[6]) with fibre a sphere of dimension $k-1$. We will use the following notation (due to Thom) for the Gysin sequence with the integers mod 2, Z_2, as coefficients:

$$\cdots \longrightarrow H^{q-k}(B) \xrightarrow{\mu} H^q(B) \xrightarrow{p^*} H^q(E) \xrightarrow{\psi} H^{q-k+1}(B) \longrightarrow \cdots$$

Recall that the homomorphism μ is multiplication by the mod 2 characteristic class: $\mu(x) = x \cdot w_k$ for any $x \in H^{q-k}(B)$. In case $w_k = 0$, then $\mu = 0$, and the Gysin sequence splits up into pieces of length three as follows:

$$0 \longrightarrow H^q(B) \xrightarrow{p^*} H^q(E) \xrightarrow{\psi} H^{q-k+1}(B) \longrightarrow$$

Moreover, if we choose an element $a \in H^{k-1}(E)$ such that $\psi(a) = 1 \in H^0(B)$, then any element $x \in H^q(E)$ may be expressed uniquely in the form

$$(1) \qquad\qquad x = p^*(u) + a \cdot p^*(v)$$

where $u \in H^q(B)$ and $v \in H^{q-k+1}(B)$ (the proof is the same as that given in § 8 of [3] except that here we are using Z_2 for coefficients). It is clear from this formula that the Steenrod squares and cup products in $H^*(E)$ are completely determined provided we know the Steenrod squares and products in $H^*(B)$, and provided we can express $Sq^i(a)$ for $1 \leqq i \leqq k-1$ in the form (1), i.e., provided the cohomology classes $\alpha_i \in H^{k-1+i}(B)$ and $\beta_i \in H^i(B)$ in the expression

$$(2) \qquad\qquad Sq^i(a) = p^*(\alpha_i) + a \cdot p^*(\beta_i)$$

are known for $1 \leqq i \leqq k-1$. Of course the classes a and α_i are not uniquely determined. If a' is any other element of $H^{k-1}(E)$ such that $\psi(a') = 1$, then by exactness of the Gysin sequence there exists an unique element $b \in H^{k-1}(B)$ such that

$$a' = a + p^*(b) .$$

Corresponding to formula (2) there is an analogous formula

$$(2') \qquad\qquad Sq^i(a') = p^*(\alpha_i') + a' \cdot p^*(\beta_i') .$$

An easy computation shows that

$$(3) \qquad\qquad \begin{cases} \alpha_i' = \alpha_i + b\beta_i + Sq^i(b) , \\ \beta_i' = \beta_i . \end{cases}$$

Thus β_i is unique; it is an invariant of the given fibre space. However, only the coset of α_i modulo the subgroup $\{\beta_i b + Sq^i b : b \in H^{k-1}(B)\}$ is unique. This coset is also an invariant of the given fibre space.

LEMMA. $\beta_i = w_i$, the ith Stiefel-Whitney class mod 2 of the fibre space.

This lemma is due essentially to Liao, [2]. See also an analogous proof in Massey, [3], § 9.

Thus β_i is identified with a standard invariant of sphere spaces. On the other hand, the coset of α_i does not seem to be related to any standard invariants. It may be thought of as a sort of " secondary characteristic class ", defined for all sphere spaces for which the mod 2 characteristic class w_k vanishes.

In view of this lemma we may write the above equations in the following form:

(4) $$Sq^i(a) = p^*(\alpha_i) + a \cdot p^*(w_i)$$

(5) $$\alpha_i' = \alpha_i + bw_i + Sq^i(b) , \quad b \in H^{k-1}(B) .$$

3. **Application to the problem of imbedding manifolds in Euclidean space.** Our method of applying the results of the preceding section to prove that certain manifolds cannot be imbedded differentiably in r-dimensional Euclidean space is essentially the same as that used in our earlier paper [3]. To save the reader the trouble of referring to this earlier paper, we give a brief summary of the essential points of this method.

Let M^n be a compact connected differentiable manifold which is imbedded differentiably in an $(n + k)$-sphere, S^{n+k}. We assume that S^{n+k} has been given a Riemannian metric. Choose a positive number ε so small that given any point $a \in S^{n+k}$ of distance $\leq \varepsilon$ from M^n, there exists a unique geodesic segment through a of length $\leq \varepsilon$ normal to M^n. Let N denote the set of all points $a \in S^{n+k}$ whose distance from M^n is $< \varepsilon$. N is an open tubular neighborhood of M^n in S^{n+k}. Let E denote the boundary of N, and $p : E \to M^n$ the projection defined by assigning to each point $a \in E$ the point $p(a) \in M^n$ where the unique geodesic segment through a of length ε normal to M^n meets M^n. Then (E, p, M^n) is a realization of the normal $(k - 1)$-sphere bundle of M^n for the given imbedding, and E is a hypersurface in S^{n+k}. Let V denote the complement of N is S^{n+k}, and let $j : E \to V$ denote the inclusion map.

For convenience we introduce the following notation: A^q denotes the image of the homomorphism $j^* : H^q(V, G) \to H^q(E, G)$, where G is the coefficient ring, and $A^* = \sum_q A^q$. Then A^* and $p^*[H^*(M^n, G)]$ are both sub-rings of $H^*(E, G)$, and they are obviously closed under any cohomology operations such as Steenrod squares and reduced pth powers. Even more, A^* and $p^*[H^*(M^n, G)]$ are " permissible sub-algebras " of $H^*(E, G)$ in the terminology of Thom, [6], p. 177. The sub-ring A^* must satisfy the following conditions:

(6) $A^\circ = H^\circ(E, G)$

(7) $H^q(E, G) = A^q + p^* H^q(M^n, G)$ $(0 < q < n + 1,$ direct sum)

(8) $A^q = 0$ for $q \geqq n + k - 1$.

The proof that conditions (6), (7), and (8) hold is based on Theorem V.14 of Thom [6]; see also § 14 of [3]. The existence of the sub-algebra A^* satisfying these conditions is a rather stringent requirement on $H^*(E, G)$.

Our program for trying to prove that a certain manifold M^n cannot be imbedded differentiably in S^{n+k} (or equivalently, in Euclidean $(n + k)$-space, R^{n+k}) may be briefly outlined as follows. Assume that such an imbedding is possible, and let $p : E \to M^n$ denote the normal $(k - 1)$-sphere bundle of this imbedding. By a well-known theorem of Seifert and Whitney, the characteristic class of the normal bundle vanishes, hence the Gysin sequence splits up into pieces of length three as described in the preceding section. Then if one can determine the structure of the cohomology ring of E and perhaps determine some other cohomology operations in E, it may be possible to prove that $H^*(E, G)$ does not admit any sub-ring A^* satisfying the conditions stated in the preceding paragraph. But this is a contradiction.

Using this method with $G = Z_2$, we will now prove our main result:

THEOREM. *If $m = 2^k$, $k > 0$, then $P_{3m-1}(R)$ cannot be imbedded differentiably in R^{4m}.*

Proof. Let x be the generator or $H^1(P_{3m-1}, Z_2)$. As is well known, the cohomology algebra $H^*(P_{3m-1}, Z_2)$ is the truncated polynomial algebra generated by x and subject to the sole relation $x^{3m} = 0$. According to a result of E. Stiefel [5], the total Stiefel-Whitney class $w = \sum_{i \geq 0} w_i$ of the tangent bundle of P_{3m-1} is given by the formula

$$w = (1 + x)^{3m} .$$

(This may be proved directly by the method of Wu [7].) Using the Whitney duality theorem, one sees that the total Stiefel-Whitney class $\bar{w} = \sum_{i \geq 0} \bar{w}_i$ of the normal bundle is given by

$$\bar{w} = (1 + x)^m$$

since

$$w\bar{w} = (1 + x)^{3m}(1 + x)^m = (1 + x)^{4m} = 1 + x^{4m} = 1$$

(Recall that $m = 2^k$). It follows that $\bar{w}_1 = 0$ and $\bar{w}_m = x^m$. Now assume that the differentiable imbedding of P_{3m-1} in S^{4m} is possible and let $p : E \to P_{3m-1}$ denote the normal bundle, whose fibre is an m-sphere.

The characteristic class $\overline{w}_{m+1} = 0$, hence we can apply the method of § 1. Choose $a \in H^m(E, Z_2)$ such that $\psi(a) = 1$. Then equations (4) of § 2 applied to this case give

$$Sq^1(a) = p^*\alpha_1$$
$$a^2 = Sq^m(a) = p^*\alpha_m + a \cdot p^*(x^m) .$$

If $a' \in H^m(E, Z_2)$, $\psi(a') = 1$, and $a' \neq a$, then of necessity $a' = a + p^*(b)$ with $b = x^m$. Therefore equation (5) of § 2 gives

$$\alpha_1' = \alpha_1 + Sq^1(x^m) = \alpha_1$$
$$\alpha_m' = \alpha_m + x^m \cdot x^m + Sq^m x^m = \alpha_m .$$

Hence α_1 and α_m are invariants, independent of the choice of a. Since $H^m(E, Z_2)$ admits the direct sum decomposition

$$H^m(E, Z_2) = A^m + p^*[H^m(P_{3m-1}, Z_2)]$$

by (7), it follows that we may choose the cohomology class a so that it belongs to A^m. From now on we assume such a choice has been made. Note also that it follows directly from the Gysin sequence that for $q > 0$.

$$\text{rank } A^q = \text{rank} H^{q-m}(P_{3m-1}, Z_2)$$

where A^q and $H^{q-m}(P_{3m-1})$ are considered as vector spaces over the field Z_2. Thus rank $A^q = 0$ or 1 for all values of q; it follows that A^q has at most one non-zero element.

First, we consider the invariant α_1. Two cases are possible: $\alpha_1 = 0$, or $\alpha_1 = p^*(x^{m+1})$. If $\alpha_1 = p^*(x^{m+1})$, then the sub-ring A^* is not closed under the operation Sq^1, which is already a contradiction. For the remainder of the proof we will assume that $\alpha_1 = 0$, i.e. $Sq^1 a = 0$, and show that this also leads to a contradiction.

Next we consider the invariant α_m. Here again two cases are possible, $\alpha_m = 0$ or $\alpha_m = x^{2m}$. First let us consider the case where $\alpha_m = 0$, i.e.., $a^2 = a \cdot p^*(x^m)$. Since $a^2 \neq 0$, it must be the unique non-zero element of A^{2m}. Let y denote the unique non-zero element of A^{2m-1}. It is clear that either

$$y = p^*(x^{2m-1}) + a \cdot p^*(x^{m-1}) \text{ or } y = a \cdot p^*(x^{m-1}) .$$

Now $a^2 y \in A^{4m-1}$, therefore $a^2 \cdot y = 0$ by equation (8). An easy calculation shows that $a^2(a \cdot p^* x^{m-1}) = a \cdot p^*(x^{3m-1}) \neq 0$. It follows that $y = p^*(x^{2m-1}) + a \cdot p^*(x^{m-1})$. Next, a computation shows that

$$Sq^1(y) = Sq^1[p^* x^{2m-1} + a \cdot p^* x^{m-1}] = p^*(x^{2m}) + a \cdot p^*(x^m) .$$

Thus $Sq^1(y)$ and a^2 are distinct non-zero elements of A^{2m} which is obviously impossible. Thus we see that the assumption $\alpha_m = 0$ leads to

a contradiction.

Next, consider the case where $\alpha_m = x^{2m}$, i.e. $a^2 = p^*(x^{2m}) + a \cdot p^*(x^m)$. The pattern of the proof in this case is the same as in the preceding paragraph. Let y be the unique non-zero element of A^{2m-1} as before, then either $y = p^*(x^{2m-1}) + a \cdot p^*(x^{m-1})$ or $y = a \cdot p^*(x^{m-1})$, and we must have $a^2 y = 0$ exactly as before. Once again an easy calculation shows that $a^2(p^* x^{2m-1} + a \cdot p^* x^{m-1}) \neq 0$, hence we must have $y = a \cdot p^* x^{m-1}$. Again, one finds that $Sq^1 y = a \cdot p^* x^m \neq a^2$, which is a contradiction. Thus we have shown that the assumption that P_{3m-1} can be imbedded differentiably in S^{4m} leads to a contradiction.

4. Some open problems. H. Hopf has proved in [1] that P_n can be imbedded differentiably in R^{2n-1} or R^{2n}, according as n is odd or even, i.e., according as P_n is orientable or not. This result, together with our main theorem and the previous results mentioned in the introduction, enables one to settle definitely the question of imbedding P_n in the lowest possible dimensional Euclidean space for $n \leq 5$: for $n \leq 5$, Hopf's result is the best possible. The first undecided case is P_6. It follows from Hopf's result that it can be imbedded in R^{12}, and from our result that it cannot be imbedded in R^8. Can it be imbedded in R^9?

The invariants α_i introduced in § 2 raise many interesting questions. Are these invariants of the normal bundle the same for any imbedding of a manifold in Euclidean space? Or, is it possible to give an example of different imbeddings of a manifold in Euclidean space which give rise to different sets of invariants α_i for the corresponding normal bundles?[2] In any case, it seems reasonable to hope that a further investigation of their properties may furnish new tools for proving non-imbeddability theorems for manifolds.

One may also carry out an analogous investigation using the integers mad p, Z_p, for any odd prime p as coefficients, and using Steenrod pth powers instead of Steenrod squares.

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[2] This question is related to the following: It is possible to give an example of an n-manifold M^n which admits two different imbeddings in R^{n+k} such that the normal bundles of these imbeddings are non-equivalent? As far as I know, the answer to this question is not known.

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