NORMAL SUBGROUPS OF SOME HOMEOMORPHISM GROUPS*

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1. Introduction. The normal subgroups of the group of all homeomorphisms of a space X have been enumerated by Fine and Schweigert [2] when X is a line, by Schreier and Ulam [3] when X is a circle, by Ulam and von Neumann [4] and Anderson [1] when X is a 2-sphere. In each of these cases there are either one or two proper normal subgroups. However, when X is an n-cell ($n > 1$), there are infinitely many. The object of this paper is to investigate the normal subgroups for a class of spaces X which includes the n-cell. Some of these normal subgroups, although not all, can be defined in terms of the family of fixed point sets of their elements, and we investigate this relationship at some length. A smallest normal subgroup is exhibited, and the corresponding quotient group is represented as a group of transformations of a related space.

2. Families of fixed point sets. Let X be a set, $\Pi(X)$ the group of all permutations of X (one-to-one mappings of X onto itself), and G a subgroup of $\Pi(X)$. Suppose that $\mathscr{F}$ is a non-empty family of subsets of X satisfying the following conditions:

(i) If $F_1, F_2 \in \mathscr{F}$, then there exists an $F_3 \in \mathscr{F}$ such that $F_3 \subset F_1 \cap F_2$,

(ii) If $F_1 \in \mathscr{F}$ and $g \in G$, then there exists an $F_2 \in \mathscr{F}$ such that $F_2 \subset g(F_1)$.

We shall call $\mathscr{F}$ ecliptic relative to G. For example, if $\mathscr{F}$ consists of the complements of all finite subsets of X, then $\mathscr{F}$ is ecliptic relative to $\Pi(X)$. If X has a topology, we denote the group of homeomorphisms of X by $H(X)$. Let X be a closed unit ball B_n in euclidean n-space and $\mathscr{F}_0$ consist of the complements in B_n of those balls which are concentric with B_n and have radius less than one. Then $\mathscr{F}_0$ is ecliptic relative to $H(B_n)$. In this connection, we note that for $h \in H(B_n)$, $h(S_{n-1}) = S_{n-1}$, where S_{n-1} is the boundary of B_n.

Let X again be an arbitrary set and G a subgroup of $\Pi(X)$. We introduce a partial ordering among the families of subsets of X as follows: $\mathscr{F} \leqq \mathscr{F}'$ provided that, for every $F \in \mathscr{F}$, there exists an $F' \in \mathscr{F}'$ such that $F' \subset F$. Evidently $\mathscr{F} \subset \mathscr{F}'$ implies $\mathscr{F} \leqq \mathscr{F}'$, where $\mathscr{F} \subset \mathscr{F}'$ means set inclusion, but the converse is false. We define equivalence of $\mathscr{F}$ and $\mathscr{F}'$ to mean $\mathscr{F} \leqq \mathscr{F}'$ and $\mathscr{F}' \leqq \mathscr{F}$, and we write $\mathscr{F} \cong \mathscr{F}'$.

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LEMMA 1. *If $\mathscr{F}, \mathscr{F}'$ are families of subsets of X, $\mathscr{F} \cong \mathscr{F}'$, and $\mathscr{F}$ is ecliptic relative to G, then $\mathscr{F}'$ is ecliptic relative to G.*

Proof. If $F'_1, F'_2 \in \mathscr{F}'$, then there exist sets $F_1, F_2, F_3 \in \mathscr{F}$ and $F'_3 \in \mathscr{F}'$ such that $F_1 \subset F'_1$, $F_2 \subset F'_2$, and $F'_3 \subset F_3 \subset F_1 \cap F_2 \subset F'_1 \cap F'_2$. Second, if $F'_1 \in \mathscr{F}'$ and $g \in G$, then we can find $F_1, F_2 \in \mathscr{F}$ and $F'_2 \in \mathscr{F}'$ such that $F_1 \subset F'_1$ and $F'_2 \subset F_2 \subset g(F_1) \subset g(F'_1)$.

To any family $\mathscr{F}$ we can adjoin all subsets of X which contain some element of $\mathscr{F}$ and thus obtain a family $\mathscr{F}^*$ which is clearly equivalent to $\mathscr{F}$ and, by Lemma 1, is ecliptic relative to G if $\mathscr{F}$ is. In fact, $\mathscr{F}^*$ has the property that $F_1^*, F_2^* \in \mathscr{F}^*$ and $g \in G$ implies $F_1^* \cap F_2^*, g(F_1^*) \in \mathscr{F}^*$. In addition, $\mathscr{F}^*$ is an upper bound, with respect to set inclusion, among the families equivalent to $\mathscr{F}$. We shall call $\mathscr{F}$ replete if it is equivalent to no larger family.

If $f \in \Pi(X)$, we set $K(f) = \{x \in X : f(x) = x\}$. For any family $\mathscr{F}$ of subsets of X, we define

$$S(\mathscr{F}, G) = \{g \in G : K(g) \supset F \text{ for some } F \in \mathscr{F}\} .$$

We note that if the empty set $\emptyset \in \mathscr{F}$, then $S(\mathscr{F}, G) = G$.

LEMMA 2. (a) *$\mathscr{F} \cong \mathscr{F}'$ implies $S(\mathscr{F}, G) = S(\mathscr{F}', G)$.*

(b) *If $\mathscr{F}$ satisfies (i), then $S(\mathscr{F}, G)$ is a subgroup of G.*

(c) *If $f \in \Pi(X)$, then*

$$f[S(\mathscr{F}, G)]f^{-1} = S(f(\mathscr{F}), fGf^{-1}) .$$

(d) *If $\mathscr{F}$ is ecliptic relative to G, then $S(\mathscr{F}, G)$ is a normal subgroup of G.*

Proof. For (a) we show that $\mathscr{F} \leqq \mathscr{F}'$ implies $S(\mathscr{F}, G) \subset S(\mathscr{F}', G)$. Indeed, if $g \in S(\mathscr{F}, G)$ and $K(g) \supset F$ for some $F \in \mathscr{F}$, we can find $F' \in \mathscr{F}'$ such that $F' \subset F \subset K(g)$, whence $g \in S(\mathscr{F}', G)$. In (b) we need merely observe that, for any $f_1, f_2 \in \Pi(X)$, $K(f_1, f_2) \supset K(f_1) \cap K(f_2)$ and $K(f_1^{-1}) = K(f_1)$. In part (c) we use the relation $K(fgf^{-1}) = f(K(g))$. If $g \in f[S(\mathscr{F}, G)]f^{-1}$, then $g = fg_1f^{-1}$, where $g_1 \in G$ and $K(g_1) \supset F$ for some $F \in \mathscr{F}$. Hence, $g \in fGf^{-1}$, $K(g) \supset f(F)$, and $g \in S(f(\mathscr{F}), fGf^{-1})$. If $g \in S(f(\mathscr{F}), fGf^{-1})$, then $g = fg_1f^{-1}$ for some $g_1 \in G$, and $K(g) \supset f(F)$ for some $F \in \mathscr{F}$. Hence, $K(g_1) \supset F$, and $g \in f[S(\mathscr{F}, G)]f^{-1}$. In part (d), let $f \in G$. From (c), $f[S(\mathscr{F}, G)]f^{-1} = S(f(\mathscr{F}), G)$. Normality will follow from (a) if we can show that $f(\mathscr{F}) \cong \mathscr{F}$. Clearly (ii) implies $f(\mathscr{F}) \leqq \mathscr{F}$. If $F_1 \in \mathscr{F}$, then there is an $F_2 \in \mathscr{F}$ such that $F_2 \subset f^{-1}(F_1)$, whence $f(F_2) \subset F_1$, and $\mathscr{F} \leqq f(\mathscr{F})$.

We shall assume, from now on, that X is a Hausdorff topological space, unless the contrary is explicitly stated. For $S(\mathscr{F}, H(X))$ we shall write $S(\mathscr{F})$, and if $\mathscr{F}$ is ecliptic relative to $H(X)$, we shall simply say that $\mathscr{F}$ is ecliptic. For any family of subsets of X, we introduce a further condition:

(iii) If $F \in \mathscr{F}$ and $U \subset X$ is open ($U \neq \emptyset$), then there exists an $h \in H(X)$ such that $h(cF) \subset U$, where cF is the complement of F in X. An ecliptic family which satisfies (iii) will be called strictly ecliptic. The family $\mathscr{F}_0$ of subsets of B_n defined above is evidently strictly ecliptic. If $\mathscr{F}$ satisfies (iii) and $\mathscr{F} \geqq \mathscr{F}'$, then clearly $\mathscr{F}'$ satisfies (iii). Since $K(h)$ is closed for every $h \in H(X)$, there is no loss of generality in assuming that the elements of any family $\mathscr{F}$ are closed, and this assumption will be made from now on, unless the contrary is stated.

LEMMA 3. *If X admits families $\mathscr{F}, \mathscr{F}'$ which satisfy* (ii) *and* (iii) *and contain more than one element, then $\mathscr{F} \cong \mathscr{F}'$.*

Proof. We may as well assume that $\mathscr{F}, \mathscr{F}'$ are replete in the closed subsets of X. If $F \in \mathscr{F}$, $F \neq X$, and $F' \in \mathscr{F}'$, then we can find $h \in H(X)$ such that $h(cF') \subset cF$. Hence, $h(F') \supset F$, $h(F') \in \mathscr{F}$, and $F' = h^{-1}(h(F')) \in \mathscr{F}$. Thus $\mathscr{F}' \subset \mathscr{F}$ and, similarly $\mathscr{F} \subset \mathscr{F}'$.

Some spaces contain no ecliptic families except $\{X\}$ and the set $\mathscr{C}(X)$ of all closed subsets of X. For, by Lemma 2, such a family defines a normal subgroup of $H(X)$; when X is a 1-sphere, Schreier and Ulam [3] showed that the only proper normal subgroup of $H(X)$ consists of the orientation-preserving elements of which some have no fixed points.

3. Minimal normal subgroups.

We shall need to know something more about $H(X)$. Rather than make specific and detailed assumptions about the existence of certain homeomorphisms, we shall assume a mildly euclidean structure for X, namely:

(iv) If $U \subset X$ is open ($U \neq \emptyset$), then there exists an open $V \subset U$ which is homeomorphic to an open ball in a euclidean space of positive dimension.

The dimension of the ball may vary for different open sets. We shall refer to V as a euclidean neighborhood in X.

THEOREM 1. *Suppose X satisfies* (iv) *and contains a strictly ecliptic family $\mathscr{F}$. If N is a normal subgroup of $H(X)$, then either $N \supset S(\mathscr{F})$ or N consists of the identity e.*

Proof. Suppose $N \neq \{e\}$ and $g_0 \in N$, $g_0 \neq e$. Then $g_0(x) \neq x$ for some $x \in X$, and we can find a neighborhood U_0 of x and a euclidean neighborhood V_0 such that $g_0(U_0) \cap U_0 = \emptyset$ and $V_0 \subset g_0(U_0)$. Let ω map V_0 homeomorphically onto an open ball in some euclidean space, let $B \subset \omega(V_0)$ be a closed unit ball of the same dimension, and set $W_0 = \omega^{-1}(\text{int } B)$, where int denotes interior. We wish to construct a homeomorphism h_0 of $\bar{W}_0$ in its relative topology with the following properties:

(a) $K(h_0)\supset \bar{W}_0\cap cW_0$,

(b) there exists an open $V\subset W_0$ such that, for all integers $n>0$, $h_0^n(\bar{V})\cap \bar{V}=\emptyset$,

(c) if $A=\bigcup_{n=0}^{\infty}h_0^n(\bar{V})$, then $\bar{A}\cap cA$ is a single point. To do this is evidently equivalent to constructing such a homeomorphism k_0 of B, for then $h_0=\omega^{-1}k_0\omega$ has the desired properties in $\bar{W}_0$. Let θ be a homeomorphism of $[0,1]$ such that $K(\theta)=\{0,1\}$ and $\theta(r)<r$ for $0<r<1$. If $p\in B$ lies at a distance r from the center of B, then we define $k_0(p)$ to be the point on the same radial line at a distance $\theta(r)$ from the center. By choosing a sufficiently small open ball in B which does not meet either the center or boundary, we can satisfy (a), (b), and (c).

We now define the function h_1 as follows: $h_1(x)=h_0(x)$ if $x\in W_0$, $h_1(x)=x$ if $x\in cW_0$ Clearly, $h_1\in H(X)$. Now $g_1=g_0h_1^{-1}g_0^{-1}h_1\in N$ since N is normal, and $g_0h_1^{-1}g_0^{-1}h_1(x)=h_1(x)$ for $x\in W_0$, since $g_0^{-1}(W_0)\subset cW_0$. Thus $g_1(x)=h_0(x)$ for $x\in W_0$. Let g be any element of $S(\mathscr{F})$. Then there exists an $F\in\mathscr{F}$ and $h_2\in H(X)$ such that $K(g)\supset F$ and $h_2(cF)\subset V$. Thus $K(h_2gh_2^{-1})\supset cV$. If we can construct an $h\in H(X)$ such that

$$g_1^{-1}hg_1h^{-1}=h_2gh_2^{-1}=f\,, \tag{1}$$

then we will have shown that $g\in N$ and $S(\mathscr{F})\subset N$, since the left member of (1) lies in N. Let us rewrite (1) as $hg_1=g_1fh$. We set

$$h(x)=\begin{cases}g_1^nfg_1^{-n}(x) & \text{for}\quad x\in g_1^n(V)\,,\quad n=1,2,\cdots,\\ x \quad \text{for}\quad x\in c\left(\bigcup_{n=1}^{\infty}g_1^n(V)\right)\,.\end{cases}$$

By property (b) above, $m\neq n$ implies $g_1^m(V)\cap g_1^n(V)=\emptyset$, whence h is single-valued. Since $K(f)\supset cV$, the restriction of f to $\bar{V}$ is a homeomorphism of $\bar{V}$, and the same holds for $g_1^nfg_1^{-n}$ and $g_1^n(\bar{V})$, $n=1,2,\cdots$. Let $\bar{A}\cap cA$ consist of the point x_0, where $A=\bigcup_{n=0}^{\infty}g_1^n(\bar{V})$. Then each $x\neq x_0$ has a neighborhood which meets at most one of the sets $g_1^n(\bar{V})$, and h,h^{-1} are evidently continuous at such points. By the construction of h_0 and V, every neighborhood of x_0 contains all but a finite number of the sets $g_1^n(\bar{V})$, whence h,h^{-1} are continuous here as well. Hence, $h\in H(X)$. If $x\in c\bar{A}$, then $g_1(x)\in K(h)$ and $hg_1(x)=g_1fh(x)$. When $x\in V$,

$$hg_1(x)=g_1fg_1^{-1}(g_1(x))=g_1f(x)=g_1fh(x)\,.$$

Finally, if $n\geqq 1$ and $x\in g_1^n(V)$, we have $g_1^n(V)\subset K(f)$, so that $g_1fg_1^n(y)=g_1^{n+1}(y)$ when $y\in V$. Hence,

$$hg_1(x)=g_1^{n+1}fg_1^{-n-1}(g_1(x))=g_1fg_1^nfg_1^{-n}(x)=g_1fh(x)\,.$$

This establishes (1) and completes the proof.

We offer the following example of a non-Hausdorff space X without euclidean neighborhoods which admits a strictly ecliptic family $\mathscr{F}$ such that $S(\mathscr{F})$ is not minimal. Let X be an infinite set in which $\mathscr{C}(X)$ consists of the finite subsets of X and X itself. Then $H(X) = \Pi(X)$. For $\mathscr{F}$ we take the collection of non-empty open sets and form $S(\mathscr{F})$. Since X is not Hausdorff, $K(h)$ need not be closed for $h \in H(X)$. Clearly $\mathscr{F}$ is strictly ecliptic, but $S(\mathscr{F})$ contains, as a proper normal subgroup, the set of $h \in H(X)$ such that $cK(h)$ is finite and h is an even permutation of $cK(h)$.

4. Normal subgroups of $H(B_n)$. As we remarked in § 2, the family $\mathscr{F}_0$ of complements of smaller, open, concentric balls in B_n is strictly ecliptic. When $\mathscr{F}_0$ is extended to a replete family, it will consist of all closed sets containing a neighborhood of the boundary S_{n-1}. In this section, we will also be concerned with the group $H_0(B_n)$ of those $h \in H(B_n)$ such that $K(h) \supset S_{n-1}$. Evidently $H_0(B_n) \supset S(\mathscr{F}_0)$, and $H_0(B_n)$ is normal in $H(B_n)$.

THEOREM 2. *If N is a normal subgroup of $H(B_n)$ which contains an element not in $H_0(B_n)$, then $N \supset H_0(B_n)$.*

Proof. We will assume, to begin with, than $n \geqq 2$. Suppose $g_0 \in N \cap cH_0(B_n)$, and choose $x \in S_{n-1}$ so that $g_0(x) \neq x$. Let W_0 be the part of an open ball with center x which lies in B_n and is small enough so that $g_0(W_0) \cap W_0 = \emptyset$. We wish to construct a homeomorphism h_0 of $\bar{W}_0$ and an open set $W \subset W_0$ such that $W \cap S_{n-1} \neq \emptyset$ and h_0, W satisfy (a), (b), (c) in the proof of Theorem 1. Let B, k_0, and V be the same as in that proof. If Π is an $(n-1)$-dimensional hyperplane which passes through the center of B and meets V, then Π divides B into two regions (including boundaries) Δ, Δ' such that $\Delta \cap \Delta' = \Pi$. The restriction of k_0 to Δ is evidently a homeomorphism of Δ. Let ψ map Δ homeomorphically onto $\bar{W}_0$ in such a way that $\psi(\Pi) = \bar{W}_0 \cap S_{n-1}$. Then $h_0 = \psi k_0 \psi^{-1}$ and $W = \psi(\Delta \cap V)$ clearly satisfy (a), (b), (c). We define $h_1(x) = h_0(x)$ for $x \in W_0$, $h_1(x) = x$ for $x \in cW_0$, so that $h_1 \in H(B_n)$. Then $g_1 = g_0 h_1^{-1} g_0^{-1} h_1 \in N$, and $g_1(x) = h_1(x)$ for $x \in W_0$, as in the proof of Theorem 1. If g is any element of $H_0(B_n)$ such that $K(g) \supset cW$, it follows from the construction in the same proof that $g \in N$.

Let $p, q \in S_{n-1}$ be antipodal, D the diameter joining them, and Π_1, $\Pi_2 \subset B_n$ two $(n-1)$-dimensional hyperplanes perpendicular to D. Now Π_1, Π_2 divide B_n into three regions (including boundaries) Δ_1, Δ_2, Δ_3 and, correspondingly, S_{n-1} into three zones (including boundaries) Z_1, Z_2, Z_3. We take Δ_2 to be the middle region, so that $\Delta_1 \cap \Delta_2 = \Pi_1$, $\Delta_2 \cap \Delta_3 = \Pi_2$, $p \in \Delta_1$, $q \in \Delta_3$. Let P, Q be arbitrary neighborhoods of p, q, respectively, such that $\bar{P} \subset \Delta_1$, $\bar{Q} \subset \Delta_3$.

Next, we construct $h_2, h_3 \in H(B_n)$ such that $h_2(\bar{P}) \supset cW$, $h_3(\bar{Q}) \supset cW$.

For example, h_2 might first expand $\bar{P}$ until its complement is quite small and then rotate the complement into W. If $g \in H_0(B_n)$ and $K(g) \supset \bar{P}$, then $K(h_2 g h_2^{-1}) \supset cW$, whence $g \in N$. Similarly, $K(g) \supset \bar{Q}$ implies $g \in N$. We now wish to construct a homeomorphic mapping θ of B_n onto $\Delta_2 \cup \Delta_3$ such that $\theta(x) = x$ for all $x \in \Delta_3$. To accomplish this, we introduce spherical coordinates $r, \phi_1, \cdots, \phi_{n-1}$ for the points $x \in B_n$ such that ϕ_1 is the angle between D and the radial line through x. Then Π_i satisfies the equation $r \cos \phi_1 = k_i$, $|k_i| < 1$ $(i = 1, 2)$. Let r, ϕ_1 be regarded as polar coordinates for the closed upper half-plane in euclidean 2-space, and let R be the set of (r, ϕ_1) such that $r \leqq 1$, $0 \leqq \phi_1 \leqq \pi$. The lines $r \cos \phi_1 = k_i$ $(i = 1, 2)$ divide R into three regions (including boundaries) R_1, R_2, R_3. Let ω be a homeomorphic mapping of R onto $R_2 \cup R_3$ such that $\omega(y) = y$ for all $y \in R_3$. We then set

$$\theta(r, \phi_1, \cdots, \phi_{n-1}) = (\omega(r, \phi_1), \phi_2, \cdots, \phi_{n-1}) .$$

Let f be any element of $H_0(B_n)$. Then $\theta f \theta^{-1} \in H(\Delta_2 \cup \Delta_3)$, and $K(\theta f \theta^{-1}) \supset \Pi_1 \cup Z_2 \cup Z_3$. We define $g_2(x) = \theta f \theta^{-1}(x)$ if $x \in \Delta_2 \cup \Delta_3$, $g_2(x) = x$ if $x \in \Delta_1$. Clearly, $g_2 \in H(B_n)$, and $K(g_2) \supset \bar{P}$, whence $g_2 \in N$. In addition, $g_2(x) = f(x)$ for $x \in \Delta_3 \cap f^{-1}(\Delta_3)$, so that $K(g_2^{-1} f) \supset \bar{Q} = \Delta_3 \cap f^{-1}(\Delta_3)$, and $g_3 = g_2^{-1} f \in N$. Hence, $f = g_2 g_3 \in N$, and $H_0(B_n) \subset N$. When $n = 1$, the constructions in the first half of the proof can not be carried out in S_0. The theorem follows, in this case, from the result obtained in [2] that the only proper normal subgroups of $H(B_1)$ are $S(\mathscr{F}_0)$ and $H_0(B_1)$.

If $G \subset \Pi(X)$ and $Y \subset X$, we denote the restrictions of the elements of G to Y by $G \mid Y$. For any orientable space X, we let $E(X)$ denote the group of all orientation-preserving homeomorphisms of X.

LEMMA 4. *If N is a normal subgroup of $H(B_n)$, then $N \mid S_{n-1}$ is a normal subgroup of $H(S_{n-1})$.*

Proof. Clearly $N \mid S_{n-1}$ is a subgroup of $H(S_{n-1})$. If $h_0 \in H(S_{n-1})$, we can extend h_0 to an element h of $H(B_n)$. Let p_0 be the center of B_n, $p \neq p_0$ a point of B_n lying on the sphere S with center p_0, and π the radial projection of S onto S_{n-1}. We define $h(p) = \pi^{-1} h_0 \pi(p)$, $h(p_0) = p_0$. Clearly $h \in H(B_n)$. Then $N \mid S_{n-1} = (hNh^{-1}) \mid S_{n-1} = h_0(N \mid S_{n-1}) h_0^{-1}$.

COROLLARY. *If N is not contained in $H_0(B_n)$ and $n \leqq 3$, then N is either $E(B_n)$ or $H(B_n)$.*

Proof. By Lemma 4, $N \mid S_{n-1}$ is a normal subgroup of $H(S_{n-1})$ different from $\{e\}$. It was proved in [3] for $n = 2$ and in [1] for $n = 3$ that the only normal subgroups of $H(S_{n-1})$ are $\{e\}$, $E(S_{n-1})$, and $H(S_{n-1})$. Hence, if $h \in E(B_n)$, there exists a $g \in N$ such that $h \mid S_{n-1} = g \mid S_{n-1}$.

Then $f = g^{-1}h \in H_0(B_n) \subset N$ by Theorem 2, and $h = gf \in N$. If $N \not\subset E(B_n)$, a similar argument shows that $N = H(B_n)$. We note that $H_0(B_1) = E(B_1)$.

5. The lattice of normal subgroups. In the first part of this section, we revert to the assumption that X is an arbitrary set. The intersection of two ecliptic families may be empty. If, for example, $\mathscr{F}_0$ is the ecliptic family defined above for B_2 and $\mathscr{F}_1$ is the family of complements of interiors of simple polygons lying entirely in the interior of B_2, then $\mathscr{F}_0 \cap \mathscr{F}_1 = \emptyset$, although $\mathscr{F}_0^* = \mathscr{F}_1^*$. However, the intersection of any collection of replete, ecliptic families is also replete, ecliptic, and non-empty, since it always contains $\{X\}$. The smallest ecliptic family (up to equivalence) which contains a given collection $\{\mathscr{F}_\alpha\}$ of ecliptic families consists of all finite intersections of elements in $\cup \mathscr{F}_\alpha$. We denote this set by $\vee \mathscr{F}_\alpha$ and set $\wedge \mathscr{F}_\alpha = \cap \mathscr{F}_\alpha$. If the $\mathscr{F}_\alpha$ are replete, then $\vee \mathscr{F}_\alpha$ is also replete. For if $F_1, \cdots, F_n \in \cup \mathscr{F}_\alpha$, and $F \supset \bigcap_i F_i$, then $F \cup F_i \in \cup \mathscr{F}_\alpha$ $(i = 1, \cdots, n)$, and $F = \bigcap_i (F \cup F_i)$.

For any collection $\{G_\alpha\}$ of subgroups of a group G, we set $\wedge G_\alpha = \cap G_\alpha$ and define $\vee G_\alpha$ as the smallest subgroup of G which contains G_α.

LEMMA 5. *If G is a subgroup of $H(X)$ and $\{\mathscr{F}_\alpha\}_{\alpha \in A}$ is a collection of replete ecliptic families relative to G, then*

$$S(\wedge \mathscr{F}_\alpha, G) = \wedge S(\mathscr{F}_\alpha, G),\ S(\vee \mathscr{F}_\alpha, G) \supset \vee S(\mathscr{F}_\alpha, G)\ .$$

Proof. If $g \in S(\wedge \mathscr{F}_\alpha, G)$, then there is an $F \in \cap \mathscr{F}_\alpha$ such that $K(g) \supset F$, whence $g \in S(\mathscr{F}_\alpha, G)$ for each $\alpha \in A$. If $g \in \wedge S(\mathscr{F}_\alpha, G)$, then, for each $\alpha \in A$, there is an $F_\alpha \in \mathscr{F}_\alpha$ such that $K(g) \supset F_\alpha$. Hence, $K(g) \supset F = \bigcup_{\alpha \in A} F_\alpha$, $F \in \mathscr{F}_\beta$ for each $\beta \in A$ since $\mathscr{F}_\beta$ is replete, and $g \in S(\wedge \mathscr{F}_\alpha, G)$. This proves the first relation. In the second, if $g \in \vee S(\mathscr{F}_\alpha, G)$, then there are sets $F_1, \cdots, F_n \in \bigcup_{\alpha \in A} \mathscr{F}_\alpha$ and elements $g_1, \cdots, g_n \in G$ such that $K(g_i) \supset F_i$ $(i = 1, \cdots, n)$ and $g = g_1 \cdots g_n$. Hence, $K(g) \supset F = \bigcap_i F_i$, $F \in \vee \mathscr{F}_\alpha$, and $g \in S(\vee \mathscr{F}_\alpha, G)$.

We now return to the case $X = B_n$.

LEMMA 6. *Let $\mathscr{G}$ be a family of* (*not necessarily closed*) *subsets of S_{n-1} which*

(a) *satisfies* (i), *or*

(b) *is ecliptic relative to $H(B_n)$. Let $\mathscr{F}$ be the family of closed subsets of B_n which contain a member of $\mathscr{G}$ in their interior* (*in the relative topology of B_n*). *Then*

(a) *$\mathscr{F}$ is ecliptic relative to $H_0(B_n)$, or*

(b) *$\mathscr{F}$ is ecliptic relative to $H(B_n)$. In either case, $\mathscr{F}$ is replete.*

Proof. If $F_0, F_0' \in \mathscr{G}$ and $F_0 \subset \operatorname{int} F$, $F_0' \subset \operatorname{int} F'$, then $F_0 \cap F_0' \subset$ int

$F \cap \operatorname{int} F' = \operatorname{int}(F \cap F')$, and $F \cap F' \in \mathscr{F}$, whence $\mathscr{F}$ satisfies (i). In part (a), if $h \in H_0(B_n)$, then $\operatorname{int} h(F) = h(\operatorname{int} F) \supset h(F_0) = F_0$, and $h(F) \in \mathscr{F}$. In part (b), if $h \in H(B_n)$, then there is an $F_0'' \in \mathscr{G}$ such that $h(F_0) \supset F_0''$, and $\operatorname{int} h(F) \supset F_0''$ as in (a), so that $h(F) \in \mathscr{F}$. Thus (ii) is verified in each case. The repleteness of $\mathscr{F}$ is obvious.

We will indicate the above relationship between $\mathscr{F}$ and $\mathscr{G}$ by saying that $\mathscr{F}$ is derived from $\mathscr{G}$. The simplest example of a derived ecliptic family relative to $H_0(B_n)$ is that in which $\mathscr{G}$ consists of a single subset of S_{n-1}. An ecliptic family relative to $H(B_n)$ is obtained by letting $\mathscr{G}$ consist of the complements in S_{n-1} of finite subsets of S_{n-1}. When $n = 2$, a family equivalent to the latter can be described as the set of complements in B_2 of interiors of simple closed curves which meet S_1 in a finite number of points. The construction can be varied by taking the set of complements of countable or first category subsets in S_{n-1}.

Returning to Lemma 5 and the case $X = B_n, G = H(B_n)$, we have not been able to determine whether equality holds in the second relation even for the case $S(\mathscr{F} \vee \mathscr{F}') \supset S(\mathscr{F}) \vee S(\mathscr{F}')$, $\mathscr{F} \vee \mathscr{F}' = \mathscr{C}(X)$. However, we do have the following result for derived families.

THEOREM 3. *Let $\mathscr{F}, \mathscr{F}'$ be derived from $\mathscr{G}, \mathscr{G}'$, respectively, where $\mathscr{G} = \{P_0\}$, $\mathscr{G}' = \{Q_0\}$, and suppose that $\bar{P}_0$, $\bar{Q}_0$ can be separated in S_{n-1} by an $(n-2)$-sphere $\sum \subset S_{n-1}$ which is tame relative to $H(B_n)$. Then*

$$(2) \qquad S(\mathscr{F} \vee \mathscr{F}', H_0(B_n)) = S(\mathscr{F}, H_0(B_n)) \vee S(\mathscr{F}', H_0(B_n)) .$$

Proof. Let Π_1 be an $(n-1)$-dimensional hyperplane passing through the center of B_n, and set $\Sigma_1 = \Pi_1 \cap S_{n-1}$. Choose $h \in H(B_n)$ such that $h(\Sigma) = \Sigma_1$. Since Π_1 and $h(\bar{Q}_0)$ are closed and disjoint, we can find a second hyperplane Π_2 parallel to Π_1 and lying between Π_1 and $h(\bar{Q}_0)$. Now Π_1, Π_2 divide B_n into three regions (including boundaries) $\Delta_1, \Delta_2, \Delta_3$ such that $h(\bar{P}_0) \subset \Delta_1$, $h(\bar{Q}_0) \subset \Delta_3$. In fact, $P_0 \subset \operatorname{int} h^{-1}(\Delta_1)$, $Q_0 \subset \operatorname{int} h^{-1}(\Delta_3)$, where int denotes interior in the relative topology of B_n. Hence, $h^{-1}(\Delta_1) \in \mathscr{F}$ and $h^{-1}(\Delta_3) \in \mathscr{F}'$. Since these sets are disjoint, $\varnothing \in \mathscr{F} \vee \mathscr{F}'$ and $S(\mathscr{F} \vee \mathscr{F}', H_0(B_n)) = H_0(B_n)$. By setting $\Delta_1 = P$, $\Delta_3 = Q$, and following the argument in the second half of the proof of Theorem 2, we can show that the group generated by those $g \in H_0(B_n)$ such that $K(g) \supset \Delta_1$ or Δ_3 is precisely $H_0(B_n)$. Since $K(g) \supset \Delta_1$ implies $g \in S(h^{-1}(\mathscr{F}), H_0(B_n))$, and $K(g) \supset \Delta_3$ implies $g \in S(h^{-1}(\mathscr{F}'), H_0(B_n))$, it follows from Lemma 2(c) that

$$\begin{aligned} H_0(B_n) &= h[S(h^{-1}(\mathscr{F}), H_0(B_n)) \vee S(h^{-1}(\mathscr{F}'), H_0(B_n))]h^{-1} \\ &= h[S(h^{-1}(\mathscr{F}), H_0(B_n))]h^{-1} \vee h[S(h^{-1}(\mathscr{F}'), H_0(B_n))]h^{-1} \\ &= S(\mathscr{F}, H_0(B_n)) \vee S(\mathscr{F}', H_0(B_n)) . \end{aligned}$$

Hence, (2) is established. When $n = 1$, the hypothesis of the theorem states that P_0 and Q_0 are the two points in S_0. The construction in the second half of the proof of Theorem 2 can evidently be carried through in this case.

6. Quotient spaces. we turn now to the problem of representing the quotient groups $H_0(B_n)/S(\mathscr{F})$, where $\mathscr{F}$ is an ecliptic family, as groups of transformations.

THEOREM 4. *Let $A \subset S_{n-1}$ have the property that the set of its neighborhoods in B_n has a countable base, and let $\mathscr{F}$ be the ecliptic family derived from $\{A\}$. Then $H_0(B_n)/S(\mathscr{F})$ can be represented as a group of order-preserving transformations of a partially ordered set Z onto itself.*

Proof. Let Y be the set of all countable sequences $\{U_k\}$ of open subsets $U_k \subset B_n$ such that $U_1 \supset U_2 \supset \cdots$, and $\{U_k\}$ is a base for the set of neighborhoods of A. We introduce a partial ordering in Y as follows: $\{U_k\} \leqq \{V_k\}$ if there exists a $k_0 > 0$ such that $k > k_0$ implies $U_k \subset V_k$. We call $\{U_k\}$ and $\{V_k\}$ equivalent if $\{U_k\} \leqq \{V_k\}$ and $\{V_k\} \leqq \{U_k\}$, and we write $\{U_k\} \equiv \{V_k\}$. Thus $\{U_k\} \equiv \{V_k\}$ means that $U_k = V_k$ for all but a finite set of k's. If $\{U_k\} \equiv \{V_k\}$ and $\{U_k\} \leqq \{W_k\}$, then clearly $\{V_k\} \leqq \{W_k\}$. Let Z be the set of equivalence classes in Y formed by the relation $\equiv$. If $u, v \in Z$, we define $u \leqq v$ to mean that the same ordering subsists between their respective equivalence classes. Moreover, $u \leqq v$ and $v \leqq u$ implies $u = v$.

If $h \in H_0(B_n)$ and $\{U_k\} \in Y$, then $\{h(U_k)\} \in Y$. Furthermore, $\{V_k\} \in Y$ and $\{U_k\} \leqq \{V_k\}$ implies $\{h(U_k)\} \leqq \{h(V_k)\}$. In particular, $\{U_k\} \equiv \{V_k\}$ implies $\{h(U_k)\} \equiv \{h(V_k)\}$. Thus, corresponding to h there is an element $\omega(h) \in \Pi(Z)$ which is order-preserving, and $g \in H_0(B_n)$ implies $\omega(gh) = \omega(g)\omega(h)$. We now show that $h \in S(\mathscr{F})$ if, and only if, $\omega(h) = i$, where i is the identity in $\Pi(Z)$. If $h \in S(\mathscr{F})$, then there is an $F \subset B_n$ such that $K(h) \supset F$ and $\operatorname{int} F \supset A$. For any $u \in Z$, let $\{U_k\}$ be a representative of u in Y. Since $\{U_k\}$ is a base for the neighborhoods of A, we can find $k_0 > 0$ such that $k > k_0$ implies $U_k \subset \operatorname{int} F$, whence $\omega(h)(u) = u$, and $\omega(h) = i$. Conversely, if $h \notin S(\mathscr{F})$, then for each $\{U_k\} \in Y$, there exists a sequence $\{x_k\}$ of points in B_n such that $x_k \in U_k$ and $h(x_k) \neq x_k$ $(k = 1, 2, \cdots)$. Setting $V_k = U_k \cap c\{h(x_k)\}$ for each k, we have $\{V_k\} \in Y$ and $\{h(V_k)\} \not\equiv \{V_k\}$. If $\{V_k\}$ is a respresentative of $v \in Z$, then $\omega(h)(v) \neq v$, and $\omega(h) \neq i$. This proves our assertion. Let θ denote the canonical mapping from $H_0(B_n)$ onto $H_0(B_n)/S(\mathscr{F})$. Then $\theta(g) = \theta(h)$ if, and only if, $\omega(g) = \omega(h)$. Hence, $\omega\theta^{-1}$ is an isomorphism between $H_0(B_n)/S(\mathscr{F})$ and $\omega(H_0(B_n))$.

If A is closed in B_n, then A is compact, and the uniform $(1/k)$-

neighborhoods of A form a base for its set of neighborhoods, so that the hypothesis of the theorem is satisfied in this case. If $A = S_{n-1}$, then $\mathscr{F} = \mathscr{F}_0$, and the construction in the proof allows us to represent $H(B_n)/S(\mathscr{F})$ as a subgroup of order-preserving elements in $\Pi(Z)$ which contains $\omega(H_0(B_n))$.

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