ON THE LINE SEGMENTS OF A CONVEX SURFACE IN E_3

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1. Introduction. For integral $n \geqq 2$ let C be a bounded open convex subset of Euclidean n-space E_n, and let C' be the boundary (surface) of C. Let B_n be the closed unit ball in E_n, that is, the set of points x in E_n with $||x|| \leqq 1$, and let $S_{(n-1)}$ be the boundary of B_n, that is, the set of points x in E_n with $||x|| = 1$. Let D be the set of directions of straight line segments lying in C', specifically, the set of points $(a - b)/||a - b||$, where a and b are distinct points of a line segment lying in C'. Thus D is contained in $S_{(n-1)}$.

V. L. Klee [2] has stated that D is an F_σ set and has raised two questions: Is D of first category in $S_{(n-1)}$? Is D of $(n - 1)$-dimensional measure zero? Both of these questions are herein answered affirmatively for the case $n = 3$. The method employed unfortunately does not generalize to $n > 3$. (For $n = 2$ the case is trivial, for then D is countable. The case is also trivial if C is of dimension less than n, for then the $(n - 2)$-dimensional measure of D cannot be greater than the $(n - 2)$-dimensional measure of $S_{(n-2)}$ which is finite. The restriction to bounded sets is only a matter of convenience, for any answers to the questions posed are easily made to serve the unbounded case.)

In one sense though, for the case $n = 3$, we show somewhat more, namely, that D is contained in the union of the ranges of a countable family of Lipschitz functions each on B_1 to S_2. By virtue of the Lipschitz nature of these functions, they possess total differentials (Lebesgue measure) almost everywhere [4; straight forward generalizations of Definition 1, V. 2.2, and Lemmas 1 and 2, V. 2.3, to cover the case of a Lipschitz function on a domain contained in E_1 to E_3] and their ranges are compact and have finite one dimensional measure [1]. The affirmative answers to Klee's questions for this case immediately follow from these last two properties.

2. Preliminaries. We assume henceforth that $n = 3$.

Let a *flat side* of C' be a two dimensional intersection of C' with a plane supporting C'. It is easy to check that the set of flat sides is countable. (Check, for instance, that relative to C', the interior of each flat side is non-vacuous and, that no two such interiors intersect.) Thus the set of directions of line segments lying in flat sides is the union of a countable family of great circles lying on S_2 and can certainly be represented as the union of the ranges of an appropriately

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chosen countable family of Lipschitz functions on B_1 to S_2.

We go on to show that the set of directions of line segments not lying in flat sides can be similarly represented.

Let $\mathscr{L}$ be the set of closed line segments each of which is the middle third line segment contained in a maximal line segment of C' not lying in a flat side. Clearly $\mathscr{L}$ is disjointed, for if any two members intersected they would be forced by the convexity of C to lie in a flat side determined by the plane containing the two line segments.

Now choose a point a in C and let 2δ be the distance from a to C'. Let $\mathscr{K}$ be the family of open right circular cylinders of radius δ extending infinitely in two directions whose axis is a line radiating out from a infinitely in two directions. Thus each member of $\mathscr{K}$ intersects C' in a set open relative to C' and having two components. Let $\mathscr{M}$ be the set of all these components corresponding to all cylinders of $\mathscr{K}$.

Since $\mathscr{M}$ forms an open covering of the compact space C' we can reduce it to a finite subcovering $\mathscr{M}'$.

Now let $\mathscr{P}$ be the family of planes each of which intersects C and perpendicularly intersects a coordinate axis in a point with rational coordinates. Let $\mathscr{Q}$ be the family of pairs of distinct parallel members of $\mathscr{P}$.

Clearly every member of $\mathscr{L}$ intersects at least one member of $\mathscr{M}'$ and every such intersection intersects both planes of at least one pair in $\mathscr{Q}$.

Since $\mathscr{M}'$ is finite and $\mathscr{Q}$ is countable, we will have achieved our aim when we have shown that corresponding to each member m of $\mathscr{M}'$ and each pair (P_1, P_2) of planes in $\mathscr{Q}$ both intersecting m there exist two Lipschitz functions each on B_1 to S_2 whose ranges together contain the set of directions of the members of $\mathscr{L}$ each of which intersects both $m \cap P_1$ and $m \cap P_2$. With m, P_1, and P_2 fixed and letting $\mathscr{L}'$ be the set of members of $\mathscr{L}$ each intersecting both $m \cap P_1$ and $m \cap P_2$, we proceed to secure the required functions.

3. The Lipschitz direction functions. Let f be the set of all pairs (x, y) such that $x \in \lambda \cap P_1$ and $y \in \lambda \cap P_2$ for some $\lambda \in \mathscr{L}'$. Let A be the domain of f. Since $\mathscr{L}'$ is disjointed and since $\lambda \cap P_1$ and $\lambda \cap P_2$ are singletons we infer that f is a function. The key to the construction of the required functions lies in the

LEMMA. *f is Lipschitz.*
Momentarily leaving aside its proof, we first show how it is used to obtain these functions.

Drawing upon the lemma, we apply a method due to McShane [3; or 4, V. 2.4, Lemma 1] to get a Lipschitz extension f^* of f on the closure of $P_1 \cap m$, that is, a Lipschitz function f^* on the closure of $P_1 \cap m$ to P_2 that agrees with f on A.

We next let h be a function that assigns to each member x of the closure of $P_1 \cap m$ one of the directions of the line connecting x to $f^*(x)$, specifically for x in the closure of $P_1 \cap m$ we let

$$h(x) = \frac{f^*(x) - x}{||f^*(x) - x||} .$$

Upon checking that the difference of two Lipschitz functions is Lipschitz and that the ratio of a Lipschitz function whose values are bounded away from the origin (in our case bounded by the distance between P_1 and P_2) with its norm is Lipschitz, we infer that h is Lipschitz. It is easy to construct a Lipschitz homeomorphism g on B_1 onto the closure of $P_1 \cap m$. So finally upon defining functions k and k' on B_1 to S_2 to be such that for x in B_1

$$k(x) = h(g(x)), \quad k'(x) = -k(x) ,$$

and noting that the composition of Lipschitz functions is Lipschitz, we conclude that k and k' are Lipschitz and furthermore that their ranges together contain the set of directions of members of $\mathscr{L}'$. These are the functions we seek.

We now turn our attention to the lemma and close our discussion with its proof.

4. Proof of the Lemma. We show that f is Lipschitz by showing that it can be represented as the composition of Lipschitz functions. To do this let us project m perpendicularly onto a plane perpendicular to the axis of the cylinder in $\mathscr{K}$ associated with m. Let m' be the projected set and let p be the projecting function. Thus p is on m onto m'. From the convexity of C and the nature of the cylinder determining m we readily check that p is a Lipschitz homeomorphism on m onto m' whose inverse is also Lipschitz. For x' in $p(A)$ let $f'(x') = p(f(p^{-1}(x')))$. For x in A clearly $f(x) = p^{-1}(f'(p(x)))$. We have only to show that f' is Lipschitz.

Let λ_1 and λ_2 be two members of $\mathscr{L}'$. Let $x_1 \in \lambda_1 \cap P_1$ and $x_2 \in \lambda_2 \cap P_1$. Let l_1 and l_2 be maximal line segments contained in C' containing respectively λ_1 and λ_2. Let l_1' and l_2' be the respective perpendicular projections of l_1 and l_2 onto the plane of m'. Clearly l_1 and l_2 fail to intersect or intersect only in an end point of both l_1 and l_2. Consequently the same is true of l_1' any l_2'. If l_1' and l_2' are parallel or, when extended, intersect on the side of P_2 opposite from P_1, then clearly

$$||p(f(x_1)) - p(f(x_2))|| \leqq ||p(x_1) - p(x_2)|| . \tag{1}$$

If, on the other hand, l_1' and l_2', when extended, intersect in a point b, on the same side of P_2 that P_1 lies on, then either an end point of l_1' lies at b or between b and P_1, or an end point of l_2' lies at b or between b and P_1. We may assume the first of these two main disjunctions without loss of generality. Now since the line segment connecting $p(x_1)$ with $p(f(x_1))$ is contained in the middle third segment of l_1', we have

$$||p(f(x_1)) - p(x_1)|| \leqq ||p(x_1) - b|| .$$

and hence

$$(2) \qquad ||p(f(x_1)) - b|| = ||p(f(x_1)) - p(x_1)|| + ||p(x_1) - b|| \leqq 2||p(x_1) - b|| .$$

As P_1 and P_2 are parallel, we may use a property of similar triangles to get

$$(3) \qquad \frac{||p(f(x_1)) - p(f(x_2))||}{||p(x_1) - p(x_2)||} = \frac{||p(f(x_1)) - b||}{||p(x_1) - b||} .$$

Combining (2) and (3) we get

$$(4) \qquad ||p(f(x_1)) - p(f(x_2)) || \leqq 2||p(x_1) - (x_2)|| .$$

Since equations (1) and (4) show that for any x_1' and x_2' in the domain of f'

$$||f'(x_1') - f'(x_2')|| \leqq 2|| x_1' - x_2' || ,$$

and hence that f' is Lipschitz, our proof is complete.

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