# AN ESTIMATE FOR DIFFERENTIAL POLYNOMIALS IN $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}$

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This article is concerned with polynomials with respect to the Cauchy-Riemann operators

$$\frac{\partial}{\partial z_1} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\Big), \cdots, \frac{\partial}{\partial z_n} = \frac{1}{2}\Big(\frac{\partial}{\partial x_n} + i\frac{\partial}{\partial y_n}\Big) .$$

We establish an $L^2$-estimate, for such polynomials, and derive from it uniqueness in a class of Cauchy problems. The estimate is quite similar to Hörmander's inequalities and, in fact, can be essentially deduced from them. However, its direct proof is very simple and leads to a constant better than the one in Hörmander's inequalities. We have therefore preferred to present it thoroughly.

The last part of the paper studies a class of Cauchy problems and applies the estimate to obtain uniqueness. There the methods are quite standard (see for instance Nirenberg [1]). The nature of the differential operators considered allows us to remove the strict convexity of the domains in which the solutions are studied, and replace it by a weaker condition.

**1. The inequality.** We consider a polynomial $P(z)$ on $C^n$. We set, for $p = (p_1, \cdots, p_n) \in N^n$:

$$P^{(p)}(z) = \Big(\frac{\partial}{\partial z_1}\Big)^{p_1} \cdots \Big(\frac{\partial}{\partial z_n}\Big)^{p_n} P(z) .$$

We shall denote by $P(D_z)$ the differential polynomial on $R^{2n}$ obtained by substituting $\partial/\partial z_j = 1/2(\partial/\partial x_j + (1/i)(\partial/\partial y_j))$ for $z_j$ $(1 \leqq j \leqq n)$ in $P(z)$.

If $S$ is a subset of $R^{2n}$, we denote by $\beta_j(S)$ the diameter of $S$ in the complex "direction" $z_j : \beta_j(s) = \sup_{z', z'' \in S} | z'_j - z''_j |$.

THEOREM 1. *Let $\Omega$ be an open set in $R^{2n}$. For all polynomials $P(z)$ on $C^n$, all functions $H(z)$ defined and holomorphic in $\Omega$, all functions $\phi(x, y) \in C_0^\infty(\Omega)$, all $p = (p_1, \cdots, p_n) \in N^n$:*

$$|| e^{H(z)} P^{(p)}(D_z)\phi ||_{L^2} \leqq \beta_1^{p_1}(\Omega) \cdots \beta_n^{p_n}(\Omega) || e^{H(z)} P(D_z)\phi ||_{L^2} .$$

It is enough to prove the inequality in Theorem 1 for $p_1 = 1$ and $p_j = 0$ for $j \geqq 2$. We shall denote by $P_1(z)$ the corresponding $P^{(p)}(z)$. On the other hand, we set, for $j = 1, \cdots, n$:

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$$H_j(z) = \frac{\partial}{\partial z_j} H(z) ,$$

$$A_j = \frac{\partial}{\partial z_j} - H_j(z) .$$

Observe that for all $1 \leqq j,\ k \leqq n$, $(\partial/\partial z_j)H_k(z) = (\partial/\partial z_k)H_j(z)$; it follows from this that the $A_j$'s all commute.

The formal adjoint of $A_j$ is $A_j^* = -\,\partial/\partial \bar{z}_j - \overline{H_j(z)}$. Observe first that the $A_j^*$'s all commute, since the $A_j$'s do. But also the $A_j^*$'s commute with the $A_k$'s, for the $\overline{H_j(z)}$'s are antiholomorphic functions of $\bar{z}$ in $\Omega$.

If $Q(z)$ is a polynomial on $C^n$, we denote by $Q(A)$ the differential operator on $R^{2n}$ obtained by substituting $A_j$ for $z_j$ $(1 \leqq j \leqq n)$ in $Q(z)$. If $\bar{Q}(z)$ is the polynomial whose coefficients are the complex conjugates of the ones of $Q(z)$, the formal adjoint of the operator $Q(A)$ is $Q^*(A) = \bar{Q}(A^*) = \bar{Q}(A_1^*, \cdots, A_n^*)$. It is easy to check that:

$$(1) \qquad (P_j)^*(A) = -\,(P^*)_j(A) = -\,[P^*(A), \bar{z}_j] .$$

Let us denote by $(\ ,\ )$ and $\|\ \ \|$ the inner product and the norm in $L^2(R^{2n})$. We may as well assume that $\beta_1(\Omega) = 2d$, with $d = \sup_{z \in \Omega} |z_1|$. If $\phi(x, y)$ has its support in $\Omega$, we can write:

$$\begin{aligned}(P^*(A)\phi, z_1(P_1)^*(A)\phi) &= (\phi, P(A)[z_1(P_1)^*(A)\phi]) \\ &= (\bar{z}_1\phi, (P_1)^*(A)P(A)\phi) + (\phi, (P_1)^*(A)P_1(A)\phi) \\ &= (P_1(A)(\bar{z}_1\phi), P(A)\phi) + \|P_1(A)\phi\|^2 \\ &= (P_1(A)\phi, z_1P(A)\phi) + \|P_1(A)\phi\|^2 .\end{aligned}$$

Hence:

$$-\|P_1(A)\phi\|^2 = (P_1(A)\phi, z_1P(A)\phi) + (\bar{z}_1P^*(A)\phi, (P^*)_1(A)\phi) ,$$

by applying (1). We get at once:

$$(2) \quad \|P_1(A)\phi\|^2 \leqq d\,\|P_1(A)\phi\| \cdot \|P(A)\phi\| + d\,\|P^*(A)\phi\| \cdot \|(P^*)_1(A)\phi\| .$$

But since the $A_j$ and the $A_k^*$ all commute with each other, $P(A)$ and $P^*(A)$ commute, and $P_1(A)$ and $(P_1)^*(A)$ do. Therefore:

$$\|P^*(A)\phi\| = \|P(A)\phi\| , \qquad \|(P_1)^*(A)\phi\| = \|P_1(A)\phi\| .$$

These relations, together with (2), lead to:

$$(3) \qquad \|P_1(A)\phi\| \leqq (2d)\,\|P(A)\phi\| .$$

In this inequality (3), let us replace $\phi$ by $e^{H(z)}\phi$; we have

$$A_j[e^{H(z)}\phi] = e^{H(z)}\frac{\partial \phi}{\partial z_j} ,$$

and hence:

$$Q(A)[e^{H(z)}\phi] = e^{H(z)}Q(D_z)\phi ,$$

for any polynomial $Q(z)$ on $C^n$. Thus, we get, from (3):

$$\| e^{H(z)}P_1(D_z)\phi \| \leqq (2d) \| e^{H(z)}P(D_z)\phi \| . \qquad \text{q.e.d.}$$

**2. Uniqueness in Cauchy problems.** We shall denote by $B_a$ $(a > 0)$ the open ball $|z| < a$ in $C^n$.

We say that an open set $\Omega$ in $R^{2n}$ is *admissible at the point* $z_0$ if $z_0$ lies on the boundary of $\Omega$, if the boundary of $\Omega$ is, near $z_0$, a piece of a $C^\infty$ hypersurface and if the following property holds:

(A) *For some* $a > 0$, *there exists a function* $F(z)$, *holomorphic in the ball* $|z - z_0| < a$, *vanishing at* $z_0$ *and such that the diameter of the set* $U_b$ *of those points* $z \in \Omega$ *which satisfy* $|z-z_0|<a$, $-b<Re\, F(z)$ *converges to* 0 *when* $b > 0$ *does.*

In the sequel, $\Omega$ will be an open set in $R^{2n}$ admissible at the origin, $a$ will be a positive number such that $(A)$ holds for $z_0 = 0$ and some function $F(z)$ holomorphic in $B_a$. Furthermore, we shall assume that the intersection of $B_a$ with the boundary of $\Omega$ is a piece $S$ of a hypersurface $C^\infty$ (passing by 0).

Let us clarify a little the geometric situation. Let us denote by $W$ the piece of the hypersurface $ReF(z) = 0$ contained in $B_a$. Since $0 \in W \cap \bar{\Omega} \subset U_b$ for every $b > 0$, we must have $W \cap \bar{\Omega} = W \cap S = \{0\}$. On the other hand, for any $b > 0$, $U_b \cup C\Omega$ is a neighborhood of 0. For, let $\varepsilon > 0$ be small so that $|z| < \varepsilon$ implies $|Re\, F(z)| < b$. If $z \in B_\varepsilon$, $z \notin U_b$ only if $z \notin \Omega$. The interior of $U_b$ is never empty. For assume it were and let $z$ belong to $U_b$; $z$ would have a neighborhood $N$ in which $Re\, F$ would still be $> -b$ and since $z \in \bar{\Omega}$, $N$ would intersect $\Omega$; obviously $N \cap \Omega$ is contained in the interior of $U_b$.

We consider a polynomial $P(z)$ on $C^n$, of degree $m \geq 1$, and a partial differential operator on $R^{2n}$ with continuous coefficients, $Q$, of order $\leqq$m $-$ 1, satisfying the condition:

$$(1) \qquad \| e^{H(z)}Qu \|_{L^2} \leqq K \sum_{p \neq 0} \| e^{H(z)}P^{(p)}(D_z)u \|_{L^2} ,$$

for all $H(z)$ holomorphic in $B_a$, all $u(x, y) \in C_0^\infty$ with support in $B_a$.

THEOREM 2. *Let* $U(x, y)$ *be a function defined and* $C^m$ *in* $\bar{\Omega}$, *with zero Cauchy data on* $S$, *satisfying*:

$$(2) \qquad |P(D_z)U| \leqq |QU| \text{ in } \bar{\Omega} .$$

*There exists a neighborhood of* 0 *in which* $U$ *vanishes identically.*

We keep our previous notations, for a, $F(z)$, etc.

Let us take a function $\beta(z)$, $C^\infty$ in $B_a$, with the following properties:

$$\beta(z) = 1 \text{ for } z \in B_a \text{ and } -2\varepsilon \leqq Re\, F(z) \leqq 0\ ;$$
$$\beta(z) = 0 \text{ for } z \in B_a \text{ and } -3\varepsilon \leqq Re\, F(z)\ ,$$

where $\varepsilon > 0$ is chosen small enough so that the support of $\beta(z)$ intersects $\Omega$ according to a compact set contained in $B_a$. That is possible because of property $(A)$; notice that the diameter of the compact set in question goes to 0 when $\varepsilon \to 0$.

We define now a function $v(z)$ as being equal to $\beta(z)U$ in $\Omega$ and to 0 elsewhere. Notice the following properties of $v$:

(i) the support of $v$ is compact (and contained in $B_a \cap \bar{\Omega}$);

(ii) $v(z)$ is $m-1$ times continuously differentiable;

(iii) $P(D_z)v = \beta P(D_z)U + RU\varphi$ in $\Omega$, $R$ being a partial differential operator with $C^\infty$ coefficients.

If one extends the definition of $RU$ by 0 outside $\Omega$, it becomes a continuous function in $B_a$ since the order of $R$ is at most $m-1$ and the Cauchy data of $U$ were 0 on $S$. On the other hand, $P(D_z)U$ vanishes also on $S$, because of (2) and of the fact that $Q$ is of order $\leqq m-1$. Hence, continuing $\beta P(D_z)U$ by 0 outside $\Omega$ leads again to a continuous function in $B_a$. We see thus that $P(D_z)v$ is a continuous function (in $R^{2n}$). This fact, together with properties (i) and (ii), allows us to extend to $v(z)$ the inequality of Theorem 1. We see that there exists a constant $A$ such that, for all holomorphic functions $H(z)$ in $B_a$,

$$(3) \qquad \sum_{p \neq 0} \| e^{H(z)} P^{(p)}(D_z)v \|_{L^2} \leqq A\delta \| e^{H(z)} P(D_z)v \|_{L^2}\ ,$$

$\delta$ being the diameter of the support of $v$. Remember that $\delta \to 0$ if $\varepsilon \to 0$. Since, on $U_{2\varepsilon}$, $v = U$, by using inequality (1) and (3), we get:

$$\int_{U_{2\varepsilon}} e^{2ReH}(|U|^2 + |QU|^2)dxdy \leqq (2AK\delta)^2 \int_{U_{2\varepsilon}} e^{2ReH} |P(D_z)U|^2 dxdy$$
$$+ (2AK)^2 \int_{CU_{2\varepsilon}} e^{2ReH} |P(D_z)v|^2 dxdy\ .$$

But since $U_{2\varepsilon} \subset \Omega$, we have the right to substitute $|QU|$ for $|P(D_z)U|$ in the first integral of the right hand side; and if we choose $\varepsilon$ small enough so that $(2AK\delta)^2 < 1/2$, we obtain finally:

$$\int_{U_\varepsilon} e^{2ReH} |U|^2 dxdy \leqq M \int_{CU_{2\varepsilon}} e^{2ReH} |P(D_z)v|^2 dxdy\ ,$$

$M$ being a constant independent of both $H(z)$ and $\varepsilon$. Observe that the integral on the right hand side is actually performed on $U_{3\varepsilon} \cap CU_{2\varepsilon}$. Let us take $H(z) = (t/2)F(z)$, $t > 0$. The nature of the domains of integration leads us to:

$$e^{-t\varepsilon}\int_{U_\varepsilon} |U|^2 dxdy \leqq Me^{-2t\varepsilon}\int_{CU_{2\varepsilon}} |P(D_z)v|^2 dxdy ,$$

or:

$$\int_{U_\varepsilon} |U|^2 dxdy \leqq M_1 e^{-t\varepsilon} ,$$

where $M_1$ does not depend on $t$; we conclude that $U = 0$ in $U_\varepsilon$, q.e.d.

We end now by a few remarks about admissible sets.

**1.** Any open set $\Omega$, strictly convex at a boundary point $z_0$ (and bounded near $z_0$ by a piece of $C^\infty$ hypersurface) is admissible at this point. For simplicity, let us assume that $z_0 = 0$, and let $H$ be an hyperplane passing by 0, such that $\bar{\Omega}$ intersects $H$ only at the origin and lies entirely on one side of $H$ (at least near 0). Let $N$ be the unit vector, orthogonal to $H$, which lies on the side of $H$ containing $\Omega$. If $N_1, \cdots, N_n$ are the complex components of $N$, we may choose, as holomorphic function $F(z)$, the hermitian product $\bar{N}_1 z_1 + \cdots + \bar{N}_n z_n$.

**2.** There are open sets, admissible at a boundary point, which are not strictly convex at this point. For instance, consider an open set $\Omega$ whose boundary contains the origin (and is a piece of $C^\infty$ hypersurface near it) and whose complement contains the cylinder $|z_1 - \alpha| < |\alpha|$, $\alpha$ being a complex number $\neq 0$. If the diameter of the intersection of $\Omega$ with the cylinder $|z_1 - k\alpha| < e^0 k|\alpha|$ $(k < 1,\ \varepsilon > 0)$ tends to 0 when $\varepsilon \to 0$, $\Omega$ will be admissible at $z = 0$. For then we may take, as holomorphic function $F(z)$, any branch of $-\log(1 - z_1/k\alpha)$. If $n = 1$, any open set whose complement contains the circle $|z_1 - \alpha| < |\alpha|$ (and whose boundary, near 0, is a piece of $C^\infty$ curve passing by 0) is admissible at $z_1 = 0$. If $n > 1$, one may still construct open sets having the desired properties, which are not strictly convex at $z = 0$.

**3.** *Let $F(z)$ be any holomorphic function of $z$ in a neighborhood $U$ of 0 in $C^n$, such that $F(0) = 0$. Let $U_+$ be the set of points $z \in U$ such that $Re\, F(z) > 0$. If $n > 1$, the set $U_+$ cannot be strictly convex at $z = 0$.*

It $U_+$ were strictly convex at 0, there should exist an hyperplane $H$, passing by 0, intersecting $\bar{U}_+$ only at this point 0 and such that $U_+$ would lie only on one side of $H$. Let $\Omega$ be the other side of $H$, and $U(b)$ be the set of $z \in U$ such that $Re\, F(z) > -b$, $(b > 0)$. After maybe shrinking $U$ we may say that the diameter of $U(b) \cap \Omega$ converges to 0 when $b \to 0$. For assume that this were not true: there would be pairs of points $z'_k$, $z''_k$ in $U(1/k)$ such that $|z'_k - z''_k| \geqq c > 0$ for every $k = 1, 2, \cdots$. We could assume that $z'_k$ converges to $z'$, $z''_k$ to $z''$, and we should have: $|z'-z''| \geqq c$, $z', z'' \in \bar{\Omega}$. But also $Re\, F(z')=0$, $Re\, F(z'')=0$, i.e., $z', z'' \in \bar{U}_{+}$. But that implies $z' = z'' = 0$, which is absurd. Hence the open set $\Omega$ is admissible at $z = 0$. But if $\Omega$ is admissible at some boundary point, the same must clearly be true for any open half space in $C^n$. And this would mean that there is uniqueness in the Cauchy problem for data on an arbitrary hyperplane and for any differential polynomial

$$P\left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right),$$

which is absurd.

## Reference

1. L. Nirenberg, *Uniqueness in the Cauchy problem* $\cdots$, Comm. Pure Appl. Math., **10** (1957), 85–105.