then $a \wedge (c(a) \wedge b) = 0$ and

$$a \vee (c(a) \vee b) = 1 \vee (a \vee b) = 1 \wedge b = b$$

and so $a \in \mathrm{Cen}(B)$. Similarly if $c(a) \in B$ then $c(a) \in \mathrm{Cen}(B)$. If $a, c(a) \in B$ then $a \vee c(a) = 1 \in B$ which is a contradiction. If $a \wedge b = 0$ then $a \notin B$ and since $b \notin \cup \{A \in \mathscr{B}(0)\colon A \neq B\}$ we have, by Lemma 7, that $B = \{x \in L;\ x \wedge a = 0\}$. Thus it follows that $c(a) \in B$. Therefore $1 = a \vee c(a) \leqq a \vee b$ which implies that $c(a) = b$ and $a = c(b)$. If $a \wedge b = b$ then $c(a) \wedge b = 0$ and so $c(a) = c(b)$ which implies that $a = b$. It follows, therefore, that $\mathrm{card}\ (\mathrm{Cen}(L)) \leqq 2\ \mathrm{card}\,(\mathrm{Cen}(B)) + 2$. Now $cd(B) \leqq k - 1$ and so $\mathrm{card}\,(\mathrm{Cen}(B)) \leqq 2^{k-1} - 2$ and so

$$\mathrm{card}\,(\mathrm{Cen}(L)) \leqq 2(2^{k-1} - 2) + 2 = 2^k - 2\,.$$

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THE UNIVERSITY OF OREGON, EUGENE, OREGON