SOME THEOREMS ON MAPPINGS ONTO

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Introduction and summary. Let $F: X \to Y$ be a continuous mapping of a topological space X into a space Y. An important problem is to find conditions[1] under which this mapping is a mapping onto: $F(X) = Y$. In the present paper, the following consideration is used in proving theorems on mappings onto.

Conditions are given under which the image $F(X)$ is closed and open in Y; hence for a connected $Y, F(X) = Y$.

This idea is not new. For instance C. Kuratowski[2] showed that, i, a subgroup G of a topological additive group X has the Baire propertyf then either G is of the first category in X or G is *both open and closed in* X, *so that* $G = X$ *if* X *is connected.*

It was also used by the author in [10], to obtain some generalizations of the Fundamental Theorem of Algebra.

In this paper the results obtained in [10] are generalized to general topological spaces.

In § I the notion of a "polynomial mapping" is introduced. Roughly speaking, a mapping $F: X \to Y$ is called a polynomial mapping if it maps every sequence which does not contain a convergent subsequence onto a sequence which also does not contain a convergent subsequence. It is proved that a polynomial mapping $F: X \to Y$ of a complete space X into a space Y maps sets closed in X onto sets closed in Y.[3]

The role of the disconnection properties in the proofs of theorems on mappings onto is discussed and a generalization of the Fundamental Theorem of Algebra to n-dimensional Euclidean spaces is obtained.

In § II some theorems on mappings onto are proved for the so-called generalized F-spaces and the Fundamental Theorem of Algebra is generalized to such spaces. Finally an application of this generalization to an existence theorem in some class of functional equations is given. For the sake of completeness, many known definitions are recalled.

I, 1. Let X be a space in which convergence satisfying the following two conditions is defined:

(a_0) if $x_n \to x$ and $x_n \to y$ then $x = y$

(a) if $x_n \to x$ and $k_1 < k_2 < \cdots$, then $x_{k_n} \to x$

The set of all convergent sequences $\{x_n\} \subset X$ will be denoted by L. Note that

Received December 21, 1959.

[1] For examples of such conditions see [2], chapter XI.

[2] See [6], p. 38 and [7], p. 81. Also [4], p. 8.

[3] According to the terminology used by Whyburn in [11], we can say that if $F: X \to Y$ is a polynomial mapping, then it is a strongly closed mapping.

(b) if $\{x_n\} \in L$, $k_1 < k_2 < \cdots$ and $x_{k_n} \to x$ then $x_n \to x$

Indeed, since $\{x_n\} \in L$, there exists a point x_0 such that $x_n \to x_0$. Hence by (a), $x_{k_n} \to x_0$ and since $x_{k_n} \to x$ we obtain by (a_0), $x_0 = x$. Thus $x_n \to x$.

In the usual way, we can introduce the notions of a subspace P of X and closedness-and-openness in X (or in P). A set $P \subset X$ is called connected if it is not a union of two non-empty disjoint sets closed in P.

Let C be a set of sequences $\{x_n\} \subset X$ such that $L \subset C$. The set C will be called the set of Cauchy (or fundamental) sequences. If $L = C$, the space is called complete. Note that since the definition of C is quite arbitrary (it is only needed that $L \subset C$) we can put $C = L$ and the space will be a complete space.

I, 2. A mapping $F: X \to Y$ of a space X into a space Y is called continuous if for every sequence $\{x_n\}_{n=1,2,\ldots} \subset X$ the condition $x_n \to x$ implies $F(x_n) \to F(x)$[4]. By "mappings" we shall in the sequel understand continuous mappings only. We introduce now the following

DEFINITION 1. A sequence $\{x_n\}_{n=1,2,\ldots}$; $x_n \in X$ is called a non-Cauchy sequence or simply a *NC* sequence if it does not contain a subsequence belonging to C.

If the set of Cauchy sequences is defined as usual, then

(c) in a *finite* dimensional Banach space the set of *NC* sequences is identical with the set of sequences $\{x_n\}_{n=1,2,\ldots}$ with $x_n \to \infty$.

Indeed, if $x_n \to \infty$, then by the completeness of the Banach space, $\{x_n\}$ cannot contain a Cauchy sequence $\{x_{k_n}\}$, since otherwise, there would be $x_{k_n} \to x$ for some x, which is impossible, by $x_n \to \infty$. On the other hand if x_n does not tend to ∞, there exists a bounded subsequence $\{x'_n\}$ of $\{x_n\}$ and since X is finite dimensional, $\{x'_n\}$ contains a convergent subsequence $\{x'_{k_n}\}$, which is a Cauchy sequence.

DEFINITION 2. A mapping $F: X \to Y$ is called a polynomial mapping[5] if the condition

$$\{x_n\} \text{ is a } NC - \text{sequence}, \ x_n \in X$$

implies that

$$\{F(x_n)\} \text{ is a } NC - \text{sequence}, \ F(x_n) \in Y.$$

By (c) we obtain that

(d) Polynomial mappings $F: X \to Y$ of a *finite* dimensional Banach space X into a finite dimensional Banach space Y are identical with

[4] In fact we should denote the convergence relation in Y by a symbol other than "$\to$" used for convergence in X, but the meaning of "$\to$" will always be clear from the text.

[5] The definition of a polynomial mapping was first introduced by the author in **[10]** for metric spaces. The definition introduced here is a generalization of this definition to general topological spaces.

those which map sequences $\{x_m\}_{m=1,2,\dots}$ tending to ∞ onto sequences $\{F(x_m)\}_{m=1,2,\dots}$ also tending to ∞.

In particular, a ($\neq$ constant) complex polynomial in the complex plane (2-dimensional Banach space) is a polynomial mapping of the plane into itself. This justifies the notion "polynomial mapping".

We prove now the following

LEMMA 1. If $F: X \to Y$ is a polynomial mapping of a complete space X into a space Y, then for every set A closed in X the image $F(A)$ is closed in Y. (In particular $F(X)$ is closed in Y).

Proof. Let $y_n \in F(A)$ be points belonging to $F(A)$ and let $y_n \to y$. We shall show that $y \in F(A)$. Indeed, there exist points $x_n \in A$ such that $y_n = F(x_n)$. Now $\{x_n\}$ cannot be a NC – sequence since $\{F(x_n)\}$ would also be a NC sequence (F being polynomial mapping) and this is impossible by $L \subset C$ and $y_n \to y$. Therefore, there exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ which belongs to C. The space X being complete, there is $x_{k_n} \to x$ for some x and $x \in A$ since A is closed. Thus by the continuity of F, $F(x_{k_n}) \to F(x)$. But $\{F(x_{k_n})\}$ is a subsequence of $\{y_n\}$ and therefore by (b) we have $y_n \to F(x)$. Hence by $y_n \to y$, $F(x) = y$ and by $x \in A$, there is $y \in F(A)$.

DEFINITION 3. A mapping $F: X \to Y$ is said to be open in the point $y_0 \in F(X)$ if there exists an open (in Y) set $U(y_0)$ containing y_0, such that $U(y_0) \subset F(X)$.

Evidently, $F(X)$ is open in Y if and only if $F: X \to Y$ is open in every point $y_0 \in F(X)$.

THEOREM 1. *If $F: X \to Y$ is a polynomial mapping of a complete space X into a connected space Y, which is open in every point $y \in F(X)$, then $F(X) = Y$.* (i.e. $F: X \to Y$ is a mapping onto).

Proof. By Lemma 1, $F(X)$ is closed in Y and by the assumption, $Y - F(X)$ is closed in Y. Hence by the connectedeness of Y there is $F(X) = Y$.

I, 3. We shall now investigate the role of the disconnection properties of subsets of Y in the proofs of theorems on mappings onto. Throughout § I, 3 we shall assume that our spaces satisfy the first countability axiom and thus all the topological relations may be expressed in terms of convergent sequences. The following Lemma is evident.

LEMMA 2. *A mapping $F: X \to Y$ is not open in the point $y \in F(X)$ if and only if $y \in Fr(F(X))$, where $Fr(F(X))$ denotes the boundary of*

$F(X)$ in Y.[6]

We prove now the following

THEOREM 2. *If $F: X \to Y$ is a polynomial mapping of a complete space X into a connected space Y which is open in every point $y \in F(X)-J$, where $J \subset F(X)$ is a set which does not disconnect the space Y and if $F: X \to Y$ is open in at least one point $y_0 \in F(X)$, then $F(X) = Y$.*[7]

Proof. Since $F: X \to Y$ is open in the point $y_0 \in F(X)$ there exists an open (in Y) set $U(y_0) \subset F(X)$. Denote by U the union of all sets open in Y, which are contained in $F(X)$. Evidently $Fr(U) \subset Fr(F(X))$. Now suppose, that there would exist a point $x_0 \in Y - F(X)$ and let x be any point belonging to U. Since the set J does not disconnect Y there exists in $Y - J$ a connected set K containing x_0 and x. But the set $Fr(U)$ disconnect the space Y between x_0 and x.[8] Therefore there exists a point $y \in [Fr(U) - J] \cap K$ and since $Fr(U) \subset Fr(F(X))$ the point $y \in [Fr(FX)) - J] \cap K \subset Fr(F(X)) - J$.

By Lemma 1, $F(X)$ is closed in Y and therefore $y \in F(X)$. But, by assumption $F: X \to Y$ is open in every point $y \in F(X) - J$, which by Lemma 2 contradicts the fact that $y \in Fr(F(X)) - J$. Thus the assumption that there exists a point $x_0 \in Y - F(X)$ leads to a contradiction.

I, 4. We prove now the

First generalization of the Fundamental Theorem of Algebra. Let $F: X \to X$ be a mapping of the n-dimensional Euclidean space X, with $n \geqq 2$ into itself defined by $\eta_i = \eta_i(\xi_1, \cdots, \xi_n)$ $i = 1, 2, \cdots n$, where the real functions η_i and their derivatives $\partial\eta_i/\partial\xi_k$ are continuous in X. If then the Jacobian $D = \partial(\eta_1, \cdots \eta_n)/\partial(\xi_1, \cdots, \xi_n) \neq 0$ in every point $x = x(\xi_1 \cdots \xi_n) \in X - J_0$, where J_0 is a countable set and if the condition $x_m(\xi_1^m, \cdots \xi_n^m) \to \infty$ implies $F(x_m) \to \infty$ for every sequence of points $x_m \in X$, then $F: X \to X$ is a mapping onto: $F(X) = X$.

Proof. Since the condition $x_m \to \infty$ implies $F(x_m) \to \infty$ we obtain by (d) that $F: X \to X$ is a polynomial mapping. Now by the countability of J_0 the set $J = F(J_0)$ is also countable and therefore 0-dimensional.[9] Hence, since the dimension n of X is $\geqq 2$, J does not disconnect X.[10] Further, if $y = F(x)$ is any point of $F(X) - J$, the mapping $F: X \to X$ is open in y, because, by the assumption $D \neq 0$ for points $x \notin J_0$, a neighbourhood (in X) of $y = F(x)$ is covered. Finally, since J_0 is countable there exists at least one point y_0 in which $F: X \to X$ is open. Thus put-

[6] If X is a space and A a subset of X, the boundary $Fr(A) = \bar{A} \cap \overline{X - A}$.

[7] This Theorem was suggested to the author by H. Hanani.

[8] S. [**3**], p. 247, also [**8**], p. 80.

[9] In the sense of Menger-Urysohn

[10] See [**5**], p. 48.

ting $Y = X$ in Theorem 2, we see that all the assumptions of this theorem hold. Therefore $F(X) = X$.

REMARK 1. If $F : X \to X$ is any ($\neq$ constant) complex polynomial defined on the complex plane X, then the mapping $F : X \to X$ is defined by two real functions $\eta_1 = \eta_1(\xi_1, \xi_2)$, $\eta_2 = \eta_2(\xi_1, \xi_2)$, where η_1 and η_2 are respectively, the real and imaginary parts of $F(x) = \eta_1 + i\eta_2$, $x = \xi_1 + i\xi_2 \in X$, $i^2 = -1$. Now for the Jacobian D there is $D = |F'(x)|^2 \neq 0$ except for a finite set of points ($F'(x)$ denotes the derivative of $F(x)$) and therefore, by the above generalization of the Fundamental Theorem of Algebra, $F(X) = X$.

REMARK 2. Our proof of the first generalization of the Fundamental Theorem of Algebra is based on Theorem 2. An essential role in Theorem 2 is played by the assumption that the set J does not disconnect Y. This assumption is satisfied because the dimension of the Euclidean space X is assumed to be $\geqq 2$. (A countable set does not disconnect an Euclidean space with dimension $\geqq 2$). This explains the role, for the Fundamental Theorem of Algebra, of the fact that the dimension of the Euclidean plane is $\geqq 2$.

II, 1. Let now X be a space (in the sense of I, 1) which is simultaneously a linear space (with multiplication by real or complex numbers). We introduce the following

DEFINITION 4. A mapping $F : X \to X$ of a linear space into itself is said to have a lower bounded rate of change in the point $y_0 \in F(X)$ if there exists a point $x_0 \in F^{-1}(y_0)$, a number (may be complex) $\lambda(x_0) \neq 0$ and an open in X set $U(y_0)$ containing y_0, such that for every $y' \in U(y_0)$ the sequence: $x'; Ax'; AAx'; \cdots$ where $Ax = x - \lambda(x_0)(F(x) - y')$ is a Cauchy sequence for some point $x' \in X$ (the point x' depends on y').

LEMMA 3. *If $F : X \to X$ is a mapping of a complete linear space X into itself, having a lower bounded rate of change in the point $y_0 \in F(X)$, then $F : X \to X$ is open in the point y_0.*

Proof. Let x'; x_0, $\lambda(x_0) \neq 0$ and $U(y_0)$ be the points, the number and the open (in X) set defined in the foregoing definition and let $y' \in U(y_0)$ be any point of $U(y_0)$. We shall show that $y' \in F(X)$. Indeed, since the sequence $x^-; Ax'; AAx'; \cdots$ is a Cauchy sequence and X is complete, it has a limit x_0'. Now by the continuity of A, we have $Ax_0' = x_0'$, i.e. $x_0' - \lambda(x_0)(F(x_0') - y') = x_0'$ and hence $F(x_0') = y'$.

II, 2. Here we shall introduce the notion of a generalized F-space and prove some theorems on mappings onto in these spaces. We begin with the definition of a generalized metric space.

DEFINITION 5. A set X of points is called a generalized metric space with metric ρ if on the Cartesian product $X \times X$ a non-negative real function $\rho(x, y)$ is defined: $0 \leqq \rho(x, y) \leqq \infty$, $x, y \in X$ which satisfies the usual axioms of any metric, i.e. $\rho(x, y) = 0$ if and only if $x = y$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) + \rho(y, z) \geqq \rho(x, z)$; x, y, z, belong to X.

Thus the difference between the definitions of a metric space and of the generalized metric space consists in the fact that the function $\rho(x, y)$ may assume the value of ∞.

EXAMPLE 1. Take the set X of all real continuous functions $x(t)$, $-\infty < t < \infty$ and define $\rho(x, y) = \sup_t | x(t) - y(t) |$. Here, X is a generalized metric space, but not a metric space.

Evidently every metric space is also a generalized metric space. In generalized metric spaces we can define convergent sequences by saying that $x_n \to x$ if $\rho(x_n, x) \to 0$ and this convergence satisfies (a_0) and (a). Thus every generalized space is a space in the sense of I, 1. If we define, as usual, a Cauchy sequence in X by saying that $\{x_n\}_{n=1,2,\cdots} \in C$ if for every $\varepsilon > 0$ there exists a $N(\varepsilon)$ such that $\rho(x_n, x_m) < \varepsilon$ for $n, m > N(\varepsilon)$ then we have $L \subset C$. The set X in Example 1 is, as is easy to see, a complete space. Let now X be a generalized metric space which is simultaneously a linear space (with multiplication by real or complex numbers) such that the following two conditions hold

(e) $\rho(x, y) = \rho(x - y, 0)$

and

(f) $(hx, 0) \leqq | h |.\ \rho(x, 0)$ for every number h and every point $x \in X$.

In such a case we shall call X a generalized F-space. For instance, the space X in Example 1 is a generalized F-space. Another example of a generalized F space is the set of all sequences $x = (\xi_1, \xi_2, \cdots)$ of real numbers ξ_i, $i = 1, 2, \cdots$ if we define $\rho(x, y) = \sup_i | \xi_i - \eta_i |$ where $y = (\eta_1, \eta_2, \cdots)$. Note that on the set on which ρ is finite, it satisfies the axioms of the so-called F-spaces.[11] This justifies the notion of generalized F-space.

Property T. Let $F : X \to X$ be a mapping of a generalized F-space into itself such that for the point $y_0 \in F(X)$ there exists a point $x_0 \in F^{-1}(y_0)$, a spherical region $S(x_0, r)$,[12] a complex function $\lambda(x_0) \neq 0$ and a real function $\alpha = \alpha(x_0)$: $0 < \alpha < 1$ such that for any two points x and y

[11] See [1], p. 35.

[12] A spherical region $S(x_0, r)$ with x_0 as centre and r as radius is defined as the set of all points $x \in X$ satisfying $\rho(x_0, x) < r$.

belonging to $S(x_0, r)$ there is

(g) $\rho[x - y - \lambda(x_0)(F(x) - F(y)), 0] \leqq \alpha\rho(x, y)$.

A mapping $F: X \to X$ for which (g) holds is said to have the property T in the point $y_0 \in F(X)$.[13]

We prove now

LEMMA 4. *If $F: X \to X$ is a mapping of a generalized F-space into itself having the property T in the point $y_0 \in F(X)$, then $F: X \to X$ has a lower bounded rate of change in the point $y_0 \in F(X)$.*

Proof. Let $S(x_0, r)$, $\lambda(x_0) \neq 0$ and $\alpha = \alpha(x_0)$ be the spherical region and the functions appearing in the definition of the property T and put $r_0 = [(\varDelta - \alpha)/|\lambda(x_0)|]\cdot r$. It suffices to show that for every $y' \in S(y_0, r_0)$ the sequence $x_0, Ax_0, AAx_0, \cdots$ where $Ax = x - \lambda(x_0)(F(x) - y')$ is a Cauchy sequence:[14]

We have for $x_1 = Ax_0$;

$$\begin{aligned}\rho(x_0, x_1) &= \rho(\lambda(x_0)(F(x_0) - y'), 0) = \rho[\lambda(x_0)(y_0 - y'), 0] \\ &\leqq |\lambda(x_0)|\,\rho(y_0, y') \leq |\lambda(x_0)|\cdot r_0 = (1 - \alpha)r .\end{aligned}$$

Thus $\rho(x_0, x_1) \leqq (1 - \alpha)r$. Now for $x, y \in S(x_0, r)$ we have by (g):

$$\text{(g)};\ \rho(Ax, Ay) = \rho(x - y - \lambda(x_0)(F(x) - F(y)), 0) \leqq \alpha\rho(x, y)$$

and therefore $\rho(Ax_0, Ax_1) \leqq \alpha\rho(x_0, x_1) \leqq (1 - \alpha)\cdot\alpha\cdot r$. Denoting $x_n = Ax_{n-1}$, $n = 1, 2, \cdots$ we obtain by induction $\rho(x_n, x_{n+1}) \leqq (1 - \alpha)\alpha^n r$, hence by $0 < \alpha < 1$ it is easily seen that $x_0, Ax_0, AAx_0, \cdots$ is a Cauchy sequence.

From Lemmas 3 and 4, and from Theorem 2, we obtain the

Second generalization of the Fundamental Theorem of Algebra. If $F: X \to X$ is a polynomial mapping of a complete generalized F-space X into itself, having the property T in every point $y \in F(X) - J$ where $J \subset F(X)$ is a set which does not disconnect the space X, and if there exists at least one point $y_0 \in F(X)$ in which $F: X \to X$ has the property T, then $F(X) = X$.

Proof. By Lemma 4, $F: X \to X$ has a lower bounded rate of change in every point $y \in F(X) - J$; hence by Lemma 3 $F: X \to X$ is open in every point $y \in F(X) - J$ where by assumption J does not disconnect X. Also, by assumption $F: X \to X$ is open in the point $y_0 \in F(X)$. Now, since a generalized F-space is connected (as a linear space) we see, by putting $Y = X$ in Theorem 2, that the assumptions of this theorem hold. Hence $F(X) = X$.

[13] For some ideas concerning this definition the author is indebted to D. Tamari.

[14] The following part of the proof is analogous to the proof of Banach's so-called theorem of contraction mappings, see **[9]** p. 47 and remark 2 p. 49.

REMARK 3. We shall now show that the above theorem is in fact a generalization of the Fundamental Theorem of Algebra. Let $F: X \to X$ be a ($\neq$ constant) polynomial, mapping the complex plane X into itself. Since every Banach space is evidently a generalized F-space, the complex plane with the usual metric $\rho(x, y) = |x - y|$ is a generalized F-space. Now take any point $x_0 \in X$ such that the derivative $F'(x_0) \neq 0$ and let $y_0 = F(x_0)$. Put $\lambda(x_0) = 1/(F'(x_0))$ and $\alpha = 1/2$. Since for $x \to x_0$ we have $(F(x_0) - F(x))/(x_0 - x) \to F'(x_0)$ there exists a spherical region $S(x_0, r)$ such that for $x, y \subset S(x_0, r), x \neq y$ there is

$$\left|\frac{F(x) - F(y)}{x - y} \cdot \frac{1}{(F'(x_0)} - 1\right| \leqq \frac{1}{2}$$

and therefore $|x - y - (1/(F'(x_0))(F(x) - F(y))| \leqq \alpha |x - y|$. But this inequality holds evidently also for $x = y$ and therefore the mapping $F: X \to X$, defined by the complex polynomial $F(x)$, $x \in X$, has the property T in every point $y_0 = F(x_0)$ for which $F'(x_0) \neq 0$. Now the set J of points $y = F(x)$ for which $F'(x) = 0$ is finite and thus it does not disconnect the complex plane X. Hence by the above second generalization of the Fundamental Theorem of Algebra there is $F(X) = X$, i.e., a complex polynomial maps the complex plane onto itself.

II, 3. It is known that

(h) a n-dimensional set does not disconnect the $n + 2$-dimensional Euclidean space.[15]

Thus

LEMMA 5. *A finite dimensional subset J of an infinite dimensional Banach space X does not disconnect X.*

Proof. Let $x_0 \in X - J$ be any fixed point and x any point of $X - J$. Suppose that the dimension of J is n and take any $(n + 2)$-dimensional plane E^{n+2} (homeomorphic with the Euclidean plane E^{n+2}) containing the points x_0 and x: $E^{n+2} \subset X$. Since the set $E^{n+2} \cap J$ is at most n-dimensional it does not disconnect E^{n+2} (by (h)) and therefore there exists a connected set $K \subset E^{n+2} - J \subset X - J$ which contains the points x_0 and x. Thus every point $x \in X - J$ may be joined with the point $x_0 \in X - J$ by a connected set $K \subset X - J$ i.e., the set $X - J$ is connected.

Let now $\|\ \|$ denote the norm in the Banach space X and define $\rho(x, y) = \| x - y \|$.

We prove the following

THEOREM 3. *Let $F: X \to X$ be a polynomial mapping of an infinite*

[15] S. [5], p. 48. The term "dimension" is used in the sense of Menger-Urysohn.

dimensional Banach space X into itself, which maps finite dimensional sets onto finite dimensional sets. If $F: X \to X$ has the property T in every point $y \in F(X - J_0)$ where J_0 is a finite dimensional set, then $F(X) = X$.

Proof. Since J_0 has a finite dimension there exists a point $x_0 \in X - J_0$ and thus the mapping $F: X \to X$ has the property T in the point $y_0 = F(x_0)$. Now, the set J of points in which the mapping does not have the property T is contained in $F(J_0)$ and since $F: X \to X$ maps finite dimensional sets onto finite dimensional sets the set J does not disconnect the space X (by Lemma 5). Hence by the second generalization of the Fundamental Theorem of Algebra there is $F(X) = X$.

Analogously to Lemma 5 it can be proved that

LEMMA 6. *A 0-dimensional set J does not disconnect a n-dimensional Banach space for $n \geqq 2$.*[16]

Hence

THEOREM 4. *Let $F: X \to X$ be a polynomial mapping of a n-dimensional Banach space X into itself, with $n \geqq 2$. If $F: X \to X$ has the property T in every point $y \in F(X - J_0)$ where J_0 is a countable set, then $F(X) = X$.*

Proof. For the proof, it suffices to note that the set $F(J_0)$ is countable and thus, by Lemma 6, does not disconnect the space X. The rest of the proof is analogous to that of Theorem 3 and may be left to the reader.

II, 4. *An application.* Let X be the generalized F-space of all real continuous functions $x(t)$ defined on the real line $-\infty < t < \infty$ with metric $\rho(x, y) = \sup_t |x(t) - y(t)|$ and let $\phi(t, u)$ be a real continuous function defined for $-\infty < t < \infty$, $-\infty < u < \infty$ satisfying the conditions:

(i) There exists a real number $m > 0$ such that for every pair $u_1 \geqq u_2$ of numbers there is $\phi(t, u_1) - \phi(t, u_2) \geqq m(u_1 - u_2)$.

(j) For each function $x_0(t) \in X$ there exist numbers $r > 0$ and M (depending on $x_0(t)$ and r) such that for $x(t)$ and $y(t) \in S(x_0, r)$ there is $|\phi(t, x(t)) - \phi(t, y(t)| \leqq M |x(t) - y(t)|$ for every t: $-\infty < t < \infty$.

Then, the mapping $F(x(t)) = \phi(t, x(t))$ maps X onto X.

Proof. We shall first show that $F: X \to X$ is a polynomial mapping. Indeed, we have by (i) $\rho(F(x_n), F(x_m)) \geqq m \cdot \rho(x_n, x_m)$ for every pair $x_n(t)$ and $x_m(t)$ of functions. Therefore, if the sequence $\{F(x_n)\}$ would contain

[16] This Lemma and Theorem 4 were suggested to the author by H. Hanani.

Cauchy subsequence $\{F(x_{k_n})\}$ the sequence $\{x_{k_n}\}$ would be a Cauchy subsequence of the sequence $\{x_n\}$. Thus $F: X \to X$ maps NC sequences onto NC sequences, i.e., it is a polynomial mapping.

Since our space is complete it suffices, by the second generalization of the Fundamental Theorem of Algebra, to prove that $F: X \to X$ has the property T in every point $y_0 = F(x_0) \in F(X)$. Indeed, take any two points $x(t)$ and $y(t)$ belonging to $S(x_0, r)$. Then for t such that $x(t) \geqq y(t)$ we have by (i) and (j) $m(x(t) - y(t)) \leqq F(x(t)) - F(y(t)) \leqq M(x(t) - y(t))$, hence $-(m/M)[x(t)-y(t)] \geqq -(1/M)\cdot[F(x(t)) - F(y(t))] \geqq -[x(t)-y(t)]$. Thus $(1 - m/M)\cdot[x(t) - y(t)] \geqq x(t) - y(t) - 1/M[F(x(t)) - F(y(t))] \geqq 0$. Therefore for any t such that $x(t) \geqq y(t)$ we have

(k) $(1 - m/M) \vee |x(t)-y(t)| \geqq |x(t)-y(t) -(1/M)[F(x(t))-F(y(t))]|$.

Analogously, for any t such that $y(t) \geqq x(t)$, (k) holds and therefore (k) holds for every t. Thus assuming that $M > m$ and putting $\lambda = 1/M$ and $\alpha = 1 - m/M$ we see by (k) that $F: X \to X$ has the property T in the point $y_0 = F(x_0)$.

Example 2. If $\phi(t, u)$ is a real continuous function defined for $-\infty < t < \infty$ and $-\infty < u < \infty$ having a continuous derivative $\phi_u(t, u)$ such that there exist constants m and $M: M > m > 0$ for which

$$m \leqq \phi_u(t, u) \leqq M$$

for every t and u, then evidently (i) and (j) hold and hence the function $F(x(t)) = \phi(t, x(t))$ maps X onto X. Such a function $\phi(t, u)$ is for instance the function $\phi(t, u) = 2u + 3t + \sin(u + t)$.

Remark 4. An analogous theorem to the above application was proved in [10] for the space X of all real continuous functions $x(t)$ defined in a *finite* interval $a \leqq t \leqq b$[17].

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[17] See [**10**], p. 162, Application of Theorem 4.

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