A CLASS OF HYPER-FC-GROUPS

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1. Introduction. An element g of an arbitrary group G is called an FC element if it has a finite number of conjugates in G. The set of all FC elements of G forms a characteristic subgroup H of G (see Baer [1]). The *upper FC-series* of G, introduced by Haimo [4] as the *FC-chain,* may be defined by

$$H_0 = \{1\} ,$$
$$H_{i+1}/H_i = H(G/H_i) ,$$

the subgroup of all FC elements of G/H_i. The upper FC-series is continued transfinitely in the usual way, by defining

$$H_\alpha = \bigcup_{\beta<\alpha} H_\beta ,$$

when α is a limit ordinal. If $H_\gamma = G$, but $H_\delta \neq G$, for all $\delta < \gamma$, we say that the group G is *hyper-FC of FC-class* γ, following McLain [7].

A group G in which the transfinite upper central series

$$\{1\} = Z_0 \leq Z_1 \leq \cdots \leq Z_\alpha \leq \cdots$$

reaches the whole group is called a ZA-group (Kurosh [6]), and we may say that G has class α if $Z_\alpha = G$, but $Z_\beta \neq G$, for all $\beta < \alpha$. Glushkov [3] and McLain [7] have given constructions for a ZA-group of any given class. The main object of this note is to construct groups of given FC-class.

2. Constructions and proofs.

Definition. We say that a group G is of type Q_α if

1. G has FC-class α, with upper FC-series
$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_\alpha = G$,
2. $H_{\gamma+1}/H_\gamma$ is infinite, for all $\gamma < \alpha$, and
3. $H_{\gamma+1}/H_\gamma$ has the unit subgroup for its centre, for all $\gamma < \alpha$.

Thus the group with only one element is of type Q_0, and, in constructing a group G of type Q_α, we may assume the existence of a group G_β of type Q_β, for each $\beta < \alpha$. If α is a limit ordinal, we define G to be the ordinary (restricted) direct product of the groups G_β, for all $\beta < \alpha$. Then G has the properties $1-3$, and thus has type Q_α. For the case $\alpha = \beta + 1$ we shall construct G by 'wreathing' the regular

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representation of G_β with a certain kind of infinite centreless FC-group of permutations of the positive integers. (For convenience, we say that a group is *centreless* if its centre consists of the unit element alone.)

DEFINITION. A faithful representation of a group G by permutations of the positive integers will be called a *special* representation of G if

(i) the stabiliser of each integer has finite index in G and
(ii) the intersection of the stabilisers of the elements of any set of all but a finite number of these integers is the unit subgroup.

DEFINITION. An infinite centreless FC-group possessing a special representation will be called a group of type F.

To construct an example of a group of type F, let $D = B_1 \times B_2 \times \cdots$ be the ordinary direct product of an infinite sequence of finite centreless groups B_i, $i = 1,2, \cdots$. Let $D_n = B_{n+1} \times B_{n+2} \times \cdots$, let k_n be the order of D/D_n and let the elements of D/D_n, in an arbitrary order, be

$$X_1^n, X_2^n, \cdots, X_{k_n}^n.$$

For each element $g \in D$ and each $n = 1, 2, \cdots$, define the permutation π_{gn} of the integers $1, 2, \cdots, k_n$ by the rule

$$\pi_{gn}(i) = j \text{ when } gX_i^n = X_j^n. \tag{1}$$

Now, for each $g \in G$, define the permutation π_g of the positive integers by the rule

$$\pi_g\Big(i + \sum_{j=1}^{n-1} k_j\Big) = \pi_{gn}(i) + \sum_{j=1}^{n-1} k_j, \tag{2}$$

for all $i = 1, 2, \cdots, k_n$, and $n = 1, 2, \cdots$. The systems of transitivity in this permutation representation of D are the sets T_n of integers m such that $\sum_{i=1}^{n-1} k_i < m \le \sum_{i=1}^{n} k_i$, for $n = 1, 2, \cdots$. If $m \in T_n$, then the subgroup D_n of D is contained in the stabiliser of m. Hence the stabiliser in D of each positive integer has finite index in D. On the other hand, suppose g is in the stabiliser in D of all but a finite number of the positive integers. Then there is a number n_0 such that g is in the stabiliser of each integer of each system T_n with $n \ge n_0$. So if i is any integer in the range $1 \le i \le k_n$, $n \ge n_0$, we know that g is in the stabiliser of $i + \sum_{j=1}^{n-1} k_j$, and this means that $gX_i^n = X_i^n$. Thus $g \in D_n$. But the subgroups D_n, with $n \ge n_0$, intersect in the unit subgroup of D. So $g = 1$. We observe also that the permutation representation of D defined by (1) and (2) is faithful. Thus we have a special representation of the infinite centreless FC-group D, which is therefore a group of type F.

LEMMA. *If G_β is a group of type Q_β and J is a group of type F,* *then a group G formed by wreathing the regular representation of G_β with a special representation R of J is a group of type $Q_{\beta+1}$.*

Proof. The wreath group G may be regarded as a semi-direct product

$$G = KE\,, \quad K \cap E = 1\,,$$

where $K = \prod_{i=1}^{\infty} A_i$ is the direct product of a sequence of groups, each isomorphic to G_β, and E is isomorphic to J. The automorphisms of K induced by elements of E permute the subgroups A_i, $i = 1, 2, \cdots$, realizing the special representation R of $J \simeq E$. Associated with G is a set of isomorphisms θ_{ij}, $i, j = 1, 2, \cdots$ such that $\theta_{ij}(A_i) = A_j$, and if $a \in A_i$, $g \in E$ and $g^{-1}A_i g = A_j$, then $g^{-1}ag = \theta_{ij}(a)$. θ_{ii} is the identity automorphism, for all i. (A brief general description of wreath groups, and further references, may be found in Hall [5].)

Let C_i be the set of all elements g in E such that $g^{-1}A_i g = A_i$. Then C_i is the centraliser in E of each element of A_i. Since the representation R is special, the subgroup C_i of E has finite index in E, for each i, and the unit element is the only element of E common to all the subgroups of any set of all but a finite number of the C's.

For all $\gamma \leq \beta$, put $H_\gamma = H_\gamma(K)$, the γth term of the upper FC-series of K. If possible, let $\tau + 1$ be the least such ordinal for which $H_{\tau+1}(G) \neq H_{\tau+1}$. Now any element k of K can be written as the product of a finite number of elements $a_{i_\nu} \in A_{i_\nu}$, $\nu = 1, 2, \cdots, n$, and the subgroup $C(k) = \bigcap_{\nu=1}^{n} C_{i_\nu}$ has finite index in E. But $C(k)$ is contained in the centraliser of k in E, so $g^{-1}kg$, with $g \in E$, is finite valued. Hence

$$H_{\tau+1}(G) \cap K = H_{\tau+1}\,.$$

Suppose $kg \in H_{\tau+1}(G)$, where $k \in K$ and $g \in E$, $g \neq 1$. Let $\sigma + 1$ be the least ordinal in the range $\tau + 1 \leq \sigma + 1 \leq \beta$ such that $k \in H_{\sigma+1}$. Now H_σ is a characteristic subgroup of K, and hence is normal in G, and both kH_σ and kgH_σ are FC elements of G/H_σ. Hence gH_σ is FC in G/H_σ.

We can choose an infinite sequence of distinct positive integers, $\mu_1, \mu_2, \cdots$, such that $g^{-1}A_{\mu_i} g \neq A_{\mu_i}$, for all $i = 1, 2, \cdots$, for otherwise g would belong to all but a finite number of the C's. Moreover, since C_i has finite index in E, for each i, we can choose the sequence μ_1, $\mu_2, \cdots$ so that distinct terms belong to distinct systems of transitivity in the representation R of E. By relabelling the subgroups A_i, $i = 1, 2, \cdots$, we may arrange that the sequence $\mu_1, \mu_2, \cdots$ is just the sequence of odd positive integers. So if n is any odd positive integer, and $g^{-1}A_n g = A_{\dot{n}}$, then $\dot{n}$ is even. Since $\sigma < \beta$, we can choose

$a_n \in A_n - H_\sigma(A_n)$, for $n = 1, 3, \cdots$. Let $a_{\dot{n}} = g^{-1} a_n g$, and define

$$c_n = g^{-1} g^{a_n} = a_{\dot{n}}^{-1} a_n, \qquad n = 1, 3, \cdots .$$

Then

$$c_n^{-1} c_m = (g^{a_n})^{-1} g^{a_m} = a_n^{-1} a_{\dot{n}} a_{\dot{m}}^{-1} a_m .$$

If $n \neq m$, the four integers $n, \dot{n}, m$ and $\dot{m}$ are all distinct and thus $(g^{a_n})^{-1} g^{a_m} \notin H_\sigma$. Thus gH_σ is not FC in G/H_σ, contrary to what we have already proved.

It follows that the upper FC-series of G is

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_\beta = K < G ,$$

for $G/K \simeq E \simeq J$, and J is an FC-group. Moreover J is infinite and centreless, and the factors $H_{\gamma+1}/H_\gamma$ are infinite and centreless, for all $\gamma < \beta$, since G_β is a group of type Q_β, and K is a direct product of groups isomorphic with G_β. Thus G is a group of type $Q_{\beta+1}$, as required.

We have now shown how to construct a group of type Q_α, given groups of type Q_β for all $\beta < \alpha$, whether α is a limit ordinal or not. So, by transfinite induction, we have:

THEOREM. *There exist groups of type Q_α, for any ordinal α.*

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REFERENCES

1. R. Baer, *Finiteness properties of groups*, Duke Math. J. **15** (1948), 1021–1032.
2. A. M. Duguid and D.H. McLain, *FC-nilpotent and FC-soluble groups*, Proc. Cambridge Philos. Soc. **52** (1956), 391-398.
3. V. M. Glushkov, *On the central series of an infinite group*, Mat. Sbornik N. S. **31** (1952), 491-495 (Russian).
4. F. Haimo, *The FC-chain of a group*, Canad. J. Math. **5** (1953), 498–511.
5. P. Hall, *Finiteness conditions in soluble groups*, Proc. London Math. Soc. (3) **4** (1954), 419-436.
6. A. G. Kurosh, *The theory of groups*, New York, 1955.
7. D. H. McLain, *Remarks on the upper central series of a group*, Proc. Glasgow Math. Soc. **3** (1956), 38-44.

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