CHAINABLE CONTINUA AND INDECOMPOSABILITY

C. E. BURGESS

This paper includes a study of continua[1] which are both linearly chainable and circularly chainable. Since there exist indecomposable continua and 2 indecomposable continua which are linearly chainable, it follows from Theorem 7 that there exist indecomposable continua and decomposable continua which have both of these types of chainability.

A *linear chain* C is a finite collection of open sets $L_1, L_2, \cdots, L_n$ such that

(1) each element of C contains an open set that does not intersect any other element of C,

(2) $\rho(L_i, L_j) > 0$ if $|i-j| > 1$, and

(3) $L_i \cdot L_j \neq 0$ if $|i-j| \leq 1$. If this is modified so that $L_1 \cdot L_n \neq 0$, then C is called a *circular chain*. Each of the sets $L_1, L_2, \cdots, L_n$ is called a *link* of C, and C is sometimes denoted by $(L_1, L_2, \cdots, L_n)$ or $C(L_1, L_2, \cdots, L_n)$. If ε is a positive number and C is a linear chain such that each link of C has a diameter less than ε, then C is called a *linear* ε-chain. A *circular ε-chain* is defined similarly.

If C is either a linear chain or a circular chain and $H_1, H_2, \cdots, H_n$ are connected sets covered by C, then these sets are said to have the *order* $H_1, H_2, \cdots, H_n$ in C provided (1) no link of C intersects two of these n sets and (2) for each $i (i < n)$, there is a linear sub-chain in C which covers $H_i + H_{i+1}$ and which does not intersect any other of the sets $H_1, H_2, \cdots, H_n$.

A continuum M is said to be *linearly chainable*[2] if for every positive number ε, there is a linear ε-chain covering M. A continuum M is said to be *circularly chainable* if for every positive number ε, there is a circular ε-chain covering M.

A *tree* T is a finite coherent[3] collection of open sets such that

(1) each element of T contains an open set that does not intersect any other element of T,

(2) each two nonintersecting elements of T are a positive distance apart, and

(3) no subcollection of T consisting of more than two elements is a circular chain. If ε is a positive number and T is a tree such that

Presented to the American Mathematical Society, August 29, 1957; received by the editors December 5, 1958. This work was supported by the National Science Foundation under G-2574 and G-5880. Most of these results were obtained while the author was a visiting lecturer at the University of Wisconsin.

[1] Throughout this paper, a connected compact metric space is called a continuum.

[2] In some places in the literature, such continua have been said to be *chainable*.

[3] A collection G of sets is said to be *coherent* if for any two subcollections G_1 and G_2 of G such that $G_1 + G_2 = G$, some element of G_1 intersects some element of G_2.

each element of T has a diameter less than ε, then T is called an *ε-tree*. A continuum M is said to be *tree-like* if for every positive number ε, there is an ε-tree covering M.

A continuum M is said to be the *essential sum* of the elements of a collection G if the sum of the elements of G is M and no element of G is a subset of the sum of the other elements of G. If n is a positive integer and the continuum M is the essential sum of n continua and is not the essential sum of $n + 1$ continua, then M is said to be *n-indecomposable*.[4]

A continuum M is said to be *unicoherent* if the intersection of each two continua having M as their sum is a continuum. A continuum M is said to be *bicoherent* if for any two proper subcontinua M_1 and M_2 having M as their sum, the set $M_1 \cdot M_2$ is the sum of two continua that do not intersect.

A continuum M is said to be a *triod* if M is the essential sum of three continua such that their intersection is a continuum which is the intersection of each two of them.

THEOREM 1. *If the continuum M is either linearly chainable or circularly chainable, then M does not contain a triod.*[5]

Proof. Since it is easy to see that every proper subcontinuum of M is linearly chainable, it will be sufficient to show that M is not a triod.

Suppose that M is a triod. Let M_1, M_2, and M_3 be three continua having M as their essential sum such that their intersection is a continuum H which is the intersection of each two of them. For each i ($i \leq 3$), let p_i be a point of M_i that is not in either of the other two of the continua M_1, M_2, and M_3. Let ε be a positive number which is less than each of the numbers $\rho(p_1, M_2 + M_3)$, $\rho(p_2, M_1 + M_3)$, and $\rho(p_3, M_1 + M_2)$. Let C be either a linear ε-chain or a circular ε-chain which covers M. Since no link of C intersects two of the sets p_1, p_2, p_3, and H, consider the case in which these four sets are in C in the order named. This would involve the contradiction that M_2 intersects either the link of C that contains p_1 or the link of C that contains p_3. A similar contradiction results from supposing any other order of the sets p_1, p_2, p_3, and H in C.

THEOREM 2. *If the unicoherent continuum M is not a triod and M_1, M_2, M_3 are three continua having M as their essential sum, then*

[4] For any such continuum M, there is a unique collection consisting of n indecomposable continua having M as their essential sum [**4**].

[5] Bing [**2**] has used the fact that no linearly chainable continuum contains a triod, but for completeness a proof is given here for both types of chainability.

some two of these continua do not intersect and the other one intersects each of these two in a continuum.

Proof. Suppose that each two of the continua M_1, M_2, and M_3 intersect. It follows from the unicoherence of M that each of the sets $M_1 \cdot (M_2 + M_3)$ and $M_2 \cdot (M_1 + M_3)$ is a continuum and their sum is a continuum. Let $N = M_1 \cdot (M_2 + M_3) + M_2 \cdot (M_1 + M_3) = M_1 \cdot M_2 + M_1 \cdot M_3 + M_2 \cdot M_3$. Hence M is the essential sum of the three continua $M_1 + N$, $M_2 + N$, and $M_3 + N$ such that N is the intersection of each two of them and the intersection of all three of them. Since this is contrary to the hypothesis that M is not a triod, it follows that some two of the continua M_1, M_2, and M_3 do not intersect. Consider the case in which M_1 and M_3 do not intersect. Then M_2 intersects both M_1 and M_3, and since $M_1 \cdot M_2 = M_1 \cdot (M_2 + M_3)$ and $M_3 \cdot M_2 = M_3 \cdot (M_2 + M_1)$, it follows from the unicoherence of M that each of the sets $M_1 \cdot M_2$ and $M_3 \cdot M_2$ is a continuum.

THEOREM 3. *If the unicoherent continuum M is circularly chainable, then M is either indecomposable or 2-indecomposable.*

Proof. Suppose that M is the essential sum of three continua M_1, M_2, and M_3. By Theorem 1, M is not a triod. Hence by Theorem 2, one of these three continua, say M_2, intersects each of the other two such that $M_1 \cdot M_2$ and $M_2 \cdot M_3$ are continua and M_1 does not intersect M_3. For each i ($i \leq 3$), let p_i be a point of M_i which is not in either of the other two of the continua M_1, M_2, and M_3. Let ε be a positive number which is less than each of the numbers $\rho(p_1, M_2 + M_3)$, $\rho(p_2, M_1 + M_3)$, $\rho(p_3, M_1 + M_2)$, and $\rho(M_1, M_3)$. Let C be a circular ε-chain which covers M. A contradiction can be obtained as follows for each of the three types of order in C for the five sets p_1, p_2, p_3, $M_2 \cdot M_1$, and $M_2 \cdot M_3$.

Case 1. If these five sets have the order $p_i, p_j, p_k, M_2 \cdot M_1, M_2 \cdot M_3$ in C, then M_j would intersect a link of C that contains one of the points p_i and p_k, contrary to the choice of ε.

Case 2. If these five sets have the order $p_1, M_2 \cdot M_1, p_i, p_j, M_2 \cdot M_3$ in C, then M_2 would intersect a link of C that contains one of the points p_1 and p_3, contrary to the choice of ε.

Case 3. If these five sets have the order $p_2, M_2 \cdot M_1, p_i, p_j, M_2 \cdot M_3$ in C, then each link of one of the linear chains of C from p_1 to p_3 would lie in $M_1 + M_3$. This would involve the contradiction that some link of C intersects both M_1 and M_3.

THEOREM 4. *If the circularly chainable continuum M is separated* *by one of its subcontinua, then M is linearly chainable.*

Proof. Let K be a subcontinuum of M which separates M. Then M is the sum of two continua M_1 and M_2 such that K is their intersection. Let p_1 and p_2 be points of $M_1 - K$ and $M_2 - K$, respectively, let ε be a positive number less than each of the numbers $\rho(p_1, M_2)$ and $\rho(p_2, M_1)$, and let C be a circular ε-chain covering M. Then each link of one of the linear chains in C from p_1 to p_2 is a subset of $M - K$. Let $L_1, L_2, \cdots, L_n$ be the links of C such that L_1 contains p_1 and there is a positive integer r such that L_r contains p_2 and no link of the linear chain $(L_1, L_2, \cdots, L_r)$ intersects K. There exist integers i and j such that L_i is the first link of $(L_1, L_2, \cdots, L_r)$ which intersects M_2 and L_j is the last link of $(L_1, L_2, \cdots, L_r)$ which intersects M_1. Then $(M_2 \cdot L_i, M_2 \cdot L_{i+1}, \cdots, M_2 \cdot L_r, L_{r+1}, \cdots, L_n, M_1 \cdot L_1, M_1 \cdot L_2, \cdots, M_1 \cdot L_j)$ is a linear ε-chain covering M.

THEOREM 5. *Every circularly chainable continuum M is either unicoherent or bicoherent. Furthermore, M is unicoherent provided some subcontinuum of M separates M, and M is bicoherent provided no subcontinuum of M separates M.*

Proof. Suppose that M is the sum of two continua H and K such that $H \cdot K$ is the sum of three mutually separated sets Y_1, Y_2, and Y_3. There exist three open sets D_1, D_2, and D_3 containing Y_1, Y_2, and Y_3, respectively, such that the closures of D_1, D_2, and D_3 are disjoint. For each i $(i \leq 3)$, there exists a subcontinuum K_i of K irreducible from Y_i to $M - D_i$. The continuum $H + K_1 + K_2 + K_3$ is a triod, and this is contrary to Theorem 1. Hence it follows that if M_1 and M_2 are two continua having M as their sum, then the set $M_1 \cdot M_2$ is either a continuum or the sum of two continua.

It follows from Theorem 4 that M is linearly chainable, and hence unicoherent [3], provided some subcontinuum of M separates M. From this and the argument in the previous paragraph, it follows that M is bicoherent provided no subcontinuum of M separates M.

THEOREM 6. *If the circularly chainable continuum M is irreducible about some finite set consisting of n points, then there is a positive integer k not greater than n such that M is k-indecomposable.*

Proof. By Theorem 5, M is either unicoherent or bicoherent. If M is unicoherent, it follows from Theorem 3 that M is either indecomposable or 2-indecomposable. If M is bicoherent, it follows from Corollary 6.1 of [5] that there is a positive integer k not greater than n such that M is k-indecomposable.

THEOREM 7. *If the continuum M is linearly chainable, then in order that M should be circularly chainable, it is necessary and sufficient that M be either indecomposable of 2-indecomposable.*

Proof of necessity. Since every lineary chainable continuum is unicoherent [3], it follows from Theorem 3 that M is either indecomposable or 2-indecomposable.

Proof of sufficiency. The case where M is indecomposable and the case where M is 2-indecomposable will be considered separately.

Case 1. Suppose M is indecomposable, and let $C(L_1, L_2, \cdots, L_n)$ be a linear ε-chain covering M. There exist two disjoint continua K_1 and K_2 of M such that each of them intersects each of the sets $L_1 - cl(L_2)$ and $L_n - cl(L_{n-1})$. If follows that there exist a positive number ε', a linear ε'-chain C' covering M, and two subchains C_1 and C_2 of C' such that

(1) each link of C' is a subset of some link of C,

(2) C_1 and C_2 have no common link, and

(3) each of the chains C_1 and C_2 has one end link in $L_1 - cl(L_2)$ and the other end link in $L_n - cl(L_{n-1})$. Let W_1 denote the set of all points of M that are covered by C_1 and let W_2 denote $M - W_1$. Then $(L_1, W_1 \cdot L_2, W_1 \cdot L_3, \cdots, W_1 \cdot L_{n-1}, L_n, W_2 \cdot L_{n-1}, W_2 \cdot L_{n-2}, \cdots, W_2 \cdot L_2)$ is a circular ε-chain covering M.

Case 2. If M is 2-indecomposable, there exist two indecomposable continua M_1 and M_2 such that M is their essential sum and $M_1 \cdot M_2$ is a continuum. Let ε be a positive number. There exists a linear ε-chain C covering M such that M_1 intersects $L_1 - cl(L_2)$ and M_2 intersects $L_n - cl(L_{n-1})$. Since each composant of $M_i (i = 1, 2)$ is everywhere dense in M_i, it follows that for each i $(i = 1, 2)$ there exist two disjoint subcontinua K_i and H_i of M_i such that

(1) each of them intersects each link of C that intersects M_i,

(2) H_i contains $M_1 \cdot M_2$,

(3) each of the continua H_1 and K_1 intersects $L_1 - cl(L_2)$, and

(4) each of the continua H_2 and H_2 intersects $L_n - cl(L_{n-1})$. Hence there exist a positive number ε', a linear ε'-chain C' covering M, and three subchains C_1, C_2, and C_3 of C' such that

(1) each link of C' is a subset of a link of C,

(2) no two of the chains C_1, C_2, and C_3 have a common link,

(3) one end link of C_1 is in $L_1 - cl(L_2)$,

(4) one end link of C_2 is in $L_n - cl(L_{n-1})$,

(5) some link of C contains a link of C_1 and a link of C_2, and

(6) C_3 has one end link in $L_1 - cl(L_2)$ and the other end link in $L_n - cl(L_{n-1})$. Let W denote the set of all points of M that are covered by C_3, and let Y denote $M - W$. Then $(L_1, W \cdot L_2, W \cdot L_3, \cdots, W \cdot L_{n-1}, L_n, Y \cdot L_{n-1}, Y \cdot L_{n-2}, \cdots, Y \cdot L_2)$ is a circular ε-chain covering M.

THEOREM 8. *If n is a positive integer and for each proper subcontinuum H of the continuum M there is a positive integer r not greater than n such that H is r-indecomposable, then there is a positive integer k not greater than n such that M is k-indecomposable.*

Proof. Suppose that M is the essential sum of $n+1$ continua $M_1, M_2, \cdots, M_{n+1}$. Some n of these continua have a connected sum, so consider the case in which $M_2 + M_3 \cdots + M_{n+1}$ is connected. There is an open set D which intersects M_1 such that the closure of D does not intersect $M_2 + M_3 + \cdots + M_{n+1}$. There is a subcontinuum M_1' of M_1 irreducible from the closure of D to $M_2 + M_3 + \cdots + M_{n+1}$. This involves the contradiction that $M_1' + M_2 + M_3 + \cdots + M_{n+1}$ is a proper subcontinuum of M and is the essential sum of $n+1$ continua.

THEOREM 9. *If every proper subcontinuum of the continuum M is circularly chainable, then every subcontinuum of M is either indecomposable or 2-indecomposable.*

Proof. Since each proper subcontinuum of M is a proper subcontinuum of another proper subcontinuum of M, it follows that every proper subcontinuum of M is linearly chainable. Hence by Theorem 7, every proper subcontinuum of M is either indecomposable or 2-indecomposable. Consequently, it follows from Theorem 8 that M itself is either indecomposable or 2-indecomposable.

EXAMPLES. A pseudo-arc [1; 6] is an example of an indecomposable continuum which satisfies the hypothesis of Theorem 9, and a continuum which is the sum of two pseudo-arcs with a point as their intersection is an example of a 2-indecomposable continuum which satisfies this hypothesis.

THEOREM 10. *If the tree-like continuum M is circularly chainable, then M is linearly chainable.*

Pooof. Let ε be a positive number, and let $C(L_1, L_2, \cdots, L_n)$ be a circular $\varepsilon/3$-chain covering M. Then M is covered by a tree T such that

(1) each element of T is a subset of a link of C,
(2) some element K_0 of T intersects only one element of C, and

(3) no element of T intersects three elements of C. A function f will be defined as follows over T. For each element K of T, there is only one linear chain $(K_0, K_1, \cdots, K_m = K)$ from K_0 to K in T. Let $f(K_0) = 0$, and suppose that for some integer i $(0 \leq i \leq m), f(K_i)$ has been defined. Then define $f(K_{i+1})$ as follows:

(1) let $f(K_{i+1}) = f(K_i) + 1$ provided K_i lies in some element L_j of C and K_{i+1} intersects $L_{j+1, \mathrm{mod}\, n}$ but K_i does not intersect this set,

(2) Let $f(K_{i+1}) = f(K_i) - 1$ provided K_{i+1} lies in some element L_j of C and K_i intersects $L_{j+1, \mathrm{mod}\, n} - L_j$ but K_{i+1} does not intersect this set, and

(3) let $f(K_{i+1}) = f(K_i)$ provided neither (1) nor (2) is satisfied. The range of f is an increasing finite sequence of consecutive integers $n_1, n_2, \cdots, n_r$. For each t $(1 \leq t \leq r)$, let M_t denote the sum of all elements X of T such that $f(X) = n_t$. Then $(M_1, M_2, \cdots, M_r)$ is a linear ε-chain covering M.

REFERENCES

1. R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific, J. Math. **1** (1951), 43-51.

2. ———, *Snake-like continua*, Duke Math. J. **18** (1951), 653-663.

3. ———, and F. B. Jones, *Another homogeneous plane continuum* Trans. Amer. Math. Soc. **90** (1959), 171-192.

4. C. E. Burgess, *Continua which are the sum of a finite number of indecomposable continua*, Proc. Amer. Math. Soc. **4** (1953), 234-239.

5. ———, *Separation properties and n-indecomposable continua*, Duke Math. J. **24** (1956), 595-600.

6. E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. **63** (1948), 581-594.

UNIVERSITY OF UTAH