INFINITELY REPEATABLE GAMES

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1. Introduction. Blackwell [1] has introduced the concept of approachability in obtaining an analog of the von Neumann minimax theorem for games with vector payoffs. This paper continues the study of this concept. Games with vector payoffs are again two person decision problems with each player having r and s pure strategies respectively but the element of the payoff matrix corresponding to the (i, j) strategy pair is a point $g(i, j)$ in Euclidean N-space. Let C_G denote the convex hull of the rs points $g(i, j)$. Then the problem studied in approachability theory can be stated briefly as follows. If a game with vector payoffs is repeated in time can player I force the average payoff to approach a preassigned closed subset S of C_G with probability approaching 1 as the number of plays becomes infinite?

Because a sequence of games is being considered the rules of play must specify to what extent a player's decision at any stage may depend on past plays. This leads to the natural question of how the class of approachable sets depends on the type of information available to player I. It is specifically this question that is considered in this paper. The problem is formulated mere precisely below.
Let

$$G = \|g(i, j)\|, \qquad 1 \leqq i \leqq r,\ 1 \leqq j \leqq s$$

be an $r \times s$ matrix each element of which is a point in Euclidean N-space and let

$$\mathscr{F} = \|e_{(i,j),k}\|\,, \quad 1 \leqq i \leqq r\,,\quad 1 \leqq j \leqq s\,,\quad 1 \leqq k \leqq t$$

denote an $rs \times t$ matrix such that $0 \leqq e_{(i,j),k}$ (for all $1 \leqq i \leqq r$, $1 \leqq j \leqq s$, $1 \leqq k \leqq t$) and $\sum_{k=1}^{t} e_{(i,j),k} = 1$ (for all $1 \leqq i \leqq r$, $1 \leqq j \leqq s$). A pair $(G, \mathscr{F})$ will determine a game as follows. By a strategy for player I is meant a sequence $f = \{f_n : n = 0, 1, 2, \cdots\}$ of functions where f_n, for $n = 1, 2, \cdots$, is a mapping from the set of n − tuples $(a_1, a_2, \cdots, a_n)$, $a_i \in \{1, 2, \cdots, t\}$, to the set $P = \{(p_1, \cdots, p_r) | 1 \leqq p_i, \sum_1^r p_i = 1\}$, and f_0 is a point in P. A strategy for player II is a sequence of vectors $h = \{h_n : n = 0, 1, 2, \cdots\}$ where $h_n \in Q = \{(q_1, \cdots, q_s) | 0 \leqq q_j, \sum_1^s$

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$q_j = 1\}$ for $n = 0, 1, 2, \cdots$. The interpretation of the play of the game $(G, \mathscr{I})$ is that player I selects a number, say i_1, from $\{1, \cdots, r\}$ according to the mixed strategy f_0 and player II selects a number, say j_1, from $\{1, \cdots, s\}$ according to the strategy h_0. The pair (i_1, j_1) is observed by a referee who employs the distribution $\{e_{(i_1,j_1)1}, \cdots, e_{(i_1,j_1),t}\}$ to choose a number, denoted a_1, from $\{1, \cdots, t\}$. The number a_1 is precisely the information told to player I at the conclusion of the first play of the game and he then chooses i_2 by means of the mixed strategy $f_1(a_1)$ while player II chooses j_2 with strategy h_1. The referee selects a_2 according to $\{e_{(i_2,j_2)},_1, \cdots, e_{(i_2,j_2)},_t\}$ and a_2 is told to player I who now employs a third mixed strategy $f_2(a_1, a_2)$ to choose i_3, etc. Thus a pair $(G, \mathscr{I})$ together with a fixed pair of strategies (f, h) defines a vector-valued stochastic process $\{Y_n : n = 1, 2, \cdots\}$ with $\{g_{(i_n, j_n)} : n = 1, 2, \cdots\}$ being a realization of the process for a particular play of the game.

Let C_G denote the convex hull of the rs points $g(i, j)$ and let S denote any closed set in C_G. S is said to be *approachable* with f^* in $(G, \mathscr{I})$ if for every h

$$P\{\lim_n \delta_n = 0\} = 1$$

where δ_n denotes the distance of the point $\sum_1^n Y_i/n$ from S and $\{Y_i : i = 1, 2, \cdots\}$ is the vector-valued process determined by f^* and h.

In § 2 necessary and sufficient conditions are obtained for a set to be approachable when player I obtains no information concerning II's choices. In § 3 sufficient conditions for approachability are given in the case when I knows nothing of his own past but is completely informed of II's past history. For convex S the condition is both necessary and sufficient. Section 4 contains necessary and sufficient conditions for the approachability of convex S in the case that the rank of $\mathscr{I}$ is equal to rs.

2. No information relevant to player II. It is clear that the minimal class of approachable sets is obtained when the rank of $\mathscr{I}$ is one (the case of no information) and the result of this section is that this class is made no larger if player I receives information concering only his own past play.

For any $p \in P$ denote by $R(p)$ the convex hull of the s points $\sum_i p_i g_{i,j}$. It is an immediate consequence of the Strong Law of Large Numbers that $R(p)$ is approachable with the strategy $f^* = \{f_n \equiv p: n=0, 1, \cdots\}$ and thus approachable in the case of rank $(\mathscr{I}) = 1$. The theorem of this section is that the collection of $R(p)$'s is essentially the totality to approachable sets when nothing is known concerning player II's past play.

THEOREM 1. *Let* rank $(\mathscr{F}) \leqq r$ *and*

$$\operatorname{rank}\left(\begin{Vmatrix} e_{(i,1),1} & \cdots & e_{(i.1),t} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ e_{(i,s),1} & \cdots & e_{(i,s),t} \end{Vmatrix}\right) = 1$$

for $i = 1, \cdots, r$. *Then a closed set* S *is approachable if and only if there exists* $p \in P$ *such that* $R(p) \subseteq S$.

Proof. The sufficiency is an immediate consequence of the Strong Law of Large Numbers.

To prove the necessity of the condition suppose that S is approachable. Let f^0 denote a strategy for player I with which S is approachable. The strategy f^0 induces a vector-valued stochastic process $\{X_n = (X_{1n}, \cdots, X_{rn}) : n = 1, 2, \cdots\}$ with

$$X_{kn} = \begin{cases} 1 \text{ if } i_n = k \\ 0 \text{ otherwise} \end{cases}$$

for $k = 1, 2, \cdots, r ; n = 1, 2, \cdots$. Consider the process

$$\{\overline{X}_n = (1/n \textstyle\sum_{i=1}^n X_{1i}, \cdots, 1/n \sum_{i=1}^n X_{ri}) : n = 1, 2, \cdots\};$$

it follows from Cantor's theorem that there exists $p_0 \in P$ such that for any $\varepsilon > 0$

$$P\{d(\overline{X}_n, p_0) < \varepsilon\, i.0.\} > 0 .$$

The proof will be completed by showing that $R(p_0) \subseteq S$.

Suppose $R(p_0) \not\subseteq S$, then there exists a positive number ε_0 and $q_0 \in Q$ such that $C(z_0, \varepsilon_0)$, the sphere of radius ε_0 and center z_0 is disjoint from S where $z_0 = \sum_{i=1}^r \sum_{j=1}^s p_{i,0}\, q_{j,0}\, g_{ij}$ and $p_0 = (p_{1,0}, \cdots, p_{r,0})$, $q_0 = (q_{1,0}, \cdots, q_{s,0})$. Let h^0 denote the strategy for player II defined by $h^0 = \{h_n^0 \equiv q_0 : n = 0, 1, 2, \cdots\}$. h^0 induces a vector process $\{W_n = (W_{1n}, \cdots, W_{sn}) : n = 1, 2, \cdots\}$ where

$$W_{kn} = \begin{cases} 1 \text{ if } j_n = k \\ 0 \text{ otherwise} . \end{cases}$$

Now it is a consequence of the Strong Law of Large Numbers that for an arbitrary positive ε

$$P\Big\{d\Big[\Big(\frac{1}{n}\sum_{i=1}^n X_{1i}\, W_{1i}, \frac{1}{n}\sum_{i=1}^n X_{2i} W_{1i}, \cdots, \frac{1}{n}\sum^n X_{ri}\, W_{si}\Big), \\ (p_{1,0}q_{1,0}, p_{2,0}\, q_{1,0}, \cdots, p_{r,0}q_{s,0})\Big] < \varepsilon\, i.\ o.\Big\} > 0 .$$

Therefore,

$$P\{\underline{\lim}\, d\Big(\frac{1}{n}\sum_{1}^{n} Y_i, C(z_0, \varepsilon_0)\Big) = 0\Big\} > 0$$

and thus $P\{\lim \delta_n = 0\} < 1$ where $\delta_n = d(1/n \sum_1^n Y_i, S)$ and $\{Y_i : i = 1, 2, \cdots\}$ is the vector-valued process determined by f^0 and h^0. Thus $R(p_0) \subseteq S$ and the proof is complete.

It is worthwhile to note that this theorem remains true if player II's class of strategies is restricted to strategies which are sequences of pure strategies, that is, if $h = \{h_n : n = 0, 1, 2, \cdots,\}$ is a strategy for II, then all components of h_n are zero with a single exception which is one. This restricted class of straregies for player II is essentially the smallest class for which the theorem remains true.

3. Complete information about player II.

If player I is informed of the complete past history of player II's choice but receives no information concerning his own past play the class of approachable sets is greatly increased.

THEOREM 2 *Let* rank $(\mathscr{F}) = s$,

$$\text{rank}\left(\left\|\begin{matrix} e_{(1,j),1} & \cdots & e_{(1,j),t} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ e_{(r,j),1} & \cdots & e_{(r,j),t} \end{matrix}\right\|\right) = 1$$

for $j = 1, 2, \cdots, s$ *and finally assume* $\sum_{k=1}^{t} e_{(i,j),k}\, e_{(u,v)\,k} = 0$ *for all* $u \neq i$ *and all* v *and* j. *Then a closed set* S *is approachable if for every* $x \notin S$, $x \in C_G$, *there exists* $p \in P$ *such that the plane through* y, *the closest point in* S *to* x, *perpendicular to the line segment* xy *separates* x *from* $R(p)$.

Proof. Let S be an arbitrary closed set satisfying the hypothesis of the theorem. The proof consists in exhibiting a strategy f^* for player I with which S is approachable. By hypothesis if $x \notin S$ there exists at least one $p \in P$ such that x is separated from $R(p)$; thus player I can associate a unique "separating p" to each x, say $p(x)$. Further, because of the structure of $\mathscr{F}$ the sequences $\{a_n : n = 1, 2, \cdots\}$ and $\{j_n : n = 1, 2, \cdots\}$ may be identified and $f_n^*(j_1, \cdots, j_n)$ will be written for $f_n^*(a_1, \cdots, a_n)$.

The strategy f^* for player I is now defined as follows:

$$f_0^* = \Big(\frac{1}{r}, \frac{1}{r}, \cdots, \frac{1}{r}\Big)$$

$$f_n^*(j_1, \cdots, j_n) = \begin{cases} \left(\frac{1}{r}, \cdots, \frac{1}{r}\right) \text{ if } \bar{z}_n \equiv \frac{1}{n} \sum_1^n r_k \in S \\ \qquad\qquad\qquad\qquad\qquad\qquad n = 1, 2, \cdots \\ p(\bar{z}_n) \quad \text{if } \bar{z}_n \notin S \end{cases}$$

where $z_k = \sum_{i=1}^r f_{k-1,i}^* g_{ij_k}$ and $f_k^*(j_1, \cdots, j_k) \equiv (f_{n,1}^*, \cdots, f_{k,r}^*) k=0, 1, \cdots$. To contruct $f_n^*(j_1, \cdots, j_n)$ player 1 has as information $(j_1, \cdots, j_n)$ and thus since a unique $p(x)$ has been associated to every $x \notin S$ it is possible for player I to reconstruct $f_0^*, \cdots, f_{n-1}^*$ and hence this strategy is well defined.

Let $\{Y_n \equiv (Y_{n,1}, \cdots, Y_{n,N})\, n = 1, 2, \cdots\}$ be the vector process generated by f^* and some arbitrary strategy h for player II. Then as mentioned previously h generates a stochastic process $\{W_n : n = 1, 2, \cdots\}$. Denote by $w = (w_1, w_2, \cdots)$ an arbitrary sample sequence of this process. The proof will be completed if it is shown that $P\{\lim_n d(1/n \sum_1^n Y_k, S) = 0 | w\} = 1$ for arbitrary w.

Now note that for fixed w the random variables $Y_{n,k}$ and $Y_{m,k}$ are stochastically independent for $n \neq m$ and $k = 1, 2, \cdots, N$ with mean values $z_{n,k}$ and $z_{m,k}$ respectively. Thus, it is an immediate consequence of the Strong Law of Large Numbers that it is sufficient to prove that $\lim d(\bar{z}_n, S) = 0$ to complete the proof of the theorem.

Suppose $\bar{z}_n \notin S$ and let u_n denote the point in S closest to $\bar{z}_n$. Then $(u_n - \bar{z}_n, z_{n+1}) \geqq (u_n - \bar{z}_n, u_n)$ and if $\delta_n \equiv d^2(\bar{z}_n, S) > 0$ it follows that

$$\delta_{n+1} \leqq |\bar{z}_{n+1} - u_u|^2 = |\bar{z}_n - u_n|^2 + 2(\bar{z}_n - u_n, \bar{z}_{n+1} - \bar{z}_n) + |\bar{z}_{n+1} - \bar{z}_n|^2.$$

However, $\bar{z}_{n+1} - \bar{z}_n = (z_{n+1} - \bar{z}_n)/n$ and thus,

$$(\bar{z}_n - w_n, \bar{z}_{n+1} - \bar{z}_n) = \frac{(\bar{z}_n - u_n, z_{n+1} - u_n) + (\bar{z}_n - u_n, u_n - \bar{z}_n)}{n+1}.$$

Further, $|\bar{z}_{n+2} - \bar{z}_n|^2 \leqq A/(n+1)^2$, where A is some constant, and thus if $\delta_{n-1} > 0$ it follows that

(a) $\delta_n \leqq (1 - 2/n)\delta_{n-1} + A/n^2$. Also since C_G is bounded

(b) $0 \leqq \delta_n \leqq B$ and

(c) $|\delta_n - \delta_{n-1}| \leqq D/n$ where B and D are constants. However, if $\{\delta_n : n = 1, 2, \cdots\}$ is a sequence of real numbers satisfying (a), (b), and (c) it is quite easy to prove that $\lim_n \delta_n = 0$. Thus the proof is complete.

THEOREM 3. *Let $T(q)$, $q \in Q$, denote the convex hull of the r points $\sum_{j=1}^s q_j g_{ij}$. Let $\mathscr{S}$ satisfy the same conditions as in Theorem 2. Then a closed cenvex set S is approachable if and only if it intersects every $T(q)$.*

The proof of this theorem is given in [1] and will be omitted.

4. rank $(\mathscr{J}) = rs$. The theorem of this section was obtained in [1] for the case of $\mathscr{J}$ equal to the identity matrix.

THEOREM 4. *Suppose rank* $(\mathscr{J}) = rs$, *then a closed convex set* S *is approachable if and only if it intersects every* $T(q)$.

Proof. The necessity is clear. If $S \cap T(q_0)$ is empty player II chooses $h^0 = \{h_n^0 \equiv q_0 : n = 0, 1, \cdots\}$ and it is clear that $P\{\lim_n d(1/n \sum_1^n Y_k, T(q_0)) = 0\} = 1$ where $\{Y_k : k = 1, 2, \cdots\}$ is generated by h^0 and any arbitrary strategy for player I. Thus since $S \cap T(q_0)$ is empty S is not approachable.

Conversely let S be an arbitrary closed convex subset of C_G satisfying the hypothesis of the theorem. Define the $r \times s$ matrix $L = ||l_{ij}||$ as follows

$$l_{ij} = (\delta_{1i}\delta_{1j}, \delta_{1j}\delta_{2i}, \cdots, \delta_{1i}\delta_{sj}, \delta_{2i}\delta_{1j}, \cdots, \delta_{2i}\delta_{sj}, \cdots, \delta_{ri}\delta_{1j}, \cdots, \delta_{ri}\delta_{sj}) \in E^{rs}$$

where $1 \leqq i \leqq r$, $1 \leqq j \leqq s$, and δ_{uv} is the Kronecker delta. Define $S_L = \{\sum_{i=1}^r \sum_{j=1}^s \alpha_{ij} l_{ij} | \sum_{i=1}^r \sum_{j=1}^s \alpha_{ij} g_{ij} \in S, 0 \leqq \alpha_{ij} \text{ and } \sum_{i=1}^r \sum_{j=1}^s \alpha_{ij} = 1\}$, then S_L is a closed convex subset of C_L and S_L intersects $T_L(q)$ for all $q \in Q$, where $T_L(q)$ is the convex hull of the r points $\sum_{j=1}^s q_j l_{ij}$. Further, it follows after some simple computations that if S_L is approachable in $(L, \mathscr{J})$ then S is approachable in $(G, \mathscr{J})$ and in fact approachable with the same strategy. Thus to complete the proof of the theorem one need only show that every closed convex subset of C_L is approachable in $(L, \mathscr{J})$ if it intersects $T_L(q)$ for all $q \in Q$.

Let S be an arbitrary closed convex subset of C_L and suppose $S \cap T_L(q)$ is nonempty for all $q \in Q$. Further, suppose $t = rs$ this can be done with no loss of generality. Define the matrix $L' = ||e_{(ij)}||$ $(1 \leqq i \leqq r, 1 \leqq j \leqq s)$ where $e_{(ij)}$ denotes the probability distribution over E^{rs} choosing l_{11} with probability $e_{(ij),1}$, l_{12} with probability $e_{(ij),2}, \cdots, l_{rs}$ with probability $e_{(ij),rs}$. Let $\bar{e}_{(ij)}$ denote the mean value of the distribution $e_{(ij)}$ and $\overline{L}'$ denote the matrix of mean values, i.e., $\overline{L}' = ||\bar{e}_{(ij)}||$. The if f and h denote respectively strategies for players I and II in the game $(L, \mathscr{J})$, the sequence $(a_1, a_2, \cdots)$ generated by $(f, h, L, \mathscr{J})$ may be taken to have been generated by $(f, h, L', \mathscr{J}^0)$ where $\mathscr{J}^0$ is the identity matrix. Finally define $S' = \{\mathscr{J}^T s | s \in S\}$, where $\mathscr{J}^T$ denotes the transpose of $\mathscr{J}$; then S' is a closed convex subset of C_L and further, $S' \cap T_{\bar{L}'}(q)$ is nonempty for all $q \in Q$. Thus, by the result of [1] R' is approachable in $(L', \mathscr{J}^0)$. Let f^0 denote a strategy for player I with which S' is approachable; the proof will be completed by showing that S is approachable with f^0.

Let h denote a strategy for player II. Let $\{Y_n : n = 1, 2, \cdots\}$ be the vector-valued process generated by (f^0, h) in L' and let $\{U_n : n = 1, 2, \cdots\}$ be the process generated by (f^0, h) in L. It remains to show that $P\{\lim_n d(\overline{U}_n, S)\} = 1$. First note that $d(\overline{U}_n, S) \leqq d(\overline{U}_n, (\mathscr{T}^T)^{-1}\overline{Y}_n) + d((\mathscr{T}^T)^{-1}\overline{Y}_n, S)$; however, $d((\mathscr{T}^T)^{-1}\overline{Y}_n, S) = d((\mathscr{T}^T)^{-1}\overline{Y}_n, (\mathscr{T}^T)^{-1}S') \leqq ||\mathscr{T}^T)^{-1}||d(\overline{Y}_n, S')$. Thus since $d(\overline{Y}_n, S') \to 0$ with probability one need only show that $d(\overline{U}_n, (\mathscr{T}^T)^{-1}\overline{Y}_n) \to 0$ with probability one to be done. However, $d(\overline{U}_n, (\mathscr{T}^T)^{-1}\overline{Y}_n) \leqq ||\mathscr{T}^T|| \, d(\overline{Y}_n, (\mathscr{T}^T)\overline{U}_n)$ and an immediate application of the Stability Theorem [2, p. 387] shows that $d(Y_n, (\mathscr{T}^T)U_n) \to 0$ with probability one; this completes the proof.

References

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