LINEAR MAPS ON SKEW SYMMETRIC MATRICES: THE INVARIANCE OF ELEMENTARY SYMMETRIC FUNCTIONS

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1. Introduction. Let S_n be the space of n-square skew symmetric matrices over the field F of real numbers. Let $E_{2k}(A)$ denote the sum of all $2k$-square principal subdeterminants of $A \in S_n$ (the elementary symmetric function of degree $2k$ of the eigenvalues of A). It is classical that if U is an n-square real orthogonal matrix and $A \in S_n$ then $UAU' \in S_n$ and moreover for each k

$$E_{2k}(UAU') = E_{2k}(A) . \tag{1.1}$$

The correspondence

$$A \to UAU' \tag{1.2}$$

for a fixed orthogonal U can then be regarded as a linear transformation on S_n onto itself that holds $E_{2k}(A)$ invariant. The question we consider here is the following: to what extent does the fact that (1.1) holds for some k characterize the map (1.2). In other words, we obtain (Theorem 3) the complete structure of those linear maps T of S_n into itself that for some $k > 1$ satisfy $E_{2k}(T(A)) = E_{2k}(A)$ for each $A \in S_n$. Our results are made to depend on the structure of linear maps of the second Grassmann product space $\bigwedge^2 U$ of a vector space U over F into itself.

K. Morita [2] examined the structure of those maps T of S_n into itself that hold invariant the dominant singular value $\alpha(A)$ of each $A \in S_n$. We recall that $\alpha(A)$ is the largest eigenvalue of the non-negative Hermitian square root of A^*A. Morita shows that if $\alpha(T(A)) = \alpha(A)$ for each $A \in S_n$ then T has essentially the form given in our Theorem 3.

2. Some definitions and preliminary results. Let U be a finite dimensional vector space of dimension n over F. Let $G_2(U)$ denote the space of all alternating bilinear functionals on the cartesian product $U \times U$ to F. Then the dual space $\bigwedge^2 U$ of $G_2(U)$ is called the second Grassmann product space of U. If x_1 and x_2 are any two vectors in U then $f = x_1 \wedge x_2 \in \bigwedge^2 U$ is defined by the equation

$$f(w) = w(x_1, x_2) , \quad w \in G_2(U) .$$

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Some elementary properties of $x_1 \wedge x_2$ are:

(i) $x_1 \wedge x_2 = 0$ if and only if x_1 and x_2 are linearly dependent.

(ii) if $x_1 \wedge x_2 = y_1 \wedge y_2 \neq 0$ then $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$ where $\langle x_1, x_2 \rangle$ is the space spanned by x_1 and x_2.

If A is a linear map of U into itself we define $C_2(A)$, the *second compound* of A, as a linear map of $\bigwedge^2 U$ into $\bigwedge^2 U$ by

$$C_2(A) x_1 \wedge x_2 = Ax_1 \wedge Ax_2 \,. \tag{2.1}$$

We remark that if $x_1, \cdots, x_n$ is a basis of U then $x_i \wedge x_j, 1 \leq i < j \leq n$ is a basis of $\bigwedge^2 U$ and hence (2.1) defines $C_2(A)$ by linear extension.

We first show that $\bigwedge^2 U$ is isomorphic in a natural way to S_n and under this isomorphism second compounds correspond to congruence transformations in S_n.

Specifically, let $\alpha_1, \cdots, \alpha_n$ be a basis of U and define φ by

$$\varphi(\alpha_i \wedge \alpha_j) = E_{ij} - E_{ji} \in S_n \tag{2.2}$$

where E_{ij} is the n-square matrix with 1 in position i, j and 0 elsewhere and extend φ linearly to all of $\bigwedge^2 U$. It is obvious that φ is an isomorphism since $E_{ij} - E_{ji}, 1 \leq i < j \leq n$ is a basis of S_n. Let T be a linear map of $\bigwedge^2 U$ into itself and define S, a linear map of S_n into itself, by

$$S(A) = \varphi T \varphi^{-1}(A), A \in S_n \,. \tag{2.3}$$

Let B be a linear map of U into itself. Then

THEOREM 1. *$T = C_2(B)$ if and only if $S(A) = B_1 A B_1'$ where B_1 is the matrix of B with respect to the ordered basis $\alpha_1, \cdots, \alpha_n$.*

Proof. Suppose $T = C_2(B)$. Then for $i < j$

$$\begin{aligned} S(E_{ij} - E_{ji}) &= \varphi T \varphi^{-1}(E_{ij} - E_{ji}) \\ &= \varphi(B\alpha_i \wedge B\alpha_j) \\ &= \varphi\Big(\sum_{k=1}^{n} b_{ki}\alpha_k \wedge \sum_{k=1}^{n} b_{kj}\alpha_k\Big) \\ &= \sum_{s,t} b_{si} b_{tj} (E_{st} - E_{ts}) \\ &= B_1(E_{ij} - E_{ji})B_1' \,. \end{aligned}$$

The implication in the other direction is similar.

Let L_{2r} denote the set of rank $2r$ matrices in S_n and let Ω_{2r} denote the set of vectors $\sum_{i=1}^{r} x_i \wedge y_i$ in $\bigwedge^2 U$ where $\dim < x_1, \cdots, x_r, y_1, \cdots, y_r > = 2r$.

THEOREM 2. $\varphi(\Omega_{2r}) = L_{2r}$

Proof. Let

$$z = \sum_{i=1}^{r} x_i \wedge y_i \in \Omega_{2r} .$$

Choose a non-singular map B of U onto U such that $B\alpha_{2j-1} = x_j$ and $B\alpha_{2j} = y_j$, $j = 1, \cdots, r$. Then

$$z = C_2(B) \sum_{j=1}^{r} \alpha_{2j-1} \wedge \alpha_{2j} ,$$

so

$$\varphi(z) = \varphi C_2(B) \sum_{j=1}^{r} \alpha_{2j-1} \wedge \alpha_{2j} . \tag{2.4}$$

Let $S(A) = B_1 A B_1'$ for $A \in S_n$ where B_1 is the matrix of B with respect to the ordered basis $\alpha_1, \cdots, \alpha_n$. Then by Theorem 1, $\varphi C_2(B)\varphi^{-1} = S$ and from (2.4) we have

$$\begin{aligned}\varphi(z) &= S\varphi \sum_{j=1}^{r} \alpha_{2j-1} \wedge \alpha_{2j} \\ &= S\Big(\sum_{j=1}^{r} (E_{2j-1,2j} - E_{2j,2j-1})\Big) \\ &= B_1\Big(\sum_{j=1}^{r} (E_{2j-1,2j} - E_{2j,2j-1})\Big)B_1' \in L_{2r} .\end{aligned}$$

The implication in the other direction is a reversal of this argument.

We see then that a map T of $\bigwedge^2 U$ into itself is a second compound of some linear map of U into itself if and only if $\varphi T \varphi^{-1}$ is a congruence map of S_n; and $T(\Omega_{2r}) \subseteq \Omega_{2r}$ if and only if $\varphi T \varphi^{-1}(L_{2r}) \subseteq L_{2r}$.

3. E_{2k} preservers. Let S be a linear map of S_n into itself such that $E_{2k}(S(A)) = E_{2k}(A)$ for all $A \in S_n$, where k is a fixed integer, $k \geq 2$. Then

LEMMA 1. *S is non-singular.*

Proof. Suppose $S(A) = 0$. Then

$$E_{2k}(A + X) = E_{2k}(S(A + X)) = E_{2k}(S(X)) = E_{2k}(X) \quad \text{for all } X \in S_n . \tag{3.1}$$

Obtain a real orthogonal P such that

$$PAP' = \sum_{i=1}^{r} \begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix} \dotplus 0_{n-2r} \tag{3.2}$$

where 0_{n-2r} is an $(n - 2r)$-square matrix of zeros and $\rho(A) = \text{rank } A = 2r$.

Here $\sum^{\cdot}$ and $\dot{+}$ indicate direct sum. Now if $\rho(A) \geq 2k$ simply set $X = 0$ and from (3.1) and (3.2) we see that

$$0 < E_{2k}(A) = E_k(\theta_1^2, \cdots, \theta_r^2) = E_{2k}(0) = 0$$

a contradiction. On the other hand, if $\rho(A) < 2k$ select $X \in S_n$ such that

$$PXP' = 0_{2r} \dot{+} \sum_1^{(k-r)\cdot} (E_{12} - E_{21}) \dot{+} 0_{n-2k}$$

where E_{12} is a 2-square matrix. Then

$$E_{2k}(A + X) = E_{2k}(PAP' + PXP') = \prod_{j=1}^{r} \theta_j^2 .$$

But $E_{2k}(PXP') = E_{2k}(X) = 0$, since $k - r < k$. Hence the proof is complete.

LEMMA 2. *If $A \in S_n$ and* deg $E_{2k}(xA + B) \leq 2$ *for all $B \in S_n$ and $A \neq 0$ then $\rho(A) = 2$.*

Proof. Suppose $\rho(A) = 2r$ and select a real orthogonal P such that PAP' has the form given in (3.2). Select B such that

$$PBP' = 0_{2r} \dot{+} \sum_2^{[\frac{n}{2}]-r\cdot} (E_{12} - E_{21}) \dot{+} C$$

where if n is even C doesn't appear and if n is odd C is a 1-square zero matrix.

Now if $k \leq r$

$$E_{2k}(xA + B) = x^{2k}E_k(\theta_1^2, \cdots, \theta_r^2) + \text{lower order terms in } x.$$

If $k > r$
$E_{2k}(xA + B) = \binom{[n/2] - r}{k - r}\theta_1^2 \cdots \theta_r^2 x^{2r}$ + lower order terms in x. Thus

$$\deg E_{2k}(xA + B) \text{ is either } 2k \text{ or } 2r.$$

But this implies $2r = 2$ and $\rho(A) = 2$.

LEMMA 3. *If $E_{2k}(S(A)) = E_{2k}(A)$ for all $A \in S_n$ then $S(L_2) \subseteq L_2$.*

Proof. Let $p(x)$ be the polynomial $E_{2k}(xA + B)$. Then if $\rho(A) = 2$ it is easy to check that $\deg p(x) \leq 2$ for all $B \in S_n$. Hence $\deg E_{2k}(xS(A) + S(B)) \leq 2$ for all $B \in S_n$. But S is non-singular by Lemma 1 and thus by Lemma 2, $\rho(S(A)) = 2$.

THEOREM 3. *If $E_{2k}(S(A)) = E_{2k}(A)$ for all $A \in S_n$, where k is a fixed integer satisfying $4 \leq 2k \leq n$ and $n \geq 5$ then there exists a real matrix P such that*

$$S(A) = \alpha PAP' \text{ for all } A \in S_n \tag{3.3}$$

where $\alpha PP' = I$ if $2k < n$ and $\alpha PP'$ is unimodular if $2k = n$. If $2k = n = 4$ then either S has the form (3.3) or

$$S(A) = \alpha P \begin{pmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{pmatrix} P' \tag{3.4}$$

where $A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$ *and $\alpha PP'$ is unimodular.*

Proof. By Lemma 1, S^{-1} exists and we check that

$$E_{2k}(S^{-1}(A)) = E_{2k}(SS^{-1}(A)) = E_{2k}(A) ,$$

for any $A \in S_n$. Hence by Lemma 3

$$S^{-1}(L_2) \subseteq L_2 \text{ and thus } S(L_2) = L_2 .$$

Now define T, a mapping of $\bigwedge^2 U$ into itself, by (2.3)

$$T = \varphi^{-1} S \varphi .$$

By Theorem 2

$$\begin{aligned} T(\Omega_2) &= \varphi^{-1} S \varphi(\Omega_2) \\ &= \varphi^{-1} S(L_2) \\ &= \varphi^{-1}(L_2) \\ &= \Omega_2 . \end{aligned}$$

At this point we invoke a theorem of Chow [**1**, pp. 38]. Let T'' be the mapping of 2-dimensional subspaces of U into themselves induced by T; that is, let $T''(\langle x, y \rangle) = \langle u, v \rangle$ whenever $T(x \wedge y) = u \wedge v$, (assuming of course that x and y are linearly independent). Then T'' is well defined and it follows from the above that it is a one-to-one onto adjacence preserving transformation: if two 2-dimensional subspaces of U intersect in a subspace of dimension 1 then their images under T'' intersect in a subspace of dimension 1. Therefore T'' is induced either by a correlation or a collineation of the subspaces of U. If $\dim U \geq 5$

T'' is induced by a collineation. If $\dim U = 4$ and if T'' is induced by a correlation then $(TT_1)''$ is induced by a collineation. Here T_1 maps $\bigwedge^2 U$ into itself and satisfies

$$T_1(x_i \wedge x_j) = x_l \wedge x_m ,\qquad \{i, j, l, m\} = \{1, 2, 3, 4\} \text{ and } i < j, l < m . \tag{3.5}$$

Now, assuming T'' is induced by a collineation we show that

$$T = \alpha C_2(P) \tag{3.6}$$

for some $\alpha \in F$ and some linear transformation $P: U \rightarrow U$. The fundamental theorem of projective geometry states that there is a one-to-one semi-linear transformation $Q: U \rightarrow U$ such that

$$T''(\langle x, y\rangle) = \langle Qx, Qy\rangle . \tag{3.7}$$

Let $x_1, \cdots, x_n$ be a basis of U and let $Qx_i = y_i$. Then

$$T(x_i \wedge x_j) = \alpha_{ij} y_i \wedge y_j \quad \alpha_{ij} \in F . \qquad 1 \leq i,\ j \leq n,\ i \neq j .$$

Then for s, k, t distinct integers in $1, \cdots, n$ and $K \in F$.

$$\begin{aligned} T((x_s + x_t) \wedge x_k) &= K(Q(x_s + x_t) \wedge Qx_k) \\ &= K(y_s + y_t) \wedge y_k , \end{aligned}$$

But

$$\begin{aligned} T((x_s + x_t) \wedge x_k) &= T(x_s \wedge x_k) + T(x_t \wedge x_k) \\ &= (\alpha_{sk} y_s + \alpha_{tk} y_t) \wedge y_k . \end{aligned}$$

Hence $\alpha_{sk} = \alpha_{tk}$ and thus $\alpha_{sk} = \alpha_{tk} = \alpha_{kt} = \alpha_{rt} = \alpha$ for any four distinct integers s, k, r, t. Hence

$$T(x_i \wedge x_j) = \alpha y_i \wedge y_j = \alpha C_2(P) x_i \wedge x_j ,$$

where $P: U \rightarrow U$ is a linear transformation with $Px_j = y_i$. Since $\{x_i \wedge x_j \mid 1 \leq i < j \leq n\}$ is a basis of $\bigwedge^2 U$, $T = \alpha C_2(P)$.
Now by Theorem 1,

$$S(A) = \alpha PAP' \text{ for all } A \in S_n$$

for $n \geq 5$ where P is an n-square non-singular matrix. If $2k = n$ then clearly $\alpha PP'$ is unimodular. Hence assume $2k < n$.
We next show that

$$\alpha PP' = I .$$

From the hypothesis,

$$E_{2k}(\alpha PAP') = E_{2k}(A),\ A \in S_n$$

and hence

$$\alpha^{2k} tr\{C_{2k}(PP')C_{2k}(A)\} = trC_{2k}(A)\ .$$

By the polar factorization theorem let $P = UB$, where U is real orthogonal and B is positive definite symmetric. Let $B = VDV'$, D diagonal with positive entries and V real orthogonal. Then since $V'AV$ runs through all of S_n as A does we have

$$\alpha^{2k} tr\{C_{2k}(D^2)C_{2k}(A)\} = trC_{2k}(A)\ . \tag{3.9}$$

We assert that any diagonal $\binom{n}{2k}$-square matrix is a linear combination of matrices $C_{2k}(A)$ for $A \in S_n$. For, let $1 \le i_1, < \cdots < i_{2k} \le n$. Let $A \in S_n$ and consider the $2k$-square principal submatrix B of A where

$$B_{\alpha\beta} = A_{i_\alpha i_\beta}\ ;$$

and suppose A has 0 entries outside of B. Then define B as follows:

$$B_{2k-\alpha,\alpha+1} = -1\ ,\quad \alpha = 0, \cdots, k-1$$
$$B_{2k-\alpha,\alpha+1} = 1\ ,\quad \alpha = k, \cdots, 2k$$

and $B_{ij} = 0$ elsewhere. Then $C_{2k}(A) = \pm E_{i_1\cdots i_{2k}}$, where $E_{i_1\cdots i_{2k}}$ is the $\binom{n}{2k}$-square matrix with the single non-zero entry 1 in the $((i_1, \cdots, i_{2k}), (i_1, \cdots, i_{2k}))$ position ordered doubly lexicographically in the indices of the rows and columns of A. Returning to (3.9) we have

$$tr\{C_{2k}(\alpha D^2)X\} = trX$$

for all $\binom{n}{2k}$-square diagonal matrices X and hence $C_{2k}(\alpha D^2) = I$, $\alpha D^2 = \pm I$. From this we easily see that

$$\alpha PP' = I\ ,$$

and (3.3) follows. The mapping T_1 on $\bigwedge^2 U$ induces the map S^1 on S_4 where

$$S^1\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{12} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{pmatrix}$$

This completes the proof.

We remark that Theorem 3 is no longer valid if $k = 1$: for consider the transformation which interchanges positions (i, j) and (j, i) in A for a fixed pair of integers $1 \le i < j \le n$. This clearly preserves $E_2(A)$ but does not have the form in Theorem 3. For example

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

is non-singular but interchanging the 1, 2 and 2, 1 entries results in a singular matrix.

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