THE H_p-PROBLEM AND THE STRUCTURE OF H_p-GROUPS

D. R. HUGHES AND J. G. THOMPSON

1. Introduction. Let G be a group, p a prime, and $H_p(G)$ the subgroup of G generated by the elements of G which do not have order p. In a research problem in the Bulletin of the American Mathematical Society, one of the authors posed the following problem: is it always true that $H_p(G) = 1$, $H_p(G) = G$, or $[G : H_p(G)] = p$? This problem is easily settled in the affirmative for $p = 2$, and a similar answer was recently given for $p = 3$ ([5]). In this paper (Section 2) we give an affirmative answer for the case that G is finite and not a p-group. Furthermore (Section 3) we are able to give a rather precise description of the structure of G in the most interesting case, when $[G : H_p(G)] = p$. This structure theorem depends heavily on the deep results of Hall and Higman ([4]) and Thompson ([6]) on finite groups. If $H\ (\neq 1)$ is a finite group and there exists a group G such that $H_p(G)$ is isomorphic to H, where $H_p(G) \neq G$, then we call H an H_p-group; it is seen that H_p-groups are natural generalizations of "Frobenius groups." By a Frobenius group we mean a finite group G possessing an automorphism σ of prime order p such that $x^\sigma = x$ if and only if $x = 1$. It is easy to show that this implies

$$x^{1+\sigma+\cdots+\sigma^{p-1}} = x(x^\sigma)\cdots(x^{\sigma^{p-1}}) = 1 ,$$

for all x in G. This last equation characterizes H_p-groups,[1] and as a generalization of Thompson's result ([6]) that Frobenius groups are nilpotent, we show that H_p-groups are solvable, among other things.

Throughout the paper, if B is a group, A a subgroup of B, then $N_B(A)$ and $C_B(A)$ mean, respectively, the normalizer and centralizer of A in B. By $Z(A)$ we mean the center of A.

2. The H_p-problem. Let G be a group, and let $H = H_p(G)$. Suppose

(1) G is finite,

(2) G is not a p-group,

(3) the index of H in G is greater than p,

(4) G is a group of minimal order satisfying (1), (2), (3). Note that every element of G which is not in H has order p.

Let q be a prime dividing $[G : 1]$, $q \neq p$, and let Q be a Sylow q-group of G; then Q is also a Sylow q-group of H. Let $N = N_G(Q)$; then

Received January 16, 1959. The first author was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under contract No. AF 18 (600)-1383.

[1] Unless the group is a p-group; see Theorem 2.

by the Frattini argument (see [1], p. 117, for instance), $G = NH$. Thus $[G:1] = [NH:1] = [N:1][H:1]/[N \cap H:1]$.

First, let us suppose $N \neq G$. Then clearly $H_p(N) \subseteq H_p(G)$, so $H_p(N) \subseteq H \cap N$. Since $Q \subseteq H_p(N)$, it follows that $H_p(N) \neq 1$, so $[N: H_p(N)] \leq p$, and hence $[N: N \cap H] \leq p$. So $p^2 = [G:H] = [G:1]/[H:1] = [N:1]/[N \cap H:1] = [N:N \cap H] \leq p$. This is impossible, so we must have $N = G$, and thus Q is normal in G.

Now let Q_1 $(\neq 1)$ be any subgroup of Q, normal in G, and consider G/Q_1. Clearly $H_p(G/Q_1) = 1$ or $H_p(G/Q_1)$ has index p in G/Q_1, unless G/Q_1 is a p-group. Indeed, it is obvious that $H_p(G/Q_1) \subseteq H/Q_1$. But $[G/Q_1 : H/Q_1] = [G:H] = p^2$, so $[G/Q_1 : H_p(G/Q_1)] \geq [G/Q : H/Q_1] = p_2$ implies $H_p(G/Q_1) = 1$. So G/Q_1 is a p-group.

LEMMA 1. *If $[G:H] = p^2$, then Q is an elementary abelian q-group, none of whose proper subgroups $(\neq 1)$ is normal in G, Q is normal in G, and $G = PQ$, where P is a Sylow p-group of G.*

Proof. We have shown that Q is normal. If Q_1 above is taken to be the Frattini subgroup of Q, then Q_1 is normal in G, since it is characteristic in Q. Since $Q_1 \neq Q$, G/Q_1 cannot be a p-group, so we must have $Q_1 = 1$. Thus Q is elementary abelian. Since G/Q is a p-group, it is clear that $G = PQ$, and the rest of the lemma follows similarly.

In what follows, P is a Sylow p-group of G and $P_0 \subseteq P$ is a Sylow p-group of H; clearly $[P:P_0] = p^2$ and P_0 is normal in P, since $P_0 = P \cap H$.

If x $(\neq 1)$ is in Q, while a is in G, not in H, and if $ax = xa$, then ax has order pq. But ax is not in H, since a is not in H, and thus ax has order p; hence $ax \neq xa$. If $P_0 = 1$, then P, of order p^2, is an automorphism group of $H = Q$ such that no non-identity element of P fixes any non-identity element of Q. But by ([2], pp. 334–335) this means that P is cyclic, whereas P is clearly elementary abelian in this case (for all its elements have order p). So $P_0 \neq 1$.

Since P_0 is normal in P, $P_0 \cap Z(P) \neq 1$ (see [3], p. 35, for instance). Let z be an element of $P_0 \cap Z(P)$, chosen to have order p, and let Z_0 be the subgroup (of order p) generated by z; note that z and Z_0 are contained in H. Let $K = Z_0Q$, and observe that $[K:1] = p[Q:1]$. Let a be an element of G, not in H, and $G_1 = \{a, K\}$ = the group generated by a and K. Then $Q \subseteq H_p(G_1) \subseteq H \cap G_1 \neq G_1$, so $[G_1 : H_p(G_1)] = p$, by induction. Hence $Z_0 \subseteq K \subseteq H_p(G_1)$, so there must be an element y in K of order pq. Then y^p is in Q and y^q is in $x^{-1}Z_0x$, for some x in K, since Z_0 is a Sylow p-group of K. By adjusting our choice of P, we can assume that y^q is in Z_0; let $u = y^p$, $v = y^q$. Then $u \neq 1$, $v \neq 1$, u is in Q, v is in Z_0, and $uv = vu$. So if $Q_1 = \{u\}$, we have $Z_0 \subseteq C_G(Q_1)$. But then $x^{-1}Z_0x \subseteq C_G(x^{-1}Q_1x)$, and if x is in P, this implies $Z_0 \subseteq C_G(x^{-1}Q_1x)$, for all x in P. But, from Lemma 1, the subgroup generated by all

$x^{-1}Q_1x$, as x ranges over P, must be Q, and so $Z_0 \subseteq C_G(Q)$. Since Z_0 is in the center of P, it follows that Z_0 is normal in G, so we consider G/Z_0. One easily sees that $H_p(G/Z_0) \subseteq H/Z_0$, and $H_p(G/Z_0)$ equals neither 1 nor G/Z_0. Hence $p^2 = [G:H] = [G/Z_0 : H/Z_0] \leq [G/Z_0 : H_p(G/Z_0)] = p$, which is a contradiction. So:

THEOREM 1. *If $H_p(G) \neq 1$ or G, and if G is finite and not a p-group, then $[G : H_p(G)] = p$.*

If G is a p-group, or is infinite, the situation seems more inaccessible; as remarked earlier, Theorem 1 still holds if $p = 2$ or 3, no matter what G is. But the proof for $p = 3$ (see [5]) utilizes the Burnside theorem (for $p = 3$) and this strongly suggests that the infinite case at least is considerably harder.

3. Structure of H_p-groups. Let us suppose that G is a finite group, and that $H = H_p(G)$ has index p in G. Then we say that H is an H_p-group.

THEOREM 2. *If H is not a p-group, then H is an H_p-group if and only if H has an automorphism σ of order p such that*

$$x^{1+\sigma+\cdots+\sigma^{p-1}} = 1 ,$$

for all x in H.

Proof. If $H = H_p(G)$, let a be in G, a not in H, and define $x^\sigma = a^{-1}xa$, for x in H. Since $(ax)^p = 1$, while $(ax)^p = a^p(x)(x^\sigma) \cdots (x^{\sigma^{p-1}})$, the equation of the theorem follows immediately.

Conversely, if σ exists satisfying the hypotheses of the theorem, then let G be the holomorph of H by the automorphism group $\{\sigma\}$. It is easy to see that $H_p(G) \subseteq H$. Since $H_p(G) \neq 1$ (for H is not a p-group), it follows that $[G : H_p(G)] = p$, from Theorem 1, so $H_p(G) = H$.

Note that if $x^\sigma = x$, then the equation of Theorem 2 implies $x^p = 1$. So if p does not divide the order of the H_p-group H, then H is even a Frobenius group, and so is nilpotent ([6]).

THEOREM 3. *If H is an H_p-group, then $H = PK$, where P is a Sylow p-group of H, K is normal in H and is nilpotent, and $P \cap K = 1$. In particular, H is solvable.*

Proof. We can assume that $P \neq 1$, and that H is not a p-group. Inductively, suppose the theorem is true for all H_p-groups whose order is less than the order of H, and (using Theorem 2) let γ be an automorphism of H, of order p, such that

$$x^{1+\gamma+\cdots+\gamma^{p-1}} = 1 , \quad \text{all } x \text{ in } H.$$

If A is a γ-invariant subgroup of H, then A is an H_p-group or is a p-group, while if B is a γ-invariant normal subgroup of H, then H/B is an H_p-group or is a p-group.

Now let B be any γ-invariant subgroup of P, B normal in P, $B \neq 1$; let $N = N_H(B)$. If $N = H$, then H/B is an H_p-group, so $H/B = (P/B)(K_1/B)$, where K_1/B is normal in H/B and is nilpotent. So K_1 is normal in H and since K_1/B is γ-invariant in H/B, so is K_1 γ-invariant in H. So K_1 is an H_p-group. If $K_1 \neq H$, then $K_1 = BK$, where K is normal in K_1 and is nilpotent, and $K \cap B = 1$. But then K is characteristic in K_1, hence is normal in H; every Sylow q-group of H, $q \neq p$, is in K. So K is characteristic in H and clearly $H = PK$, $P \cap K = 1$.

If $K_1 = H$ for every such B, then $B = P$ is the only γ-invariant normal subgroup of P, other than 1. Hence in particular P is elementary abelian. Then H/P is an H_p-group, and even a Frobenius group, so is nilpotent. Furthermore (since H is then solvable), $H = PK$, where K is isomorphic to H/P. Let $K = Q_1Q_2 \cdots Q_t$, where Q_i is a Sylow q_i-group of K (and of H) for distinct primes $q_1, q_2, \cdots, q_t$.

Now let G be the holomorph of H with the group $\{\gamma\}$. Then, by the Frattini argument, $N_G(Q_i) \cap H \neq N_G(Q_i)$, so by an appropriate choice of γ_i in G, γ_i not in H, we can assume that Q_i is γ_i-invariant. Thus PQ_i is γ_i-invariant and so it is an H_p-group (it is straightforward to check that any element of G, not in H, can play the role of γ).[2]

If $t > 1$, then PQ_i has order smaller than H, so Q_i is normal in PQ_i. Thus both P and K are contained in $N_H(Q_i)$, so Q_i is normal in H, hence K, which is the direct product of the Q_i, is normal in H, so we are done.

If $t = 1$, let $Q = Q_1$, and as above, choose γ in G, not in H, so that Q is γ-invariant. If $Q_0 \neq 1$ is a γ-invariant normal subgroup of Q, then PQ_0 is an H_p-group, smaller than $H = PQ$ if $Q_0 \neq Q$; thus P normalizes Q_0, so Q_0 is normal in H. Then by considering H/Q_0, we find that Q/Q_0 is normal, so Q is normal in H, and again we are done. Thus we can assume that Q is elementary abelian with only trivial γ-invariant normal subgroups.

Now we consider the holomorph G again. The maximal normal p-group of G is P, since $\{\gamma\}$ (as part of G) is not normalized modulo P by Q. Then G/P is a solvable (and in particular, p-solvable) group of automorphisms of the elementary abelian group P, and G/P has no normal p-group ($\neq 1$). Furthermore, this representation of G/P as a linear transformation group on P is faithful, since $C_H(P) \cap Q = 1$ (otherwise $C_H(P) \cap Q$ would be a non-trivial γ-invariant normal subgroup of Q). Thus we can utilize Theorem B of Hall and Higman ([4]); since Q is abelian, Theorem B asserts that γ, as a linear transformation of P, has the minimal

[2] In these references to the holomorph G, we are not making a distinction between an element as an automorphism of H and as an element of G; the automorphism is actually identified with an element of G which induces the prescribed automorphism in H.

polynomial $(x-1)^p$. But in fact, γ has a minimal polynomial which divides $1+x+\cdots+x^{p-1}$, since

$$b^{1+\gamma+\cdots+\gamma^{p-1}}=1\ ,$$

for all b in P. Thus we have a contradiction, and so Q is normal in H, and we are done.

Now we must consider the case that if $B\ (\neq 1)$ is any γ-invariant subgroup of P, normal in P, then $N=N_H(B)$ is never equal to H. Hence N, being γ-invariant, is an H_p-group or is a p-group, so $N=P_1K_1$, where P_1 is a Sylow p-group of N, K_1 is normal in N and is nilpotent, and $K_1\cap P_1=1$. Since B is normal in N, K_1 is contained in $C_N(B)$, and thus contained in $C_H(B)$, so $N_H(B)/C_H(B)$ is a p-group (i.e., is isomorphic to P_1/P_0, for some subgroup P_0 of P_1). But then, since this holds for all such B, Thompson's theorem ([**6**]) asserts that P has a normal complement K in H; i.e., $H=PK$, where $P\cap K=1$ and K is normal in H. Since K consists exactly of the elements of H whose order is prime to p, K is characteristic. Thus K is an H_p-group (even a Frobenius group) and is nilpotent, so we are done.

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