THE PALEY-WIENER THEOREM IN METRIC LINEAR SPACES

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1. Introduction. By a *basis* in a topological linear space $\mathscr{T}$ we mean a sequence $\{x_n\}$ of points of $\mathscr{T}$ such that to every x in $\mathscr{T}$ there corresponds a unique sequence $\{a_n\}$ of scalars for which

$$x = \sum_{n=1}^{\infty} a_n x_n \, .$$

Denoting the coefficient functionals here by φ_n, we can rewrite this as

$$x = \sum_{n=1}^{\infty} \varphi_n(x) x_n \, . \tag{1.1}$$

If it happens that all φ_n are continuous on $\mathscr{T}$, the basis will be referred to as a *Schauder basis*. Every basis in a Fréchet space [**14**, pp. 59, 110] is known to be a Schauder basis (see Newns [**21**], pp. 431–432), and it will be shown here that the same holds for bases in an arbitrary complete metric linear space over the real or complex field.

The classical Paley-Wiener theorem asserts that for $\mathscr{T}$ a Banach space, all sequences which sufficiently closely approximate bases must themselves be bases. A more precise statement of the theorem is obtained by replacing $\mathscr{M}$ in Theorem 1 by a Banach space $\mathscr{B}$.

The bibliography at the end of the present paper includes a chronological listing of articles on the Paley-Wiener theorem, and we give now a brief résumé of its history. As originally presented in 1934 by Paley and Wiener [**1**, p. 100], the theorem was derived specifically for the Hilbert space L^2. Then, in applying the theorem to the Pincherle basis problem [**2**, p. 469], Boas observed in 1940 that the proof of Paley and Wiener remains valid for Banach spaces. Boas also succeeded in simplifying a portion of the proof. However, the first really elementary proof of the theorem was published in 1949 by Schafke [**8**], to whom conclusion (3) is due. The remaining articles on the Paley-Wiener theorem deal mainly with various generalizations of condition (2.1) for Hilbert spaces.

From the viewpoint of modern functional analysis, the key to theorems of Paley-Wiener type lies in the inversion of an operator $I+T$ by means of a geometric series in T. This crucial observation was made by Buck [**15**, p. 410] in 1953.[1]

Received July 28, 1958, and in revised form June 22, 1959. The research reported upon here was supported in part by the National Science Foundation.

[1] The same technique was used also in [**9**], the author having been unaware of the earlier remarks of Buck. A further application (to generalized bases) appears in [**12**].

Our purpose in the present note is to utilize the operator technique in deriving a number of variants of the Paley-Wiener theorem. For reference, we begin by sketching a proof of the theorem itself for complete metric linear spaces. The ensuing variants then have in common the hypothesis that $\{x_n\}$ be a Schauder basis and $\{y_n\}$ a sequence triangular with respect to $\{x_n\}$. This is evidently motivated by the case of Pincherle bases in spaces of analytic functions (see, for example, [**9**]), and we conclude, in fact, with a generalization to arbitrary Fréchet spaces of the theorem of Boas [**2**, p. 447, Theorem 4.1] on Pincherle bases.

The author is indebted to Professor Robert C. James for reading the manuscript and suggesting a number of important simplifications. In particular, Theorem 2 replaces a weaker theorem of the original manuscript.

2. The proof for metric linear spaces.

In what follows, we shall denote by $\mathscr{M}$ a complete metric linear space over the real or complex field and employ the notation of Banach:

$$\| x \| = \rho(x, 0) \qquad (x \in \mathscr{M}),$$

where ρ is the metric on $\mathscr{M}$. It will be assumed further that ρ is translation invariant.[2]

With these conventions the Paley-Wiener theorem can be formulated as

THEOREM 1. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathscr{M}$, and let λ be a real number $(0 < \lambda < 1)$ such that*

$$\left\| \sum_{n=1}^{m} a_n(y_n - x_n) \right\| \leq \lambda \left\| \sum_{n=1}^{m} a_n x_n \right\| \tag{2.1}$$

holds for all finite sequences $a_1, a_2, \cdots, a_m$ of scalars. Then

(1) *if $\{x_n\}$ is total in $\mathscr{M}$, so is $\{y_n\}$;*

(2) *if $\{x_n\}$ is a basis in $\mathscr{M}$, so is $\{y_n\}$, and the coefficients in any expansion $\sum b_n y_n$ satisfy*

$$\left\| \sum_{n=1}^{\infty} b_n x_n \right\| \leq \frac{1}{1-\lambda} \left\| \sum_{n=1}^{\infty} b_n y_n \right\| ; \tag{2.2}$$

(3) *if $\{x_n\}$ is a basis in $\mathscr{M}$, there exists an automorphism[3] A on $\mathscr{M}$ such that $y_n = Ax_n$ $(n = 1, 2, \cdots)$.*

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[2] A translation-invariant metric yielding the original topology always exists (see, for example, [**19**, p. 34]).

[3] The term *automorphism* is used to designate any linear homeomorphic mapping of the space onto itself. By the open mapping theorem [**13**, p. 41, Theorem 5] every one-to-one continuous linear mapping of $\mathscr{M}$ onto itself is an automorphism on $\mathscr{M}$.

Proof. For convenience we consider first the case in which $\{x_n\}$ is a basis. Condition (2.1) then allows us to define a continuous linear operator T on $\mathscr{M}$ as

$$Tx = \sum_{n=1}^{\infty} \varphi_n(x) \cdot (y_n - x_n) \tag{2.3}$$

and yields the inequality

$$\| T^n x \| \leq \lambda^n \| x \| \qquad (n = 0, 1, \cdots).$$

By comparison with the corresponding geometric series in λ we infer convergence of the operator series

$$U = \sum_{n=0}^{\infty} (-T)^n \tag{2.4}$$

and obtain the inequality

$$\| Ux \| \leq (1 - \lambda)^{-1} \| x \| . \tag{2.5}$$

Hence, the linear operator U is continuous on $\mathscr{M}$.

For any x in $\mathscr{M}$ the element y of $\mathscr{M}$ defined by $y = Ux$ has the evident property that $x = (I + T)y$, where I is the identity operator. From $y = \sum b_n x_n$ it therefore follows that $x = \sum b_n y_n$, and this proves that $\{y_n\}$ spans $\mathscr{M}$ in the infinite-series sense. That $\{y_n\}$ is linearly independent in the infinite-series sense can then be seen by rewriting (2.5) in the form (2.2). Assertions (2) and (3) are thereby established, the latter with A taken as $I + T\,(= U^{-1})$.

No essential change in the above argument is required to prove (1). We can clearly presume the x_n to be finitely linearly independent and replace the infinite series in (2.3) by corresponding finite sums. Thus defined on a dense subset of $\mathscr{M}$, T is then extended to all of $\mathscr{M}$ in the usual fashion.

It should perhaps be mentioned that the automorphism A in (3) is uniquely determined by the way it correlates the basis elements x_n and y_n. In fact,

$$Ax = \sum_{n=1}^{\infty} \varphi_n(x) y_n . \tag{2.6}$$

3. Coefficient functionals and coordinate subspaces. We recall that a Fréchet space is defined [14, pp. 59, 110] as a metrizable, complete, locally convex topological linear space over the real or complex field. Generalizing a theorem of Banach, Newns has shown [21, pp. 431-432] that for bases in Fréchet spaces the coefficient functionals φ_n are always continuous. This can, however, be carried one step farther by discarding the hypothesis of local convexity.

Such is the content of

THEOREM 2. *Every basis in $\mathscr{M}$ is a Schauder basis.*

Proof. As observed in footnote 2, there is no loss of generality in taking the metric ρ on $\mathscr{M}$ to be translation invariant. Having done this, we can conveniently make use of the functional $\| x \| = \rho(x, 0)$.

Let $\{x_n\}$ be a basis in $\mathscr{M}$, so that for each x in $\mathscr{M}$ we have the expansion (1.1), or equivalently

$$\lim_{m\to\infty} \left\| x - \sum_{n=1}^{m} \varphi_n(x)x_n \right\| = 0 .$$

Since this yields boundedness in m (for fixed x) of

$$\left\| \sum_{n=1}^{m} \varphi_n(x)x_n \right\| ,$$

the quantity

$$\| x \|' = \sup_{m\geq 1} \left\| \sum_{n=1}^{m} \varphi_n(x)x_n \right\| \tag{3.1}$$

is always finite. Thus, $\rho'(x, y) = \| x - y \|'$ defines a translation-invariant metric ρ' on $\mathscr{M}$ with the property that $\rho(x, y) \leq \rho'(x, y$ for all x, y in $\mathscr{M}$.

It is immediate from (3.1) that

$$\| \varphi_n(x)x_n \| \leq 2 \| x \|' \qquad (n = 1, 2, \cdots), \tag{3.2}$$

and the corollary to Proposition 2, pp. 25–26, of Bourbaki [**14**] then ensures that each φ_n is continuous in the metric ρ'. The proof will be completed by showing that ρ and ρ' define the same topology on $\mathscr{M}$.

We establish, first of all, that $\mathscr{M}$ is complete in the metric ρ'. To this end, let $\{z_k\}$ be a Cauchy sequence in the metric ρ'. From (3.2) and the result of Bourbaki just cited it follows that, for each n, $\{\varphi_n(z_k)\}_{k=1}^{\infty}$ is a Cauchy sequence of scalars and therefore converges to some scalar c_n. Now, given $\varepsilon > 0$, there exists a positive integer N such that $\| z_j - z_k \|' \leq \varepsilon$ for $j, k > N$. For arbitrary positive integers m and $m' \leq m$ we thus have

$$\left\| \sum_{n=m'}^{m} [\varphi_n(z_j) - \varphi_n(z_k)]x_n \right\| \leq 2\varepsilon \qquad (j, k > N), \tag{3.3}$$

which yields in the limit as $j \to \infty$

$$\left\| \sum_{n=m'}^{m} c_n x_n \right\| \leq 2\varepsilon + \left\| \sum_{n=m'}^{m} \varphi_n(z_k)x_n \right\| \qquad (k > N).$$

The ρ-convergence of $\sum \varphi_n(z_k)x_n$ (for fixed k) gives rise to a Cauchy condition on its partial sums and thereby on the partial sums of $\sum c_n x_n$. Hence, $\sum c_n x_n$ converges (ρ) to some point z of $\mathscr{M}$. Taking $m' = 1$ in (3.3) and passing to the limit on j, we arrive at

$$\| z - z_k \|' = \sup_{m \geq 1} \left\| \sum_{n=1}^{m} [c_n - \varphi_n(z_k)] x_n \right\| \leq 2\varepsilon \, (k > N) \, .$$

That is, $\{z_k\}$ converges to z in the metric ρ'.

The remainder of the proof involves simply a routine application of a corollary of the open mapping theorem [**13**, p. 41, Theorem 6] to conclude that ρ and ρ' define the same topology on $\mathscr{M}$.

Relative to a given basis $\{x_n\}$, a *coordinate subspace* of $\mathscr{M}$ is defined as a subspace of the form $\{x\colon \varphi_n(x) = 0 \text{ for } n \in K\}$, where K is some set of positive integers. The coordinate subspaces arising when K consists of the first $k - 1$ positive integers are of special interest in the sequel, and we denote them by $\mathscr{M}_k$. That is, for each positive integer k, $\mathscr{M}_k$ is the set of all elements of $\mathscr{M}$ expressible as infinite linear combinations of the basis elements $x_k, x_{k+1}, \cdots$. For convenience, $\mathscr{M}_k$ will be referred to as a *terminal coordinate subspace* (or, more precisely, as *the* kth *terminal coordinate subspace*) of $\mathscr{M}$ relative to $\{x_n\}$.

Since coordinate subspaces relative to Schauder bases are necessarily closed, we have

COROLLARY 2.1. *All coordinate subspaces of $\mathscr{M}$ are closed.*

4. Some variants of the Paley-Wiener theorem. A sequence $\{y_n\}$ in $\mathscr{M}$ will be called *triangular* with respect to a basis $\{x_n\}$ provided that each y_n has the representation

$$y_n = x_n + \sum_{i=n+1}^{\infty} \varphi_i(y_n) x_i \, . \tag{4.1}$$

In the present section we shall be concerned with the problem of determining conditions under which $\{y_n\}$ will itself be a basis in $\mathscr{M}$. This arises as a natural analogue of the Pincherle basis problem, and our methods here have much in common with those of [**9**].

We take advantage of the following special properties of triangular sequences.

LEMMA 1. *Let $\{x_n\}$ be a basis in $\mathscr{M}$, and let $\mathscr{M}_k$ be a terminal coordinate subspace of $\mathscr{M}$ relative to $\{x_n\}$. If $\{y_n\}$ is a sequence in $\mathscr{M}$ triangular with respect to $\{x_n\}$, then*

(1) *$\{y_n\}$ is linearly independent in the infinite-series sense, and*

(2) *for $\{y_n\}_{n=k}^{\infty}$ to be a basis in $\mathscr{M}_k$ it is necessary and sufficient that $\{y_n\}_{n=1}^{\infty}$ be a basis in $\mathscr{M}$.*

Proof. To show that $\{y_n\}$ is linearly independent in the infinite-series sense, we suppose that

$$\sum_{n=1}^{\infty} b_n y_n = 0 .$$

Then, from (4.1) and the fact that $\mathscr{M}_2$ is closed, it is immediate that $b_1 x_1 + z_2 = 0$, where z_2 is some point in $\mathscr{M}_2$. Hence $b_1 = 0$, and an obvious inductive argument establishes $b_n = 0$ $(n = 1, 2, \cdots)$.

The second assertion is dealt with similarly. Let $\{y_n\}_{n=k}^{\infty}$ be a basis in $\mathscr{M}_k$, and let y be any element of $\mathscr{M}$. It is evident that, for a suitably chosen scalar b_1, the point $y - b_1 y_1$ will lie in $\mathscr{M}_2$. Proceeding inductively, we then see that there exist scalars $b_1, b_2, \cdots, b_{k-1}$ yielding

$$y - \sum_{n=1}^{k-1} b_n y_n \in \mathscr{M}_k .$$

Consequently, $\{y_n\}_{n=1}^{\infty}$ spans $\mathscr{M}$ in the infinite-series sense and is therefore a basis in $\mathscr{M}$. The converse in (2) is trivial.

This leads to our first variant of Theorem 1.

THEOREM 3. *Let $\{x_n\}$ be a basis in $\mathscr{M}$ and $\{y_n\}$ a sequence triangular with respect to $\{x_n\}$. If there exist a positive number $\lambda < 1$ and a positive integer k such that*

$$\left\| \sum_{n=k}^{m} a_n (y_n - x_n) \right\| \le \lambda \left\| \sum_{n=k}^{m} a_n x_n \right\| \tag{4.2}$$

holds for all finite sequences $a_k, a_{k+1}, \cdots, a_m$ of scalars, then

(1) *$\{y_n\}$ is a basis in $\mathscr{M}$, and*

(2) *there exists an automorphism A on $\mathscr{M}$ such that $y_n = Ax_n$ $(n = 1, 2, \cdots)$.*

Proof. For conclusion (1) we apply Theorem 1 to infer that $\{y_n\}_{n=k}^{\infty}$ is a basis in $\mathscr{M}_k$, and then invoke (2) of Lemma 1. Theorem 1 shows also that the mapping

$$A_k x = \sum_{n=k}^{\infty} \varphi_n(x) y_n$$

is an automorphism on $\mathscr{M}_k$. We can obviously extend A_k to a mapping A of $\mathscr{M}$ into itself by setting

$$Ax = \sum_{n=1}^{\infty} \varphi_n(x) y_n ,$$

and from the fact that $\{y_n\}$ is a basis in $\mathscr{M}$ it is then clear that A maps $\mathscr{M}$ onto itself in one-to-one fashion. There remains simply to observe that the continuity of A_k implies continuity of A, so that A is an automorphism on $\mathscr{M}$.

A further variant of the main theorem is at hand when $\{x_n\}$ is an *absolutely ρ-convergent basis* in $\mathscr{M}$, that is, when $\{x_n\}$ is a basis for which all $x \in \mathscr{M}$ satisfy

$$\sum_{n=1}^{\infty} \| \varphi_n(x)x_n \| < +\infty .$$[4]

THEOREM 4. *Let $\{x_n\}$ be an absolutely ρ-convergent basis in $\mathscr{M}$ and $\{y_n\}$ a sequence triangular with respect to $\{x_n\}$. If*

$$\limsup_{n\to\infty} \left\{ \sup_{a\neq 0} \frac{\sum_{i=n+1}^{\infty} \| a\varphi_i(y_n)x_i \|}{\| ax_n \|} \right\} < 1 \qquad (a, \text{ scalar}), \tag{4.3}$$

then

(1) *$\{y_n\}$ is an absolutely ρ-convergent basis in $\mathscr{M}$, and*

(2) *there exists an automorphism A on $\mathscr{M}$ such that $y_n = Ax_n$ $(n = 1, 2, \cdots)$.*

Proof. We first remetrize $\mathscr{M}$ by setting $\rho'(x, y) = \| x - y \|'$, where

$$\| x \|' = \sum_{n=1}^{\infty} \| \varphi_n(x)x_n \| \tag{4.4}$$

for all x in $\mathscr{M}$. Obviously ρ' is translation invariant, and $\| x \| \leq \| x \|'$.

Condition (4.3) can now be restated as follows: there exist a positive number $\lambda < 1$ and a positive integer k such that

$$\| a(y_n - x_n) \|' \leq \lambda \| ax_n \|' \tag{4.5}$$

holds for $n \geq k$ and all scalars a. This, together with (4.4), yields the inequality (4.2) in the metric ρ'. Hence, $\{y_n\}$ is a basis in $\mathscr{M}$, and there exists an automorphism A on $\mathscr{M}$ such that $y_n = Ax_n$ $(n = 1, 2, \cdots)$. It follows that, for arbitrary scalar sequences $\{b_n\}$, convergence of the series $\sum b_n y_n$ implies convergence (and thereby absolute ρ-convergence) of the series $\sum b_n x_n$. Since (4.5) results in

$$\| b_n y_n \| \leq (1 + \lambda) \| b_n x_n \|$$

for $n \geq k$, we see that $\{y_n\}$ is, in fact, an absolutely ρ-convergent basis in $\mathscr{M}$. This completes the proof.

As noted in the derivation, there is no real loss of generality in requiring that

$$\left\| \sum_{n=1}^{\infty} \varphi_n(x)x_n \right\| = \sum_{n=1}^{\infty} \| \varphi_n(x)x_n \|$$

[4] In metric linear spaces the notion of absolute ρ-convergence coincides with that of absolute convergence as defined by Day [**16**, pp. 11, 59] in terms of the Minkowski functional. Here, absolutely convergent bases are defined only for Fréchet spaces (see § 5).

for all x in $\mathscr{M}$. Whenever the metric ρ and the basis $\{x_n\}$ are interrelated in this fashion, condition (4.3) assumes the simpler form

$$\limsup_{n\to\infty}\left\{\sup_{a\neq 0}\frac{\|a(y_n-x_n)\|}{\|ax_n\|}\right\}<1 \qquad (a,\ \text{scalar}). \tag{4.3'}$$

5. The case of Fréchet spaces. Proposition 6, p. 97, of [14] ensures that the topology on a Fréchet space $\mathscr{F}$ can be described by a sequence $\{\|\ \ \|_q\}$ of continuous semi-norms, and with no loss of generality this sequence will be taken as monotone increasing (a condition automatically fulfilled in spaces of analytic functions). Thus, $\|x\|_p \leq \|x\|_q$ for $q > p$ and all $x \in \mathscr{F}$; and convergence in $\mathscr{F}$ is equivalent to convergence with respect to each of the semi-norms $\|\ \ \|_q$. The topology on $\mathscr{F}$ is then that of the translation-invariant metric

$$\rho(x, y) = \sum_{q=1}^{\infty}\frac{1}{2^q}\frac{\|x-y\|_q}{1+\|x-y\|_q}\,. \tag{5.1}$$

As we proceed to show, the Paley-Wiener theorem and its variants can be generalized in the case of Fréchet spaces by replacing the inequalities on the metric by corresponding inequalities on the semi-norms.

THEOREM 5. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Fréchet space $\mathscr{F}$, and let $\{\lambda_q\}$ be a sequence of real numbers $(0 < \lambda_q < 1)$ such that*

$$\left\|\sum_{n=1}^{m}a_n(y_n-x_n)\right\|_q \leq \lambda_q\left\|\sum_{n=1}^{m}a_nx_n\right\|_q \qquad (q = 1, 2, \cdots)$$

holds for all finite sequences $a_1, a_2, \cdots, a_m$ of scalars. Then

(1) *if $\{x_n\}$ is total in $\mathscr{F}$, so is $\{y_n\}$;*

(2) *if $\{x_n\}$ is a basis in $\mathscr{F}$, so is $\{y_n\}$, and the coefficients in any expansion $\sum b_ny_n$ satisfy*

$$\left\|\sum_{n=1}^{\infty}b_nx_n\right\|_q \leq \frac{1}{1-\lambda_q}\left\|\sum_{n=1}^{\infty}b_ny_n\right\|_q \qquad (q = 1, 2, \cdots);$$

(3) *if $\{x_n\}$ is a basis in $\mathscr{F}$, there exists an automorphism A on $\mathscr{F}$ such that $y_n = Ax_n$ $(n = 1, 2, \cdots)$.*

THEOREM 6. *Let $\{x_n\}$ be a basis in a Fréchet space $\mathscr{F}$ and $\{y_n\}$ a sequence triangular with respect to $\{x_n\}$. If there exist positive integers k_q and positive numbers $\lambda_q < 1$ such that*

$$\left\|\sum_{n=k_q}^{m}a_n(y_n-x_n)\right\|_q \leq \lambda_q\left\|\sum_{n=k_q}^{m}a_nx_n\right\|_q \qquad (q = 1, 2, \cdots) \tag{5.2}$$

holds for all finite sequences $a_{k_q}, a_{k_q+1}, \cdots, a_m$ of scalars, then

(1) *$\{y_n\}$ is a basis in $\mathscr{F}$, and*

(2) *there exists an automorphism* A *on* $\mathscr{F}$ *such that* $y_n = Ax_n$ $(n = 1, 2, \cdots)$.

The proof of Theorem 5 duplicates that of Theorem 1. The proof of Theorem 6 would likewise duplicate that of Theorem 3 if we knew that $\{k_q\}$ were bounded (so that in effect we could replace it by a single number k). Failing this, we use the following argument, based directly on the properties of the transformation T of (2.3).

Convergence of the series

$$Tx = \sum_{n=1}^{\infty} \varphi_n(x)(y_n - x_n)$$

is ensured by condition (5.2). In fact, if x lies in the k_qth terminal coordinate subspace $\mathscr{F}_{k_q}$, we have

$$\| Tx \|_q \leq \lambda_q \| x \|_q \qquad (q = 1, 2, \cdots). \tag{5.3}$$

Since the complementary subspace corresponding to each $\mathscr{F}_{k_q}$ is finite dimensional, it follows that T is continuous on $\mathscr{F}$.[5]

Now, taking account of the fact that $\mathscr{F}_k$ is closed, we verify at once that x in $\mathscr{F}_{k-1}$ implies Tx in $\mathscr{F}_k$. Hence, for arbitrary x in $\mathscr{F}$, the point $T^{k-1}x$ must lie in $\mathscr{F}_k$ $(k = 1, 2, \cdots)$. This result, combined with (5.3), leads to the inequality

$$\| T^n x \|_q \leq (\lambda_q)^{n-k_q} \| T^{k_q} x \|_q \qquad (q = 1, 2, \cdots)$$

for $n \geq k_q$ and all x in $\mathscr{F}$. As in the proof of Theorem 1, it follows that the operator series

$$U = \sum_{n=0}^{\infty} (-T)^n$$

converges and that $U = (I + T)^{-1}$. From this we conclude that $A = I + T$ is an automorphism on $\mathscr{F}$ carrying x_n into y_n $(n = 1, 2, \cdots)$, and that $\{y_n\}$ is a basis in $\mathscr{F}$.

To frame an analogue of Theorem 4, we first define an *absolutely convergent basis* in the Fréchet space $\mathscr{F}$ as a basis $\{x_n\}$ such that

$$\sum_{n=1}^{\infty} \| \varphi_n(x) x_n \|_q < +\infty \qquad (q = 1, 2, \cdots)$$

for all x in $\mathscr{F}$.[6]

[5] Any x in $\mathscr{F}$ can be expressed as $x = x' + x''$, where x' is the projection of x on the complementary subspace to $\mathscr{F}_{k_q}$ and x'' is the projection of x on $\mathscr{F}_{k_q}$. By continuity of the coefficient functionals, $x \to 0$ implies $x' \to 0$ and thereby $x'' \to 0$. Then $Tx' \to 0$ and $Tx'' \to 0$, so that $Tx \to 0$.

[6] It is evident from [**14**, p. 101] that this definition is independent of the choice of semi-norm sequence from among those defining the topology on $\mathscr{F}$.

THEOREM 7. *Let $\mathscr{F}$ be a Fréchet space, $\{x_n\}$ an absolutely convergent basis in $\mathscr{F}$, and $\{y_n\}$ a sequence triangular with respect to $\{x_n\}$. If*

$$\limsup_{n\to\infty} \frac{\sum_{i=n+1}^{\infty} \| \varphi_i(y_n)x_i \|_q}{\| x_n \|_q} < 1 \qquad (q = 1, 2, \cdots), \tag{5.4}$$

then

(1) *$\{y_n\}$ is an absolutely convergent basis in $\mathscr{F}$, and*

(2) *there exists an automorphism A on $\mathscr{F}$ such that $y_n = Ax_n$ $(n = 1, 2, \cdots)$.*

Proof. Setting

$$\| x \|_q' = \sum_{n=1}^{\infty} \| \varphi_n(x)x_n \|_q \qquad (q = 1, 2, \cdots) \tag{5.5}$$

for all x in $\mathscr{F}$, we observe that $\{\| \ \|_q'\}$ is an increasing sequence of semi-norms on $\mathscr{F}$. It thus defines a metric ρ' on $\mathscr{F}$ according to (5.1), and there is no difficulty in showing that ρ' yields the same topology as ρ.[7] Then, to each index q there correspond a positive number $\lambda_q < 1$ and a positive integer k_q such that

$$\| y_n - x_n \|_q' < \lambda_q \| x_n \|_q'$$

holds for $n > k_q$. The additivity property (5.5) assures us also that (5.2) holds for the primed semi-norms, and the proof is completed just as in the case of Theorem 4.

Again we note that the semi-norms on $\mathscr{F}$ can be required to have the additivity property

$$\left\| \sum_{n=1}^{\infty} \varphi_n(x)x_n \right\|_q = \sum_{n=1}^{\infty} \| \varphi_n(x)x_n \|_q \qquad (q = 1, 2, \cdots)$$

relative to the absolutely convergent basis $\{x_n\}$. In terms of a "natural" sequence of semi-norms of this sort, condition (5.4) reduces to

$$\limsup_{n\to\infty} \frac{\| y_n - x_n \|_q}{\| x_n \|_q} < 1 \qquad (q = 1, 2, \cdots). \tag{5.4$'$}$$

It is readily seen that the coefficients for an element in a given basis are finite linear combinations of the coefficients in a basis triangular with respect to the given one. We have, in fact,

LEMMA 2. *Let $\{x_n\}$ be a basis in $\mathscr{M}$ and $\{y_n\}$ a basis triangular with respect to $\{x_n\}$. If x is an element of $\mathscr{M}$ having expansions in*

[7] This argument appears also in the proof of Lemma 4 of [**11**].

the two bases as

$$x = \sum_{n=1}^{\infty} a_n x_n \qquad \textit{and} \qquad x = \sum_{n=1}^{\infty} b_n y_n ,$$

then

$$a_1 = b_1 \qquad \textit{and} \qquad a_n = b_n + \sum_{k=1}^{n-1} b_{n-k}\varphi_n(y_{n-k}) \qquad (n \geq 2).$$

Proof. The expansion of x in the basis $\{y_n\}$ appears as

$$\begin{aligned} x = b_1[x_1 + \varphi_2(y_1)x_2 + \varphi_3(y_1)x_3 + \cdots] \\ + b_2[x_2 + \varphi_3(y_2)x_3 + \cdots] \\ + b_3[x_3 + \cdots] \\ + \cdots . \end{aligned}$$

Since $\mathscr{M}_2$ is closed, it follows from the linear independence of $\{x_n\}$ that $a_1 = b_1$. The fact that $\mathscr{M}_3$ is closed then results in $a_2 = b_2 + \varphi_2(y_1)$, and the general formula is obtained by induction. (Note that the proof in no way depends on rearrangement of the series.)

Using this lemma, we show how certain inequalities on the coefficients a_n give rise to corresponding inequalities on the coefficients b_n. The underlying space will be taken as a Fréchet space $\mathscr{F}$, and $\{y_n\}$ will again be assumed to be a basis triangular with respect to the basis $\{x_n\}$.

Thus, let x be an element of $\mathscr{F}$ having the expansion

$$x = \sum_{n=1}^{\infty} a_n x_n ,$$

and for each q let M_q be a constant such that

$$|a_n| \leq \frac{M_q}{\|x_n\|_q} \qquad (n = 1, 2, \cdots).$$

Constants of this sort always exist if the basis $\{x_n\}$ is absolutely convergent, since we can, for example, put $M_q = \sum \|a_n x_n\|_q$. (In spaces of analytic functions, where we have access to the Cauchy inequalities, the maximum modulus of course yields a better choice for M_q.) In similar fashion $H_q(y_n)$ will be taken as any constant for which

$$(5.6) \qquad |\varphi_i(y_n)| \leq H_q(y_n)\frac{\|x_n\|_q}{\|x_i\|_q} \qquad (i \geq n + 1).$$

Absolute convergence of $\{x_n\}$ again suffices to ensure the existence of such a constant, for example

$$H_q(y_n) = \frac{\sum_{i=n+1}^{\infty} \| \varphi_i(y_n)x_i \|_q}{\| x_n \|_q},$$

and our remark on the case of analytic function spaces carries over.

Combined with the identities on the coefficients given in Lemma 2, the above inequalities provide the estimates

$$| b_1 | \cdot \| x_1 \|_q \leq M_q ,$$
$$| b_n | \cdot \| x_n \|_q \leq M_q + \sum_{k=1}^{n-1} H_q(y_{n-k}) \, | b_{n-k} | \cdot \| x_{n-k} \|_q \qquad (n \geq 2).$$

We apply now a procedure based on the techniques (due to Narumi [20]) used in proving Theorem 5 of [10]. With $\{B_n\}$ defined inductively according to the equations

$$B_1 \| x_1 \|_q = M_q ,$$
$$B_n \| x_n \|_q = M_q + \sum_{k=1}^{n-1} B_{n-k} H_q(y_{n-k}) \| x_{n-k} \|_q \qquad (n \geq 2)$$

it is readily verified that

$$B_n \| x_n \|_q - B_{n-1} \| x_{n-1} \|_q = B_{n-1} H_q(y_{n-1}) \| x_{n-1} \|_q .$$

Thus, for $n \geq 2$

$$B_n \| x_n \|_q = [1 + H_q(y_{n-1})] B_{n-1} \| x_{n-1} \|_q ,$$

so that

$$B_1 \| x_1 \|_q = M_q ,$$
$$B_n \| x_n \|_q = M_q \prod_{k=1}^{n-1} [1 + H_q(y_k)] \qquad (n \geq 2).$$

There follows

THEOREM 8. *Let $\mathscr{F}$ be a Fréchet space, x an element of $\mathscr{F}$, $\{x_n\}$ a basis in $\mathscr{F}$, and $\{y_n\}$ a basis triangular with respect to $\{x_n\}$. Let us suppose further that there exist constants M_q such that the coefficients in the expansion*

$$x = \sum_{n=1}^{\infty} a_n x_n$$

satisfy

$$| a_n | \leq \frac{M_q}{\| x_n \|_q} \qquad (n = 1, 2, \cdots)$$

for each index q, and that there exist constants $H_q(y_n)$ for which (5.6) *holds. Then the coefficients in the expansion*

$$x = \sum_{n=1}^{\infty} b_n y_n$$

satisfy the inequalities

$$|b_1| \leq \frac{M_q}{\|x_1\|_q}, \qquad |b_n| \leq \frac{M_q}{\|x_n\|_q} \prod_{k=1}^{n-1} [1 + H_q(y_k)] \qquad (n \geq 2).$$

If in addition there exist constants J_q such that

$$\limsup_{n\to\infty} H_q(y_n) < J_q \qquad (q = 1, 2, \cdots), \tag{5.7}$$

then there also exist constants K_q such that

$$|b_n| < (1 + J_q)^n \frac{K_q}{\|x_n\|_q} \qquad (n = 1, 2, \cdots)$$

for all q, and the constants K_q are independent of q whenever the same is true of M_q, $H_q(y_n)$, and J_q. In particular, condition (5.4) *implies*

$$|b_n| < 2^n \frac{K_q}{\|x_n\|_q}.$$

6. Concluding remarks. We begin by making explicit the specialization of Theorem 7 to spaces of analytic functions.

Thus, let Ω be a non-empty plane region, and fix $\{\Omega_q\}$ as any sequence of non-empty subregions of Ω such that $\overline{\Omega}_q$ is a compact subset of Ω_{q+1} $(q = 1, 2, \cdots)$ and $\Omega = \cup\, \Omega_q$. The family of all functions f analytic on Ω, topologized by the sequence of semi-norms

$$M_q(f) = \max_{\overline{\Omega}_q} |f|,$$

is a Fréchet space which we shall denote by $\mathscr{A}(\Omega)$.

Applied to $\mathscr{A}(\Omega)$, Theorem 7 yields the following variant of Theorem 2, p. 117, of Evgrafov [17].[8]

THEOREM 9. *Let $\{\alpha_n\}$ be an absolutely convergent basis in $\mathscr{A}(\Omega)$, and let $\{\beta_n\}$ be the triangular sequence defined by*

$$\beta_n(z) = \alpha_n(z) + \sum_{k=1}^{\infty} A_{nk}\alpha_{n+k}(z),$$

where the A_{nk} are any complex numbers for which the indicated series converge. If

[8] Evgrafov's theorem, stated in terms of total systems, is given only for Ω simply connected and all α_n bounded on Ω. Also, our condition of absolute convergence is replaced in the hypotheses of Evgrafov by the existence of a rather special sort of biorthogonal system.

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} |A_{nk}| \frac{M_q(\alpha_{n+k})}{M_q(\alpha_n)} < 1 \qquad (q = 1, 2, \cdots), \tag{6.1}$$

then $\{\beta_n\}$ *is an absolutely convergent basis in* $\mathscr{A}(\Omega)$, *and there exists an automorphism on* $\mathscr{A}(\Omega)$ *carrying* α_n *into* β_n *for each* n.

A further specialization results in the theorem of Boas (cited in § 1) on Pincherle bases in spaces of functions analytic on the discs $N_R(0) = \{z: |z| < R\}$ $(0 < R \leq +\infty)$. It is convenient here to let the index n start with 0 and to put $\delta_n(z) = z^n$ $(n = 0, 1, \cdots)$.

COROLLARY 9.1. (*Boas*). *Let*

$$\alpha_n(z) = z^n\Big(1 + \sum_{k=1}^{\infty} A_{nk} z^k\Big) \qquad (n = 0, 1, \cdots),$$

where the A_{nk} *are complex numbers, define a sequence in* $\mathscr{A}(N_R(0))$. *If*

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} |A_{nk}| r^k < 1 \tag{6.2}$$

for each $r < R$, *then* $\{\alpha_n\}$ *is an absolutely convergent basis in* $\mathscr{A}(N_R(0))$, *and there exists an automorphism* A *on* $\mathscr{A}(N_R(0))$ *such that* $\alpha_n = A\delta_n$ $(n = 0, 1, \cdots)$.[9]

Returning to the case of general Fréchet spaces, we observe that the results of § 5 remain valid if we assume only that the conditions on the semi-norms are satisfied for infinitely many indices q. In fact, the topology on $\mathscr{F}$ obviously is not affected if we replace the initial sequence of semi-norms by any subsequence of it.

Finally, we remark that when the underlying space is a Banach space, Theorems 4 and 7 coalesce. The common theorem is, however, somewhat restricted in scope, since every Banach space admitting an absolutely convergent basis is isomorphic to the space l^1 of absolutely summable sequences (see Karlin [**18**, p. 974]).

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