BOOLEAN ALGEBRAS OF PROJECTIONS OF FINITE MULTIPLICITY

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Introduction. The multiplicity theory in Banach spaces has been developed recently by Dieudonne [2] and Bade [1]. In [6] we studied the algebra of bounded operators, in a given Hilbert space, that commute with all projections of a given Boolean algebra of self adjoint projections. By using Bade's paper [1], we propose to generalize these results to Banach spaces. The notation of [1] will be used. Let X be a complex Banach space. Let the Boolean algebra of projections be given as follows:

On the compact Hausdorff space Ω, let a measure $E(\cdot)$ be defined for every Borel set, such that:

1. For every Borel set α, $E(\alpha)$ is a projection on X.
2. For every $x \in X$, the vector valued set function $E(\cdot)x$ is countable additive.
3. If α and β are Borel sets then

$$E(\alpha)E(\beta) = E(\alpha \cap \beta) .$$

4. There exists a constant M such that $|E(\alpha) \leqq M|$ for every Borel set α.
5. The Boolean algebra of projections $E(\cdot)$ is complete. (See [1] for definition of completeness.)

In [1] the space Ω was defined to be the Stone space of the Boolean algebra. In the above form it is easier to find examples. Bade's results remain true for this slightly generalized version.

Throughout the paper we assume that the Boolean algebra has uniform multiplicity $n, n < \infty$. (Definition 3.2 of [1]). Thus the following is proved in [1]:

There exist n vectors $x_1, x_2, \cdots, x_n$ and n bounded functionals $x_1^*, x_2^*, \cdots, x_n^*$ such that:

1. $$X = \bigvee_{i=1}^{n} sp(E(\alpha)x_i, \ \alpha \text{ a Borel set})$$
2. Let $x_i^* E(\cdot)x_i = \mu_i(\cdot)$. The measures $\mu_i i = 1, \cdots, n$ are equivalent.
3. For every Borel set e, $\mu_i(e) \geqq 0$ and $\mu_i(e) = 0$ and only if $E(e) = 0$.
4. If $i \neq j$ then $x_i^* E(e) x_j = 0$.
5. For every $x \in X$ there exists n functions $f_1(\omega), \cdots, f_n(\omega)$ defined on Ω such that:
   a. $f_i(\omega) \in L(\Omega, \mu_i)$.
   b. For every Borel set e,

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$$x_i^* E(e)x = \int_e f_i(\omega)\mu_i(d\omega) \ .$$

c. Let $e_m = \left\{\omega \| f_i(\omega)| \leqq m, i = 1, \cdots, n\right\}$
then

$$x = \lim_{m\to\infty} \sum_{i=1}^{n} \int f_i(\omega)E(d\omega)x_i \ .$$

d. The transformation T from X to $\sum_{i=1}^{n} L(\mu_i)$ given by

$$Tx = \begin{pmatrix} f_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\omega) \end{pmatrix}$$

is continuous. The functions $f_1(\omega), \cdots, f_n(\omega)$ are uniquely defined by x, up to sets of measure zero.

These results are proved in 5.1 and 5.2 of [1]. Instead of writing $Tx = \begin{pmatrix} f_1(\omega) \\ \vdots \\ f_n(\omega) \end{pmatrix}$ let us use the notation $x \sim \begin{pmatrix} f_1(\omega) \\ \vdots \\ f_n(\omega) \end{pmatrix}$.

Let $\mathfrak{A}$ be the algebra of bounded operators on X, which commute with all the projections $E(\alpha)$. The purpose of this paper is to study $\mathfrak{A}$.

Representation of the Algebra $\mathfrak{A}$

Let $A \in \mathfrak{A}$, and let

$$Ax_i \sim \begin{pmatrix} a_{1,i}(\omega) \\ \cdot \\ \cdot \\ \cdot \\ a_{n,i}(\omega) \end{pmatrix} \qquad i = 1, \cdots, n \ .$$

Denote this correspondence by $A \sim (a_{i,j}(\omega))$. The functions $a_{i,j}(\omega)$ satisfy by 5.b.

2.1 $$x_i^* E(e)Ax_j = x_i^* AE(e)x_j = \int_e a_{i,j}(\omega)\mu_i(d\omega)$$

and

$$a_{i,j}(\omega) \in L(\mu_i) \ .$$

Equation 2.1 defines the functions $a_{i,j}(\omega)$ uuiquely (a.e.).

Now let $x \in X$ and $x \sim \begin{pmatrix} f_1(\omega) \\ \vdots \\ f_n(\omega) \end{pmatrix}$. If e is a Borel set on which the functions $f_i(\omega)$, $a_{i,j}(\omega)$ are bounded then:

$$E(e)x = \sum_{i=1}^{n} \int_e f_i(\omega)E(d\omega)x_i$$

and

$$\begin{aligned} AE(e)x = E(e)Ax &= \sum_{i=1}^{n} \int_e f_i(\omega)E(d\omega)(Ax_i) \\ &= \sum_{i=1}^{n} \int_e f_i(\omega)E(d\omega)(E(e)Ax_i) . \end{aligned}$$

But

$$E(e)Ax_i = \sum_{j=1}^{n} \int_e a_{j,i}(\omega)E(d\omega)x_j .$$

Hence

$$E(e)Ax = \sum_{1 \leqq i,j \leqq n} \int_e f_i(\omega)E(d\omega)\Big(\int_e a_{j,i}(\lambda)E(d\lambda)x_j\Big) .$$

From condition 3 of the introduction it follows that

$$\begin{aligned} E(e)Ax &= \sum_{1 \leqq i,j \leqq n} \int_e a_{j,i}(\omega)f_i(\omega)E(d\omega)x_j \\ &= \sum_{j=1}^{n} \int_e \Big(\sum_{i=1}^{n} a_{j,i}(\omega)f_i(\omega)\Big)E(d\omega)x_j . \end{aligned}$$

Therefore

$$x_j^* E(e)Ax = \int_e \Big(\sum_{i=1}^{n} a_{j,i}(\omega)f_i(\omega)\Big)\mu_j(d\omega) .$$

This equation means

$$Ax \sim (a_{i,j}(\omega))\begin{pmatrix} f_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\omega) \end{pmatrix} .$$

REMARK. Equation 5.b. of the introduction was proved here, for only some Borel sets. But we know that

$$Ax \sim \begin{pmatrix} g_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ g_n(\omega) \end{pmatrix}$$

for some functions $g_1(\omega), \cdots, g_n(\omega)$. The above argument shows that

$$(a_{i,j}(\omega))\begin{pmatrix} f_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\omega) \end{pmatrix} = \begin{pmatrix} g_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ g_n(\omega) \end{pmatrix} \text{a.e.}$$

THEOREM 2.1. *For every operator $A \in \mathfrak{A}$ there corresponds a matrix of measurable functions $a_{i,j}(\omega)$, $1 \leqq i, j \leqq n$, such that:*

1. $a_{i,j}(\omega) \in L(\mu_i)$.
2. *If*

$$x \sim \begin{pmatrix} f_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\omega) \end{pmatrix}$$

then

$$Ax \sim (a_{i,j}(\omega)) \begin{pmatrix} f_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\omega) \end{pmatrix} .$$

3. *If a matrix of functions, $(b_{i,j}(\omega))$, satisfies condition 2 then $a_{i,j}(\omega) = b_{i,j}(\omega)$ a.e.*

The matrix of the sum or product of two operators is the sum or product of the matrices. If A^{-1} exists and is bounded then

$$A^{-1} \sim (a_{i,j}(\omega))^{-1} .$$

The functions $a_{i,j}(\omega)$ are determined by equation 2.1.

Proof. The existence of a representing matrix was proved above. The other parts of the theorem follow from the uniqueness assertion given in condition 3.

COROLLARY. *Let $A \in \mathfrak{A}$. If $B \in \mathfrak{A}$ and $AB = I(BA = I)$ then $BA = I(AB = I)$.*

Proof. If $AB = I$ then

$$(a_{i,j}(\omega))(b_{i,j}(\omega)) = (\delta_{i,j}) \text{ a.e.}$$

Hence

$$(b_{i,j}(\omega))(a_{i,j}(\omega)) = (\delta_{i,j}) \text{ a.e.}$$

Thus by Theorem 2.1 $BA = I$.

THEOREM 2.2. *Let A_m, $A \in \mathfrak{A}$. If the sequence $\{A_m\}$ converges strongly to A then sequence of functions $\{a_{i,j}^{(m)}(\omega)\}$ converges in measure to $a_{i,j}(\omega)$, for each $1 \leqq i, j \leqq n$. (It does not matter with respect to what measure, because the measures are finite and equivalent).*

Conversely, the sequence $\{A_m\}$ *converges strongly to* A *if:*

1. *The sequence* $\{a_{i,j}^{(m)}\}$ *converges in measure to* $a_{i,j}(\omega)$.
2. *The sequence* $\{|A_m|\}$ *is bounded.*
3. $\bigcup_{K=1}^{\infty}\{\omega \,|\, |a_{i,j}^{(m)}(\omega)| \leqq K.\ \ 1 \leqq i, j \leqq n,\ m = 1, 2, \cdots\} = \Omega$.

Proof. If for each $x \in X$

$$\lim_{m\to\infty} A_m x = Ax$$

then for every Borel set e

$$\left|\int_e (a_{i,j}^{(m)}(\omega) - a_{i,j}(\omega))\mu_i(d\omega)\right| = |x_i^* E(e)(A_m x_j - Ax_j)| \leqq M|A_m x_j - Ax_j| \to 0 \quad m \to \infty$$

where M does not depend on e. Thus the sequence $\{a_{i,j}^{(m)}(\omega)\}$ converges in measure to $a_{i,j}(\omega)$.

On the other hand, if conditions 1, 2 and 3 are satisfied and e is a Borel set, on which the functions $a_{i,j}^{(m)}(\omega)$ are uniformly bounded, then

$$A_m E(e)x_i = \sum_{j=1}^{n}\int_e a_{j,i}^{(m)}(\omega)E(d\omega)x_j$$

and by the Lebesque Theorem, [5] IV.10.10

$$\lim_{m\to\infty} A_m E(e)x_i = \sum_{j=1}^{n}\int_e a_{j,i}(w)E(dw)x_j\ .$$

Now, by condition 3, the set of linear combinations of $E(e)x_i$, $1 \leqq i \leqq n$ and e as defined above, is dense. Thus the sequence $\{A_m x\}$ has a limit for x in a dense subset of X, and by condition 2 it has a limit for every $x \in X$. Let A be the strong limit of $\{A_m\}$ then

$$AE(e)x_i = \sum_{j=1}^{n}\int_e a_{j,i}(\omega)E(d\omega)x_j\ .$$

Thus the matrix of A is $(a_{i,j}(\omega))$. (See Remark before Theorem 2.1).

In order to develop further the theory, let us borrow the following results from [6].

LEMMA 2.1. *Let* $(a_{i,j}(\omega))$ *be a matrix of measurable finite functions. There exists a decomposition of the form*

$$(a_{i,j}(\omega)) = \sum_{k=1}^{n} z_k(\omega)\varepsilon_k(\omega) + N(\omega) \tag{2.2}$$

where $z_1(\omega), \cdots, z_n(\omega)$ *are measurable functions and* $\varepsilon_1(\omega), \cdots, \varepsilon_n(\omega)$, $N(\omega)$ *are matrices of measurable functions satisfying:*

$$\varepsilon_i^2(\omega) = \varepsilon_i(\omega),\ if\ i \neq j\ then\ \varepsilon_i(\omega)\varepsilon_j(\omega) = 0,\ \sum_{i=1}^{n} \varepsilon_i(\omega) = (\delta_{i,j}) .$$

Also

$$\varepsilon_i(\omega)N(\omega) = N(\omega)\varepsilon_i(\omega), \qquad (N(\omega))^n = 0 .$$

Moreover, there exist n Borel sets $\beta_1, \cdots, \beta_n$ *whose union is* Ω *such that on* β_i *the numbers* $z_1(\omega), \cdots, z_i(\omega)$ *are different while* $z_{i+1}(\omega) = \cdots = z_n(\omega) = 0$.

The proof is given in Lemma 3.1, 3.2 and Theorem 3.1 of [6].

THEOREM 2.3. *Let* $A \in \mathfrak{A}$. *There exists a sequence of Borel sets,* $\{\alpha_m\}$ *such that:*

1. *The sequence* $\{\alpha_m\}$ *increases to* Ω.
2. *The operator* $AE(\alpha_m)$ *is spectral. (For definition of spectral operators see* [3]).

Thus A is a strong limit of a sequence of spectral operators.

Proof. Let $A \sim (a_{i,j}(\omega)) = \sum_{k=1}^{n} z_k(\omega)\varepsilon_k(\omega) + N(\omega)$, where the right side of the equation is defined in Lemma 2.1. Let α be a Borel set such that

a. On the set α the functions $z_k(\omega)$ are bounded.

b. If $\chi_\alpha(\omega)$ is the characteristic function of α, then $\chi_\alpha(\omega)\varepsilon_k(\omega)$ and $\chi_\alpha(\omega)N(\omega)$ are representing matrices of the operators $E_{k,\alpha}$ and N_α respectively in $\mathfrak{A}$.

Then, by Theorem 2.1,

$$AE(\alpha) = \sum_{i=1}^{n} \Big(\int_\alpha z_i(\omega)E(d\omega)\Big)E_{i,\alpha} + N_\alpha \tag{2.3}$$

where $E_{i,\alpha}$ are disjoint projections and N_α is a nilpotent of order n commuting with them.

Thus, for such α, the operator $AE(\alpha)$ is spectral, and the resolution of the identity (see [3]) of A restricted to $E(\alpha)X$ is

$$\sum E(z_i^{-1}(\cdot))E_{i,\alpha} .$$

In order to prove the theorem, we have to find a sequence of Borel sets, satisfying conditions a, b and 1. Also with no loss of generality, we may study the operator A on $E(\beta_i)X$ (Lemma 2.1). Thus we may assume that at each point ω, the matrix $(a_{j,k}(\omega))$ has exactly i eigenvalues.

Define

$$\alpha_m = \Big\{\omega \,\big|\, |z_j(\omega)| \leqq m \text{ and } |z_j(\omega) - z_k(\omega)| \geqq \frac{1}{m},\ 1 \leqq k < j \leqq i\Big\} .$$

On the set α_m the matrix $\varepsilon_1(\omega)$ can be calculated as follows:

Let $Q(z)$ be the polynomial

$$Q(z) = b_0 + b_1 z + \cdots + b_{i(n+1)-1} z$$

such that:

$$Q(z_1(\omega)) = 1, \qquad Q(z_j(\omega)) = 0, \qquad 2 \leqq j \leqq i ,$$
$$Q^{(P)}(z_j(\omega)) = 0, \qquad 1 \leqq j \leqq i, \qquad 1 \leqq p \leqq n$$

then

$$Q(a_{i,j}(\omega)) = \varepsilon_1(\omega) \text{ [see [4] p. 188].}$$

These equations have a unique solution $b_j = b_j(\omega)$, which are measurable and bounded (on α_m) functions of ω. Thus

$$\begin{aligned} &\chi_{\alpha_m}(\omega)\varepsilon_1(\omega) \\ &\quad = \chi_{\alpha_m}(\omega)[b_0(\omega) + b_1(\omega)(a_{i,j}(\omega)) + \cdots + b_{i(n+1)-1}(a_{i,j}(\omega))^{i(n+1)-1}] \end{aligned}$$

and this matrix represents the operator $E_{1,m}$, in $\mathfrak{A}$, where

$$\begin{aligned} E_{1,m} = E(\alpha_m)\Big[&\int b_0(\omega)E(d\omega) \\ &+ A\int b_1(\omega)E(d\omega) + \cdots + A^{i(n+1)-1}\int b_{i(n+1)-1}(\omega)E(d\omega)\Big]. \end{aligned}$$

Similarly the matrices $\chi_{\alpha_m}(\omega)\varepsilon_j(\omega)$ represent the operators $E_{j,m}$ in $\mathfrak{A}$, and by equation 2.2 the matrix $\chi_{\alpha_m}(\omega)N(\omega)$ represents a nilpotent of order n, N_m, in $\mathfrak{A}$ where

$$AE(\alpha_m) = \sum_{j=1}^{i} E_{j,m}\int_{\alpha_m} z_j(\omega)E(d\omega) + N_m .$$

COROLLARY. *Let $A \in \mathfrak{A}$ be a generalized nilpotent (see [3] for definition) then*

$$A^n = 0 .$$

Proof. By equation 2.3 and Theorem 8 of [3]

$$AE(\alpha_m) = N_m .$$

Hence for every $x \in X$

$$E(\alpha_m)A^n x = 0$$

therefore

$$A^n x = 0 .$$

LEMMA 2.2. *Let $A \in \mathfrak{A}$. If $A \sim (a_{i,j}(\omega))$ and $z_k(\omega)$, $k = 1, 2, \cdots, n$ are the functions defined in equation 2.2 then*

$$|z_k(\omega)| \leqq |A| \text{ a.e.}$$

Proof. Let us assume, to the contrary, that for some i and $\varepsilon > 0$ the set

$$\gamma = \{\omega \,|\, |z_i(\omega)| \geqq |A| + \varepsilon\}$$

is not of measure zero. Let $\{\alpha_m\}$ be the sequence defined in Theorem 2.3, for some $mE(\gamma \cap \alpha_m) \neq 0$. Now, on $\gamma \cap \alpha_m$ $|z_i(\omega)| \geqq |A| + \varepsilon > 0$ hence $\varepsilon_i(\omega) \neq 0$. Thus $E(\gamma \cap \alpha_m)E_{i,\alpha_m} \neq 0$, where E_{i,α_m} is defined in Theorem 2.3. If the operator B is the restriction of A to $E(\gamma \cap \alpha_m)E_{i,\alpha_m}X$ then

$$B = \int_{\gamma \cap \alpha_m} z_i(\omega)E(d\omega) + M$$

where M is a nilpotent. Thus, if $|\mu| \leqq |A|$ then $|\mu| \leqq |z_1(\omega)| - \varepsilon$, $\omega \in \gamma \cap \alpha_m$, and $\mu \notin \sigma(B)$. Also, if $|\mu| > |A|$ then $|\mu| > |B|$ and $\mu \notin \sigma(B)$. This shows that $\sigma(B)$ is empty which is impossible.

THEOREM 2.4. *Let $(a_{i,j}(\omega))) \sim A \in \mathfrak{A}$. If the number $\lambda \notin \sigma(A)$ then for some $\varepsilon > 0$*

$$\operatorname{dist}(\lambda, \sigma(a_{i,j}(\omega))) \geqq \varepsilon \text{ a.e.}$$

Proof. Let $\lambda \in \rho(A)$. The matrix of $(\lambda I - A)^{-1}$ has the form

$$\sum_{i=1}^{n} \left(\frac{\varepsilon_i(\omega)}{\lambda - z_i(\omega)} - \frac{N(\omega)}{(\lambda - z_i(\omega))^2} + \cdots + \frac{(-N(\omega))^{n-1}}{(\lambda - z_i(\omega))^n} \right).$$

Thus by Lemma 2.2

$$\frac{1}{\operatorname{dist}(\lambda, \sigma(a_{i,j}(\omega)))} = \max_k \frac{1}{|\lambda - z_k(\omega)|} \leqq |(\lambda I - A)^{-1}| \text{ a.e}.$$

THEOREM 2.5. *Let $(a_{i,j}(\omega)) \sim A \in \mathfrak{A}$ and let $f(z)$ be regular in a neighborhood of $\sigma(A)$. Then the matrix $f((a_{i,j}(\omega)))$ exists a.e. and it is the matrix corresponding to $f(A)$.*

Proof. Let e be a Borel subset of Ω then

$$x_k^* E(e) f(A) x_j = x_k^* E(e) \frac{1}{2\pi i} \int_C f(\lambda) R(\lambda, A) x_j d\lambda$$

where C is a finite collection of Jordan curves surrounding $\sigma(A)$. Now $R(\lambda; A) \sim (r_{k,j}(\omega, \lambda)) = (\lambda\delta_{k,j} - a_{k,j}(\omega))^{-1}$ thus

$$x_k^* E(e) f(A) x_j = \frac{1}{2\pi i}\int_C f(\lambda)(x_k^* E(e) R(\lambda, A) x_j) d\lambda$$
$$= \frac{1}{2\pi i}\int_C f(\lambda)\left[\int_e r_{k,j}(\omega;\lambda)\mu_k(d\omega)\right] d\lambda$$

by equation 2.1. The functions $r_{k,j}(\omega, \lambda)$ can be computed by Cramer's rule. By Theorem 2.4 and the compactness of C there exists a positive constant δ such that if $\lambda \in C$ then

$$\operatorname{dist}(\lambda, \sigma(a_{k,j}(\omega))) \geqq \delta \text{ a.e.}$$

Now, if e is a Borel set on which the functions $a_{i,j}(\omega)$ are bounded, then the functions $r_{k,j}(\omega, \lambda)$ are measurable and bounded on $e \times C$. For such Borel sets e, we may use Fubini's theorem to conclude that

$$x_k^* E(e) f(A) x_j = \int_e \frac{1}{2\pi i}\left(\int_C f(\lambda) r_{k,j}(\omega, \lambda) d\lambda\right)\mu_k(d\omega) .$$

From this equation it follows that the components of the matrix of $f(A)$ are given by

$$(*) \qquad \frac{1}{2\pi i}\int_C f(\lambda) r_{k,j}(\omega, \lambda) d\lambda \text{ a.e.}$$

Now by the argument of Lemma 2.1 in [6] the matrix $f((a_{k,j}(\omega)))$ exists a.e. and its components are, thus, given by $(*)$.

THEOREM 2.6. *Let $A \in \mathfrak{A}$ be a compact operator. If $A \sim (a_{i,j}(\omega))$ and*

$$(a_{i,j}(\omega)) = \sum_{k=1}^{n} z_k(\omega)\varepsilon_k(\omega) + N(\omega)$$

is the decomposition given in Lemma 2.1, *then there exists a sequence* $\{\omega_v\}$, *of points in* ω, *such that:*

1. $E(\{\omega_v\}) \neq 0$
2. $z_k(\omega) = 0$ a.e. *for* $\omega \neq \omega_v$ $\qquad v = 1, 2, \cdots$
3. $\lim_{v\to\infty} z_k(\omega_v) = 0.$

Proof. Let β_i and α_m be the sets defined in Lemma 2.1 and Theorem 2.3. It is enough to prove the theorem for points in β_i, thus we assume that the matrix $(a_{j,k}(\omega))$ has exactly i eigenvalues. Define

$$e_{m,p} = \alpha_m \cap \left\{\omega \,\|\, z_k(\omega) \,| \geqq \frac{1}{p},\ k = 1, \cdots, i\right\} .$$

The operator A restricted to $E(e_{m,p})X$ is compact and, by Theorem 2.3. has a bounded inverse. Thus the space $E(e_{m,p})X$ has a finite dimension. Therefore there exists a finite set of points, $\omega_1^{m,p}, \cdots, \omega_j^{m,p}$, such that

$$E(\{\omega_k^{m,p}\}) \neq 0$$

and

$$E(e_{m,p} - \{\omega_1^{m,p}, \cdots, \omega_j^{m,p}\}) = 0 .$$

By letting $m, p \to \infty$ we get a sequence ω_v satisfying conditions 1 and 2. In order to prove 3, let us assume that for some $\varepsilon > 0$ there are infinitely many points, ω_v such that

$$|z_{k_v}(\omega_v)| \geqq \varepsilon .$$

The operator A is compact, hence $\sigma(A)$ has only zero as a limit point. By theorem 2.4 $z_{k_v}(\omega_v) \in \sigma(A)$. Thus for some constant $b \neq 0$

$$z_{k_v}(\omega_v) = b$$

for infinitely many points, ω_v. Let

$$G(b, A) = \frac{1}{2\pi i}\int_C R(\lambda; A)d\lambda$$

where C is a circle around b which does not contain any other point of $\sigma(A)$. The operator $G(b; A)$ is a compact projection. The matrix of $G(b; A)$ is, according to Theorem 2.5,

$$G(b; (a_{i,j}(\omega))) .$$

Thus

$$G(b; A)E(\{\omega_v\}) \neq 0$$

whenever $z_{k_v}(\omega_v) = b$, because the matrix of the product is not zero at ω_v. This contradicts the fact that $G(b; A)$ is a projection into a finite dimensional space, and thus condition 3 is proved.

EXAMPLES. The following two examples are designed to show that some of the theorems, proved in [6] for Hilbert spaces, are false for Banach spaces. Notice that the examples are simble because there exist projections on

$$sp\{E(\alpha)x_i, \ \alpha \text{ a Borel set}\} .$$

1. Let μ be the Lebesque measure on (0, 1). Let f be a monotone increasing function such that

$$f(0) = 1, \qquad f(1) = \infty, \qquad f \in L(0, 1) .$$

Define

$$\mu_1(e) = \int_e f(t)\mu(dt) .$$

The Banach space X will be $L_1(\mu) \oplus L_1(\mu_1)$. Each $x \in X$ has the form

$$x = \begin{pmatrix} g_1(\omega) \\ g_2(\omega) \end{pmatrix}$$

$$|x| = \int |g_1| d\mu + \int |g_2| f d\mu .$$

Let

$$E(e)x = \begin{pmatrix} \chi_e(\omega) g_1(\omega) \\ \chi_e(\omega) g_2(\omega) \end{pmatrix} .$$

It follows that the Boolean algebra is complete and has uniform multiplicity 2. Let

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_1^* \begin{pmatrix} g_1(\omega) \\ g_2(\omega) \end{pmatrix} = \int g_1 d\mu, \; x_2^* \begin{pmatrix} g_1(\omega) \\ g_2(\omega) \end{pmatrix} = \int g_2 d\mu .$$

If $A \sim \mathfrak{A}$ then $A \sim \begin{pmatrix} a_{1,1}(\omega), & a_{1,2}(\omega) \\ a_{2,1}(\omega), & a_{2,2}(\omega) \end{pmatrix}$ and

$$|A| \int |g_2| f d\mu \geqq \left| A \begin{pmatrix} 0 \\ g_1 \end{pmatrix} \right| = \left| \begin{pmatrix} a_{1,2}(\omega) g_2(\omega) \\ a_{2,2}(\omega) g_2(\omega) \end{pmatrix} \right| \geqq \int |a_{1,2} g_2| d\mu$$

for every $g_2 \in L_1(\mu_1)$. Thus

$$\int_e |a_{1,2}(\omega)| d\mu \leqq |A| \int_e f d\mu .$$

Hence $|a_{1,2}(\omega)| \leqq |A| f(\omega)$ a.e., or

$$a_{1,2}(\omega) = b_{1,2}(\omega) f(\omega) \text{ and } |b_{1,2}(\omega)| \leqq |A| .$$

Similarly

$$|A| \int |g_1| d\mu \geqq \left| A \begin{pmatrix} g_1(\omega) \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} a_{1,1}(\omega) g_1(\omega) \\ a_{2,1}(\omega) g_1(\omega) \end{pmatrix} \right| \geqq \int |a_{2,1} g_1| f d\mu$$

Hence

$$|a_{2,1}(\omega) f(\omega)| \leqq |A| \text{ a.e.}$$

or

$$a_{2,1}(\omega) = \frac{b_{2.1}(\omega)}{f(\omega)} \text{ and } |b_{2,1}(\omega)| \leqq |A| .$$

Every operator in $\mathfrak{A}$ is given, thus, by a matrix of the form:

$$\begin{pmatrix} b_{1,1}(\omega), & b_{1,2}(\omega) f(\omega) \\ \dfrac{b_{2,1}(\omega)}{f(\omega)}, & b_{2,2}(\omega) \end{pmatrix}$$

where the functions $b_{i,j}(\omega)$ are measurable and bounded. Also, every such matrix defines a bounded operator.

This example shows that Theorem 2.2 of [6] can not be generalized to Banach spaces:

The two topologies on $\mathfrak{A}$ given by the norms $|A|$ and

$$\max_{i,j} \operatorname{ess\,sup}_{\omega} |a_{i,j}(\omega)|$$

are not equivalent.

2. Let $X = C_0 \oplus l_1$. Every $x \in X$ has the form

$$x = (x_1, y_1, x_2, y_2. \cdots, x_n, y_n, \cdots)$$

where

$$\lim_{n\to\infty} x_n = 0, |x| = \max |x_i| + \sum_{i=1}^{\infty} |y_i| .$$

Define

$$E_n(x_1, y_1, \cdots, x_n, y_n, \cdots) = (0, \cdots, 0, x_n, y_n, 0 \cdots) .$$

The Boolean algebra, generated by E_n, has uniform multiplicity 2. Let the projection F be defined by

$$\begin{aligned} &F(x_1, y_1, \cdots, x_n, y_n, \cdots) \\ &\qquad = \frac{1}{2}(x_1 + y_1, x_1 + y_1, \cdots, x_n + y_n, x_n + y_n, \cdots) . \end{aligned}$$

The projection F is not bounded but $|FE_n| = 1$. Let the operator B be defined by

$$B(x_1, y_1, \cdots, x_n, y_n, \cdots) = \left(\frac{x_1}{2}, \frac{y_1}{2}, \cdots, \frac{x_n}{2^n}, \frac{y_n}{2^n}, \cdots\right)$$

and let $A = BF$. The operator A is bounded and compact, for if $|x| = |(x_1, y_1, \cdots, x_n, y_n, \cdots)| \leqq 1$ then

$$\begin{aligned} &\left|Ax - \frac{1}{2}\left(\frac{x_1 + y_1}{2}, \frac{x_1 + y_1}{2}, \cdots, \frac{x_n + y_n}{2^n}, \frac{x_n + y_n}{2^n}, 0 \cdots 0 \cdots\right)\right| \\ &\quad = \left|\frac{1}{2}\left(0, \cdots, 0, \frac{x_{n+1} + y_{n+1}}{2^{n+1}}, \frac{x_{n+1} + y_{n+1}}{2^{n+1}}, \cdots\right)\right| \\ &\quad \leqq \frac{1}{2^{n+1}}\left[\left(\frac{\sup |x_n| + \sup |y_n|}{2}\right)\right. \\ &\quad \left. + \sum_{i=1}^{\infty} \frac{1}{2^n} + \sum_{i=1}^{\infty} |y_i|\right] \leqq \frac{1}{2^{n+1}}(1 + 2 + 1) \to 0 . \end{aligned}$$

Thus A is the uniform limit of compact operators. Now, $\sigma(B) = \left\{\frac{1}{2^n}, n = 1, 2, \cdots\right\}$. If $0 \neq \lambda \in \sigma(A)$ then for some $x \in X$, $\lambda x = Ax$. Hence $x = Fx$ and $\lambda x = \lambda Fx = BFx = Bx$. Therefore

$$\sigma(A) = \left\{0, \frac{1}{2^n}, n = 1, 2, \cdots\right\} .$$

Let us compute $G\left(\frac{1}{2^n}, A\right)x$ for $x \in X$.

$$G\left(\frac{1}{2^n}; A\right)x = \sum_{k=1}^{\infty} G\left(\frac{1}{2^n}; A\right)E_k x .$$

Now on $E_k X$, $\sigma(A) = \left\{0, \frac{1}{2^k}\right\}$, hence

$$G\left(\frac{1}{2^n}; A\right)E_k x = 0 \text{ for } k \neq n$$

and

$$G\left(\frac{1}{2^n}; A\right)E_n x = FE_n x .$$

Therefore

$$G\left(\frac{1}{2^n}; A\right)x = FE_n x$$

and

$$\sum_{v=1}^{n} G\left(\frac{1}{2^v}; A\right)x = F(E_1 + \cdots + E_n)x .$$

The last equation shows that A is not spectral, and the preceding equation shows that Theorem 4.4 of [6] is false for Banach spaces:

There exists a compact operator A in $\mathfrak{A}$ that is not spectral though the projections $G(\xi; A)$ are uniformly bounded.

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