INTEGRAL CLOSURE OF DIFFERENTIAL RINGS

EDWARD C. POSNER

We prove that a commutative differentiably simple ring of characteristic zero finitely generated over its field of constants is integrally closed in its field of quotients. (A ring is differentiably simple if it has non-trivial multiplication and has no ideal invariant under a given family of derivations; i.e., has no differential ideals other than (0). The field of constants is the subring of the ring annihilated by each derivation of the family of derivations.) The result of the first sentence is used to obtain a condition that the powers of an element of a function field in one variable form an integral basis. The following results from [1] will be used: A commutative differentiably simple ring of characteristic zero is an integral domain whose ring of constants is a field. Crucial is the following lemma:

LEMMA. *Let F be a field of characteristic zero; $x_1, \cdots, x_n$ be n independent transcendentals over F; $y_1, \cdots, y_q$ be integral over $x_1, \cdots, x_n$; and d an F-derivation of $F[x, y]$ into itself. Then d (or rather its natural extension to $F(x, y)$) sends O_x (the set of elements of $F(x, y)$ integral over $x_1, \cdots, x_n$) into itself.*

Proof. In general any F-derivation of $F(x, y)$ into itself can be written as

$$d = \sum_{i=1}^{n} A_1 \frac{\partial}{\partial x_i} ,$$

A_i elements of $F(x, y)$, $1 \leqq i \leqq n$, Further, d maps $F[x, y]$ into itself if and only if $d(x_i)$ is in $F[x, y]$ for each i and $d(y_j)$ is in $F[x, y]$ for each j. The first set of conditions is equivalent to the condition that A_i be in $F[x, y]$ for each i.

In order to be able to use power series, we assume that F is algebraically closed. For if not, let $\bar{F}$ be its algebraic closure. Let d also be the extension of d to $\bar{F}(x, y)$. Since d sends $\bar{F}[x, y]$ into itself, d send $\bar{O}_x$ into itself, where $\bar{O}_x$ denotes the ring of integral functions of $\bar{F}(x, y)$. A fortiori, d sends O_x into $\bar{O}_x$. But $\bar{O}_x \cap F[x, y] = O_x$ so actually d sends O_x into itself as required.

Let P be a place of $F(x, y)$ over F which has residue field F and which is finnite on $x_1, \cdots, x_n$. We will prove that if g, in $F(x, y)$, is finite at P, $d(g)$ is finite at P. Let a_i denote the residue of x_i at P; then there exist uniformizing parameters $t_1, \cdots, t_n$ at P such that

Received August 13, 1959, and in revised form January 23, 1960.

$x_i - a_i$ is a positive integral power of t_i, say $x_i - a_i = t_i^{p_i}$. Every element B of $F(x, y)$ finite at P has a power series in $t_1, \cdots, t_n$ with coefficients in F. We call the smallest power of t_i occurring in this series the i-order of B at P, and denote it by $\operatorname{ord}_{P,i} B$; the definition of $\operatorname{ord}_{P,i} B$ extends to arbitrary elements B of $F(x, y)$ in an obvious way. Fixing i, we see that if $\operatorname{ord}_{P,i} d(B) \geqq \operatorname{ord}_{P,i} B$ for every B finite at P then $\operatorname{ord}_{P,i} d(B) \geqq 0$ for every such B. Suppose there exists some B finite at P with $\operatorname{ord}_{P,i} d(B) < \operatorname{ord}_{P,i} B$. Then $\alpha_i - p_i < 0$, where $\alpha_i = \operatorname{ord}_{P,i} A_i$, so that $r_i = p_i - \alpha_i > 0$, and $\operatorname{ord}_{P,i} B = r_i + \operatorname{ord}_{P,i} dB$ for every (B) in $F(x, y)$ with $\operatorname{ord}_{P,i} B \neq 0$. Since d maps $F[x, y]$ into itself, the only values which $\operatorname{ord}_{P,i} B$ can have when B is in $F[x, y]$ are integral multiples of r_i, for otherwise some element of $F[x, y]$ would have negative i-order. Since $t_1, \cdots, t_n$ are uniformizing parameters, it follows that $r_i = 1$, for otherwise we could replace t_i by $t_i^{r_i}$. Thus, d drops positive i-orders by 1, so that $\operatorname{ord}_{P,i} d(B) \geqq 0$ for every B finite at P. Since this holds for every i, $d(B)$ is finite at P whenever B is. Since this holds for every P, we conclude that d maps O_x into itself.

THEOREM 1. *Let F be a field of characteristic zero, $A = F[z_1, z_2, \cdots, z_k]$ a commutative finitely generated ring extension of F. Let D be a (finite or infinite) family of derivations of A into itself over F. Let A be differentiably simple under D. Then A is integrally closed in its quotient field K.*

Proof. A is an integral domain by (1). By Noether's Normalization Lemma, we can write $A = F[x_1, \cdots, x_n; y_1, \cdots, y_q]$, with n the transcendence degree of K/F and $y_1, \cdots, y_q$ interal over $x_1, \cdots, x_n$. To prove $A = O_x$, let I denote the conductor of O_x, that is, the set of elements u of $F[x, y]$ such that $u \cdot O_x \subset F[x, y]$; by [3], pp. 271–2, prop. 6, I is a non-zero ideal of $F[x, y]$. To prove I differential under D, let d be in D, h be in I, g be in O_x. Then $h \cdot g$ is in $F[x, y]$, $d(h \cdot g)$ is in $F[x, y]$, $d(h)g + hd(g)$ is in $F[x, y]$. Now $d(g)$ is in O_x by the lemma so $hd(g)$ is in $F[x, y]$ since h is in I. Then $d(h)g$ is in $F[x, y]$, I is differential under D. Then $I = F[x, y]$ so $1 \cdot O_x \subset F[x, y]$, $O_x = F[x, y]$ as promised.

REMARK. D can always be taken to be finite since the derivations of $F[x, y]$ into itself form a finite $F[x, y]$-module.

The converse of Theorem 1 is false, i.e., there are integrally closed finitely generated domains which are not differentiably simple under any family of F-derivations. For example, let $y^2 = x_1^3 + x_2^3$. Then $F[x, y] = O_x$ but is not differentiably simple over F. In fact, the ideal (x_1, x_2, y) of $F[x, y]$ is differential for any derivation, as is easy to see. But when $n = 1$, we do have the converse. (For background material, see [2], pp. 83–88.)

THEOREM 2. *Let K be a function field in one variable over a field F of characteristic zero, and let x be an element of K transcendental over F. Let O_x denote the set of elements of K integral over x. Then O_x is differentiably simple with field of constants F under a family of two or fewer derivations.*

Proof. First we shall specify the derivations. O_x is a Dedekind ring, i.e., every ideal of O_x is invertible. Let $K = F(x, y)$ with y integral over x and let $f(x, y) = 0$ be the irreducible monic for y. Define d on K by

$$d(g(x, y)) = \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial x}\,.$$

This is well-defined, and d sends O_x into itself by the lemma. Let J be the ideal of O_x generated by the values of d of integral elements. J is invertible, so there exist $h_i(x, y)$ in K, $1 \leqq i \leqq q$, such that $h_i d$ sends O_x into itself and such that there exist u_i in O_x, $1 \leqq i \leqq q$, with $\sum_{i=1}^{q} h_i d(u_i) = 1$. ($q$ can be taken to be 2. For J is generated by f_x and f_y, since $d(M(x, y)) = f_y M_x - f_x M_y$ for M in K. q can be taken to be 1 if and only if J is principal, which need not occur.) The family D is $\{h_1 d, \cdots, h_q d\}$. To prove O_x differentiably simple under D, suppose the contrary. As in the preceding and following theorems, F may be assumed to be algebraically closed. If O_x has a non-zero differential ideal, it has a maximal differential ideal I, since O_x has a unit. O_x^2 is not contained in I, so by Theorem 4 of [1], I is prime. But every prime ideal of O_x is maximal; in fact, if w belongs to O_x, there is a λ in F with $w - \lambda$ in I. Since I is differential for D, $h_i d(w) - h_i d(\lambda)$ is in I, $1 \leqq i \leqq q$, $h_i d(w)$ is in I, $1 \leqq i \leqq q$. That is, $h_i d(w)$ is in I for all w in O_x. Then $\sum_{i=1}^{q} h_i d(u_i) = 1$, 1 is in I, $I = O_x$. This contradiction proves that O_x has no differential ideals. Its field of constants is F. For if u is in $F(x, y)$ and $d(u) = 0$ then $(d/dx)(u) = 0$, so that u belongs to F.

THEOREM 3. *Let K, F, x, O_x be as in the hypothesis of Theorem 2. Let R be an order of O_x and let y be an element of K integral over x with irreducible monic f such that $K = F(x, y)$. Then $R = O_x$ if and only if y belongs to R and the ideal J in R generated by f_x and f_y is invertible.*

Proof. If $R = O_x$, then y belongs to R and every ideal in R is invertible. Conversely, suppose that y belongs to R and that J, the ideal generated in R by the values of d, is invertible. (Here d is the same derivation as in Theorem 2.) That is, assume that there exist h_i

in K, $1 \leqq i \leqq q$, with $h_i d$ sending R into itself, and elements v_i in R, $1 \leqq i \leqq q$, with $1 = \sum_{i=1}^{q} h_i d(v_i)$. We shall prove R differentiably simple under $D = \{h_i d, \cdots, h_q d\}$. It is known that every prime ideal of R is maximal; it fact, if I is a prime ideal of R, and w is an element of R, there is a λ in F with $w - \lambda$ in I. If R has a differential ideal, it has a maximal differential ideal, and one proceeds as in Theorem 2. So R is differentiably simple under D. By Theorem 1, R is integrally closed in K, i.e., $R = O_x$ as required.

As an illustration, let $K = F(x, y)$ with $f(x, y) = y^n - P(x) = 0$, $n \geqq 1$, P a polynomial in x with no repeated roots. Here $R = F[x, y]$. Let us examine the ideal in $F[x, y]$ generated by f_x and f_y, i.e., by $P'(x)$ and y^{n-1}. This ideal contains $y^{n-1}y = y^n = P(x)$ and $p'(x)$. $P(x)$ and $P'(x)$ have no common factor, so there are polynomials $Q(x)$ and $S(x)$ with $QP + SP' = 1$. Then the ideal generated by f_x and f_y is $F[x, y]$ and so is trivially invertible. We conclude $F[x, y] = O_x$.

BIBLIOGRAPHY

1. E. C. Posner, *Differentiably simple rings*, Proc. Amer. Math. Soc., **11** (1960), 337343.

2. B. L. Van der Waerden, *Modern algebra*. Volume II. New York: Frederick Ungar Publishing Company, 1950.

3. A. Weil, *Foundations of algebraic geometry*, New York: American Mathematical Society, 1946.

THE UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN