TRANSFORMATIONS ON TENSOR PRODUCT SPACES

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1. Introduction. Let U and V be m- and n-dimensional vector spaces over an algebraically closed field F of characteristic 0. Then $U \otimes V$, the tensor product of U and V, is the dual space of the space of all bilinear functionals mapping the cartesian product of U and V into F. If $x \in U$, $y \in V$ and w is a bilinear functional, then $x \otimes y$ is defined by: $x \otimes y(w) = w(x, y)$. If $e_1, \cdots, e_m$ and $f_1, \cdots, f_n$ are bases for U and V, respectively, then the $e_i \otimes f_j$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, form a basis for $U \otimes V$.

Let $M_{m,n}$ denote the vector space of $m \times n$ matrices over F. Then $U \otimes V$ is isomorphic to $M_{m,n}$ under the mapping ψ where $\psi(e_i \otimes f_j) = E_{ij}$, and E_{ij} is the matrix with 1 in the (i, j) position and 0 elsewhere. An element $z \in U \otimes V$ is said to be of rank k if $z = \sum_{i=1}^{k} x_i \otimes y_i$, where $x_1, \cdots, x_k$ are linearly independent and so are $y_1, \cdots, y_k$. If $R_k = \{z \in U \otimes V \mid \operatorname{rank}(z) = k\}$, then $\psi(R_k)$ is the set of matrices of rank k, in $M_{m,n}$. In view of the isomorphism any linear map T of $U \otimes V$ into itself can be considered as a linear map of $M_{m,n}$ into itself.

In [2] and [3], Hua and Jacob obtained the structure of any mapping T that preserves the rank of every matrix in $M_{m,n}$ and whose inverse exists and has this property (coherence invariance). (In [3] F is replaced by a division ring, and T is shown to be semi-linear by appealing to the fundamental theorem of projective geometry.) In [4] we obtained the structure of T when $m = n$, T is linear and T preserves rank 1, 2 and n. Specifically, there exist non-singular matrices M and N such that $T(A) = MAN$ for all $A \in M_{nn}$, or $T(A) = MA'N$ for all A, where A' designates the transpose of A. Frobenius (cf. [1], p. 249) obtained this result when T is a a linear map which preserves the determinant of every A. In [5] it was shown that this result can be obtained by requiring only that T be linear and preserve rank n. In the present paper we show that rank 1 suffices (Theorem 1), or rank 2 with the side condition that T maps no matrix of rank 4 or less into 0 (Theorem 2). Thus our hypothesis will be that T is linear and $T(R_1) \subseteq R_1$. We remark that T may be singular and still its kernel may have a zero intersection with R_1; e.g., take $U = V$ and $T(x \otimes y) = x \otimes y + y \otimes x$.

2. Rank one preservers. Throughout this section T will be a linear transformation (l.t.) of $U \otimes V$ into $U \otimes V$ such that $T(R_1) \subseteq R_1$. Here

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U and V are m- and n-dimensional vector spaces over F. Let $e_1, \cdots, e_m$ aud $f_1 \cdots, f_n$ be fixed bases for U and V, and set

$$T(e_i \otimes f_j) = u_{ij} \otimes v_{ij}\ , \qquad i = 1, \cdots, m;\ j = 1, \cdots, n. \tag{1}$$

Note that no u_{ij} or v_{ij} can be zero. We shall show, in case $m \neq n$ that there exist vectors u_i and v_j such that $T(e_i \otimes f_j) = u_i \otimes v_j$, and hence that the l.t. T is a tensor product of transformations on U and V separately. In case $m = n$ it will be shown that a slight modification of T is a tensor product.

Denote by $L(x_1, \cdots, x_t)$ the subspace spanned by the vectors $x_1, \cdots, x_t$, and let $\rho(x_1, \cdots, x_t)$ be the dimension of $L(x_1, \cdots, x_t)$.

LEMMA 1. *Let* $x_1, \cdots, x_r, w_1, \cdots, w_s$ *be vectors in* U, *and let* $y_1, \cdots, y_r, z_1, \cdots, z_s$ *be vectors in* V. *Let*

$$\sum_{i=1}^{r} (x_i \otimes y_i) = \sum_{j=1}^{s} (w_j \otimes z_j)\ . \tag{2}$$

If $\rho(x_1, \cdots, x_r) = r$, *then* $y_i \in L(z_1, \cdots, z_s)$, $i = 1, \cdots, r$; *and similarly if* $\rho(y_1, \cdots, y_r) = r$, *then* $x_i \in L(w_1, \cdots, w_s)$, $i = 1, \cdots, r$.

Proof. Suppose that $\rho(x_1, \cdots, x_r) = r$. Let θ be a linear functional on U such that $\theta(x_1) = 1$, $\theta(x_i) = 0$, $i \neq 1$, and let α be an arbitrary linear functional on V. For $x \in U$, $y \in V$, define

$$g(x, y) = \theta(x)\alpha(y)\ . \tag{3}$$

Applying (2) to g, we get

$$\alpha(y_1) = \sum_{i=1}^{s} \theta(w_j)\alpha(z_j) = \alpha\Big(\sum_{j=1}^{s} \theta(w_j)z_j\Big)$$

where each $\theta(w_j)$ is a scalar. Since α is arbitrary, y_1, and similarly $y_2, \cdots, y_r$, are contained in $L(z_1, \cdots, z_s)$. The second part of the lemma is proved in the same way.

LEMMA 2. *If* $T(R_1) \subseteq R_1$, *and* T *satisfies* (1), *then for* $i = 1, \cdots, m$, *either*

$$\rho(u_{i1}, \cdots, u_{in}) = n \quad \textit{and} \quad \rho(v_{i1}, \cdots, v_{in}) = 1\ , \tag{4}$$

or

$$\rho(u_{i1}, \cdots, u_{in}) = 1 \quad \textit{and} \quad \rho(v_{i1}, \cdots, v_{in}) = n\ . \tag{5}$$

Similarly, for $j = 1, \cdots, n$, *either*

$$\rho(u_{1j}, \cdots, u_{mj}) = m \quad \textit{and} \quad \rho(v_{1j}, \cdots, v_{mj}) = 1\ , \tag{6}$$

or

$$(u_{1j}, \cdots, u_{mj}) = 1 \quad \textit{and} \quad (v_{1j}, \cdots, v_{mj}) = m \,. \tag{7}$$

Proof. Suppose that $u_{i\alpha}$ and $u_{i\beta}$ are independent. Then

$$T(e_i \otimes (f_\alpha + f_\beta)) = (u_{i\alpha} \otimes v_{i\alpha}) + (u_{i\beta} \otimes v_{i\beta})$$

must be a tensor product $u \otimes v$. By Lemma 1, $v_{i\alpha}$, $v_{i\beta} \in L(v)$. Since all $v_{ij} \neq 0$, $L(v_{i\alpha}) = L(v_{i\beta})$. For $\gamma \neq \alpha, \beta$, $L(v_{i\gamma}) = L(v_{i\alpha})$, since $u_{i\gamma}$ must be independent of at least one of $u_{i\alpha}$, $u_{i\beta}$. We have shown that if $\rho(u_{i1}, \cdots, u_{in}) \geq 2$, then $\rho(v_{i1}, \cdots, v_{in}) = 1$.

Suppose next that $\rho(u_{i1}, \cdots, u_{in}) = 1$, viz., $u_{i\alpha} = c_\alpha u_{i1}$, $c_\alpha \neq 0$, $\alpha = 1, \cdots, n$. If

$$\rho(v_{i1}, \cdots, v_{in}) < n \,, \quad \text{let} \quad \sum_{\alpha=1}^{n} a_\alpha v_{i\alpha} = 0$$

be a non-trivial dependence relation. Then

$$T\Big(e_i \otimes \Big(\sum_{\alpha=1}^{n} \frac{a_\alpha}{c_\alpha} f_\alpha\Big)\Big) = \sum_{\alpha=1}^{n} \Big(c_\alpha u_{i1} \otimes \frac{a_\alpha}{c_\alpha} v_{i\alpha}\Big) = u_{i1} \otimes \Big(\sum_{\alpha=1}^{n} a_\alpha v_{i\alpha}\Big) = 0 \,,$$

which is impossible by the nature of T. Hence $\rho(u_{i1}, \cdots, u_{in}) = 1$ implies $\rho(v_{i1}, \cdots, v_{in}) = n$.

It follows by a similar argument that if $\rho(v_{i1}, \cdots, v_{in}) = 1$, then $\rho(u_{i1}, \cdots, u_{in}) = n$. Hence either (4) or (5) must hold. The second part of the lemma is proved similarly.

We remark that if $m < n$ (or $n < m$), then (4) (or (7)) cannot hold.

LEMMA 3. *Either* (4) *and* (7) *hold for all* i, j; *or* (5) *and* (6) *hold for all* i, j.

Proof. We show first that either (4) or (5) holds uniformly in i. Suppose that for some i and k, $1 \leq i \leq k \leq m$, $\rho(u_{i1}, \cdots, u_{in}) = n$ while $\rho(u_{k1}, \cdots, u_{kn}) = 1$. Then for some α, $1 \leq \alpha \leq n$, $\rho(u_{i\alpha}, u_{k\alpha}) = 2$. For $\beta \neq \alpha$ consider

$$\begin{aligned} \eta &= T[(e_i + e_k) \otimes (cf_\alpha + f_\beta)] \\ &= c(u_{i\alpha} \otimes v_{i\alpha}) + (u_{i\beta} \otimes v_{i\beta}) + c(u_{k\alpha} \otimes v_{k\alpha}) + (u_{k\beta} \otimes v_{k\beta}) \,, \end{aligned}$$

where c is an arbitrary scalar.

By hypothesis and Lemma 2, $v_{i\alpha} = av_{k\alpha}$ and $v_{i\beta} = b_1 v_{i\alpha} = bv_{k\alpha}$ for suitable non-zero scalars a and b, while $\rho(v_{k\alpha}, v_{k\beta}) = 2$. Thus $\eta = (acu_{i\alpha} + bu_{i\beta} + cu_{k\alpha}) \otimes v_{k\alpha} + (u_{k\beta} \otimes v_{k\beta})$, and by Lemma 1, $\rho(acu_{i\alpha} + bu_{i\beta} + cu_{k\alpha}, u_{k\beta}) = 1$ for all scalars c. Since $\rho(u_{k\alpha}, u_{k\beta}) = 1$, this implies that $\rho(cu_{i\alpha} + u_{i\beta}, u_{k\beta}) = 1$ for all c. This is impossible, since $\rho(u_{i\alpha}, u_{i\beta}) = 2$. Thus either (4) is true for all i, or (5) is true for all i. A similar argument applies to (6) and (7).

If (4) and (6) hold for all i and j, then there exist non-zero scalars c_{ij} such that $v_{ij} = c_{ij}v_{11}$, $i = 1, \cdots, m$, $j = 1. \cdots, n$. For a_j, b scalars, consider

$$T\left[\left(\sum_{i=1}^{m} a_i e_i\right) \otimes (f_1 - bf_2)\right] = \left(\sum_{i=1}^{m} a_i c_{i1} u_{i1} - b \sum_{i=1}^{m} a_i c_{i2} u_{i2}\right) \otimes v_{11} .$$

Let $z_1, \cdots, z_m$ and $w_1, \cdots, w_m$ be the m-column vectors which are respectively the representations of $u_{11}, \cdots, u_{m1}$ and $u_{12}, \cdots, u_{m2}$ with respect to the basis $e_1, \cdots, e_m$. Let C be the m-square matrix whose columns are $c_{11}z_1, \cdots, c_{m1}z_m$ and let W be the m-square matrix whose columns are $c_{12}w_1, \cdots, c_{m2}w_m$. Then with respect to the basis $e_1, \cdots, e_m$ the vector $\sum_{i=1}^{m} a_i c_{i1} u_{i1} - b \sum_{i=1}^{m} a_i c_{i2} u_{i2}$ has the representation $(C - bW)a$ where a is the column m-tuple $(a_1, \cdots, a_m)$. Now C and W are non-singular since $\rho(u_{11}, \cdots, u_{m1}) = \rho(u_{12}, \cdots, u_{m2}) = m$, so choose b to be an eigenvalue of $W^{-1}C$ and choose a to be the corresponding eigenvector. Then $(C - bW)a = 0$ and hence there exist scalars $a_1, \cdots, a_m$ not all 0 and b such that

$$T\left(\sum_{i=1}^{m} a_i e_i \otimes (f_1 - bf_2)\right) = 0 ,$$

a contradiction since $T(R_1) \subseteq R_1$.

Hence (4) and (6) cannot hold for all i and j. Similarly both (5) and (7) cannot hold for all i and j. This completes the proof of the lemma.

In view of the remark preceding this lemma, (5) and (6) must hold when $m \neq n$.

THEOREM 1. *Let U and V be m- and n-dimensional vector spaces respectively. Let T be a linear transformation on $U \otimes V$ which maps elements of rank one into elements of rank one. Let T_1 be the* l.t. *of $V \otimes U$ into $U \otimes V$ which maps $y \otimes x$ onto $x \otimes y$. If $m = n$, let φ be any non-singular* l.t. *of U onto V. Then if $m \neq n$, there exist non-singular* l.t.'s *A and B on U and V, respectively, such that $T = A \otimes B$. If $m = n$, there exist non-singular A and B such that either $T = A \otimes B$ or $T = T_1(\varphi A \otimes \varphi^{-1}B)$.*

Proof. By (1) and Lemma 3, $T(e_i \otimes f_j) = u_{ij} \otimes v_{ij}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, where either (5) and (6) hold or (4) and (7) hold. Suppose first that the former is the case; in particular, $\rho(u_{i1}, \cdots, u_{in}) = 1$ for $i = 1, \cdots, m$ and $\rho(v_{1j}, \cdots, v_{mj}) = 1$ for $j = 1, \cdots, n$. Then there exist non-zero scalars s_{ij}, t_{ij} such that $u_{ij} = s_{ij}u_{i1}$ and $v_{ij} = t_{ij}v_{1j}$. Thus

$$T(e_i \otimes f_j) = c_{ij} u_i \otimes v_j , \tag{8}$$

where $u_i = u_{i1}$, $v_j = v_{1j}$, and $c_{ij} = s_{ij}t_{ij}$. For $i = 2, \cdots, n$,

$$T\left[(e_1 + e_i) \otimes \left(\sum_{j=1}^{n} f_j\right)\right] = u_1 \otimes \sum_{j=1}^{n} c_{1j}v_j + u_i \otimes \sum_{j=1}^{n} c_{ij}v_j$$

must be a direct product $x \otimes w$. By (6) and Lemma 1, $\sum_{j=1}^{n} c_{ij}v_j = d_i \sum_{j=1}^{n} c_{1j}v_j$ for some constant d_i. By (5), $c_{ij} = d_i c_{1j}$. Hence

$$T(e_i \otimes f_j) = x_i \otimes y_j \,, \tag{9}$$

where $x_i = d_i u_i$ and $y_j = c_{1j}v_j$. Since the $\{x_i\}$ and $\{y_j\}$ are each linearly independent sets, there non-singular linear transformations A and B such that $x_i = Ae_i$ and $y_j = Bf_j$. Then $T = A \otimes B$.

When $m = n$, (4) and (7) may hold; in particular,

$$\rho(v_{i1}, \cdots, v_{in}) = 1 \text{ and } \rho(u_{1j}, \cdots, u_{nj}) = 1 \quad \text{for} \quad i, j = 1, \cdots, n.$$

As in the preceding case, there exist linearly independent sets $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ such that

$$T(e_i \otimes f_j) = x_j \otimes y_i \,. \tag{10}$$

There exist non-singular transformations A and B of U and V, respectively, such that $Ae_i = \varphi^{-1}y_i$ and $Bf_j = \varphi x_j$, $i, j = 1, \cdots, n$. Thus $T_1^{-1}T(e_i \otimes f_j) = \varphi Ae_i \otimes \varphi^{-1}Bf_j$. Q.E.D.

In matrix language we have the following.

COROLLARY. *Let T be a* l.t. *on the space M_{nn} of n-square matrices. If the set of rank one matrices is invariant under T, then there exist non-singular matrices A and B such that either $T(X) = AXB$ for all $X \in M_{nn}$ or $T(X) = AX'B$ for all $X \in M_{nn}$.*

3. **Rank two preservers.** In this section T will be a l.t. of $U \otimes V$ such that $T(R_2) \subseteq R_2$. We shall show that under certain conditions $T(R_1) \subseteq R_1$.

LEMMA 4. *If W is a subspace of $U \otimes V$ such that, for some integer r, $1 \leq r \leq \min(m, n)$,*

$$\dim W \geq mn - r \max(m, n) + 1 \,, \tag{11}$$

then $W \cap \bigcup_{j=1}^{r} R_j \neq \phi$.

Proof. Suppose that $m = \max(m, n)$. The products $e_i \otimes f_j$, $i = 1, \cdots, m$, $j = 1, \cdots, r$, are linearly independent and span a space W_1 of dimension mr. Furthermore, $W_1 \subseteq \bigcup_{j=1}^{r} R_j$. Then $\dim(W_1 \cap W) = \dim W_1 + \dim W - \dim(W_1 \cup W) \geq mr + (mn - rm + 1) - mn = 1$. The result follows, since $W_1 \cap W \subseteq \bigcup_{j=1}^{r} R_j \cap W$.

LEMMA 5. *If $T(R_2) \subseteq T(R_2) \subseteq R_2$, then $T(R_1) \subseteq R_1 \cup R_2$.*

Proof. Suppose $x_1 \otimes y_1 \in R_1$, and choose $x_2 \otimes y_2 \in R_1$ such that $\rho(x_1, x_2) = \rho(y_1, y_2) = 2$. Then $\alpha = sT(x_1 \otimes y_1) + tT(x_2 \otimes y_2) \in R_2$ for all non-zero scalars s, t. Now suppose that $T(x_1 \otimes y_1) = \sum_{j=1}^{p} u_j \otimes v_j$, where $\rho(u_1, \cdots, u_p) = \rho(v_1, \cdots, v_p) = p$, and that $T(x_2 \otimes y_2) = \sum_{j=1}^{q} z_j \otimes w_j$, where $\rho(z_1, \cdots, z_q) = \rho(w_1, \cdots, w_q) = q$. Let $u_{p+1}, \cdots, u_m$ be a completion of $u_1, \cdots, u_p$ to a basis for U. It follows that

$$\sum_{j=1}^{q} z_j \otimes w_j = \sum_{j=1}^{m} u_j \otimes h_j$$

for some vectors $h_j \in V$, $j = 1, \cdots, m$. Then

$$\begin{aligned}\alpha &= \sum_{j=1}^{p} u_j \otimes sv_j + \sum_{j=1}^{p} u_j \otimes th_j + \sum_{j=p+1}^{m} u_j \otimes th_j \\ &= \sum_{j=1}^{p} u_j \otimes (sv_j + th_j) + \sum_{j=p+1}^{m} u_j \otimes th_j .\end{aligned}$$

Since $\alpha \in R_2$, it follows by Lemma 1 that

$$\rho(sv_1 + th_1, \cdots, sv_p + th_p) \leq 2 \quad \text{for} \quad st \neq 0 .$$

The vectors $sv_1 + th_1, \cdots, sv_p + th_p$ are linearly independent when $s = 1$ and $t = 0$. By continuity, they remain independent for small values of t. Hence $p \leq 2$ and $T(x_1 \otimes y_1) \in R_1 \cup R_2$.

THEOREM 2. *If $T(R_2) \subseteq R_2$ and $0 \notin T(\bigcup_{j=1}^{4} R_j)$, then $T(R_1) \subseteq R_1$.*

Proof. Suppose $x_1 \otimes y_1 \in R_1$ and $T(x_1 \otimes y_1) \notin R_1$. By Lemma 5, $T(x_1 \otimes y_1) \in R_2$, since $0 \notin T(R_1)$. Thus $T(x_1 \otimes y_1) = (u_1 \otimes v_1) + (u_2 \otimes v_2)$, where $\rho(u_1, u_2) = \rho(v_1, v_2) = 2$. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be bases for U and V respectively. Then for $st \neq 0$

$$sT(x_1 \otimes y_1) + tT(x_i \otimes y_j) \in R_1 \cup R_2 \quad \text{for } i = 1, \cdots, m,\ j = 1, \cdots, n. \tag{12}$$

At this point it seems simpler to regard the images $T(x_i \otimes y_j)$ as elements of M_{mn}. It is clear that there is no loss in generality in taking $T(x_1 \otimes y_1) = E_{11} + E_{22}$.

Let i and j be fixed for this discussion, and let $A = T(x_i \otimes y_j)$. Let $a_1, \cdots, a_n$ be the m-dimensional vectors which are the columns of A, and let ε_k be the unit vector with 1 in the kth position. It follows from (12) that

$$\rho(s\varepsilon_1 + ta_1, s\varepsilon_2 + ta_2, ta_3, \cdots, ta_n) = 2 \tag{13}$$

for $st \neq 0$. The Grassmann products

$$(s\varepsilon_1 + ta_1) \wedge (s\varepsilon_2 + ta_2) \wedge ta_k , \quad 3 \leq k \leq n \tag{14}$$

must be zero for $st \neq 0$. In the expansion of (14) the coefficient of s^2t is 0; that is, $\varepsilon_1 \wedge \varepsilon_2 \wedge a_k = 0$.

Thus the matrix A has non-zero entries only in the first two rows and columns. It follows immediately that the dimension of the range of $T \leq 2(m+n) - 4$. Hence the dimension of the kernel of $T \geq mn - 2(m+n) + 4 > mn - 4\max(m, n) + 1$.

By Lemma 4, there exists an element of $\bigcup_{j=1}^{4}$ whose image is zero. This contradicts the hypothesis; hence $T(R_1) \subseteq R_1$.

We see then that the form of T satisfying Theorem 2 is given in the conclusions of Theorem 1.

REMARK. We feel that the hypothesis $0 \notin T(\bigcup_{j=1}^{4} R_j)$ of Theorem 2 should not be necessary, but we have not been able to prove the theorem without it. More generally, we conjecture that $T(R_k) \subseteq R_k$ for some fixed k, $1 \leq k \leq n$, should suffice to prove that T is essentially a tensor product.

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