NOTE ON ALDER'S POLYNOMIALS

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1. Alder's polynomial $G_{M,t}(x)$ may be defined by means of

$$(1) \qquad 1+\sum_{s=1}^{\infty}(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}}(1-kx^{2s})\frac{(kx)_{s-1}}{(x)_s} = \prod_{n=1}^{\infty}(1-kx^n)\sum_{t=0}^{\infty}\frac{k^t G_{M,t}(x)}{(x)_t},$$

where M is a fixed integer ≥ 2 and

$$(a)_t = (1-a)(1-ax)\cdots(1-ax^{t-1}),\ (a)_0 = 1 .$$

Alder [1] obtained the identities

$$(2) \qquad \prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-M})(1-x^{(2M+1)n-M-1})(1-x^{(2M+1)n})}{1-x^n} = \sum_{t=1}^{\infty}\frac{G_{M,t}(x)}{(x)_t},$$

$$(3) \qquad \prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-1})(1-x^{(2M+1)n-2M})(1-x^{(2M+1)n})}{1-x^n} = \sum_{t=0}^{\infty}\frac{x^t G_{M,t}(x)}{(x)_t}$$

thus generalizing the well-known Rogers-Ramanujan identities. Singh [2, 3] has further generalized (2), (3); he showed that

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-s})(1-x^{(2M+1)n-2M-1+s})(1-x^{(2M+1)n})}{1-x^n} = \sum_{t=0}^{\infty}\frac{A_s(x,\ t)G_{m,t}(x)}{(x)_t},$$

where the $A_s(x,\ t)$ are polynomials in x.

In a recent paper [4] Singh has proved that

$$(4) \qquad G_{M,t}(x) = x^t \qquad (t \leq M-1) .$$

In the present note we give another proof of (4) and indeed obtain the explicit formula

$$(5) \qquad G_{M,t}(x) = \sum_{\substack{Ms\leq t\\ s\geq 0}}(-1)^s\frac{(x)_t}{(x)_s(x)_{t-Ms}}x^{\frac{1}{2}s(s-1)+st}(1-x^s+x^{t-Ms+s})$$

valid for all t.

2. Since

$$(1-kx^{2s})(kx)_{s-1} = (kx)_s + kx^s(1-x^s)(kx)_{s-1} ,$$

the left member of (1) is equal to

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$$\begin{aligned}1 &+ \sum_{s=1}^{\infty}(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}}\left\{\frac{(kx)_s}{(x)_s} + kx^s\frac{(kx)_{s-1}}{(x)_{s-1}}\right\}\\ &= \sum_{s-0}^{\infty}(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s=1\}}\frac{(kx)_s}{(x)_s}\\ &- \sum_{s=0}^{\infty}(-1)^s k^{M(s+1)+1}x^{\frac{1}{2}(s+1)\{(2M+1)(s+1)-1\}+(s+1)}\frac{(kx)_s}{(x)_s}\\ &= \sum_{s=0}^{\infty}(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}}\frac{(kx)_s}{(x)_s}\{1 - k^{M+1}x^{(M+1)(2s+1)}\} .\end{aligned}$$

Thus (1) becomes

$$\begin{aligned}\sum_{t=0}^{\infty}\frac{k^t G_{M,t}(x)}{(x)_t} &= \sum_{s=0}^{\infty}(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}} \cdot \frac{1-k^{M1}x^{(M+1)(2s+1)}}{(x)_s}\prod_{j=1}^{\infty}(1-kx^{s+j})^{-1}\\ &= \sum_{s=0}^{\infty}(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}} \cdot \frac{1-k^{M+1}x^{(M+1)(2s+1)}}{(x)_s}\sum_{j=0}^{\infty}\frac{k^j x^{sj+j}}{(x)_j} .\end{aligned} \tag{6}$$

For $t < M$, it is clear that the coefficient of k^t on the right is simply $x^t/(x)_t$. This proves Singh's result (4).

For $t = M$ we get

$$\frac{G_{M,M}(x)}{(x)_M} = -\frac{x^M}{1-x} + \frac{x^M}{(x)_M} ,$$

so that

$$G_{M,M}(x) = x^M - x^M\frac{(x)_M}{1-x} ,$$

which also was found by Singh.

For $t = M + 1$, similarly, we have

$$\frac{G_{M,M+1}(x)}{(x)_{M+1}} = \frac{x^{M+1}}{(x)_{M+1}} - x^{M+1} - \frac{x^{M+2}}{(1-x)^2} ,$$

so that

$$\begin{aligned}G_{M,M+1}(x) &= x^{M+1}\left\{1 - (x)_{M+1} - x\frac{(x)_{M+1}}{(1-x)^2}\right\}\\ &= x^{M+1}\{1 - (1 + x^3)(x^3)_{M-1}\} .\end{aligned} \tag{7}$$

also due to Singh.

3. For arbitrary $t \geq M + 1$, it follows from (6) that

$$\begin{aligned}G_{M,t}(x) &= \sum_{Ms\leq t}(-1)^s\frac{(x)_t}{(x)_s(x)_{t-Ms}}x^{\frac{1}{2}s\{(2M+1)s-1\}(s+1)(t-Ms)}\\ &- \sum_{M(s+1)\leq t}(-1)^s\frac{(x)_t}{(x)_s(x)_{t-M(s+1)-1}}x^{e_s} ,\end{aligned}$$

where

$$e_s = \tfrac{1}{2}s\{(2M+1)s-1\} + (s+1)\{t-M(s+1)-1\}(M+1)(2s+1)\,.$$

This simplifies to

$$(8)\qquad \begin{aligned} G_{M,t}(x) &= x^t \sum_{Ms\le t} (-1)^s \frac{(x)_t}{(x)_s(x)_{t-Ms}} x^{\frac{1}{2}s(s-1)+s(t-M)} \\ &+ \sum_{0<Ms<t} (-1)^s \frac{(x)_t}{(x)_{s-1}(x)_{t-Ms-1}} x^{\frac{1}{2}s(s-1)+st}\,, \end{aligned}$$

or if we prefer

$$(9)\qquad G_{M,t}(x) = \sum_{\substack{Ms\le t\\ s\ge 0}} (-1)^s \frac{(x)_t}{(x)_s(x)_{t-Ms}} x^{\frac{1}{2}s(s-1)+st}(1-x^s+x^{t-Ms+s})\,.$$

For example (9) reduces to

$$(10)\qquad G_{M,t}(x) = x^t\left\{1 - \frac{(x)_t}{(x)_1(x)_{t-M}}(1-x+x^{t-M+1})\right\}$$

for $M+1 \le t \le 2M-1$. When $t = M+1$, it is easily verified that (9) reduces to (7). Singh [4] conjectured the truth of (10) for $t \le 2(M-1)$.

REFEFENCES

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