ORDERED SEMIGROUPS IN PARTIALLY ORDERED SEMIGROUPS

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In this note we establish a local version of the following result: a locally compact connected partially ordered non-degenerate semigroup S with unit contains a non-degenerate linearly ordered local subsemigroup (containing the unit). This is an extension of a result of Gleason [2; 664] who proved a similar theorem under the additional hypotheses that

(1) S is a semigroup with right invariant uniform structure and

(2) for any compact neighborhood U of the identity there are nets $\{x_i\}$ in S and $\{n_i\}$ integers such that $x_i \to e$ and $x_i^{n_i} \notin U$. A consequence of our theorem is the fact that a nondegenerate compact connected partially ordered semigroup with unit contains a standard thread joining the unit to the minimal ideal.

By a local semigroup S we mean a Hausdorff space with an open subset U and a multiplication $m: U \times U \to S$ which is continuous and associative insofar as is meaningful. A unit is an (unique, if it exists) element u of U satisfying $ux = xu = x$ for all $x \in U$. A local subsemigroup of S is a subset L containing the unit such that for some open set V about the unit, $(V \cap L)^2 \subset L$. We say that the local semigroup S is partially ordered if the relation $\leqq$ defined by $a \leqq b$ if and only if $a = bc$ is reflexive and antisymmetric. In case S is a semigroup, S is partially ordered if and only if each principal right ideal has a unique generator, i.e. (assuming a unit) that $aS = bS$ implies $a = b$. In this case, $\leqq$ is also transitive.

Closure is denoted by *, the null set by $\square$, the boundary of V by $F(V)$, and the complement of B in A by $A \backslash B$.

As in [4] we use the following topolopy for the space $\mathscr{S}(X)$ of non-empty closed subsets of the space X: for open sets U and V of X, let $N(U, V) = \{A \mid A \in \mathscr{S}(X), A \subset U, A \cap V \neq \square\}$; take $\{N(U, V) \mid U, V \text{ open}\}$ for a sub-basis for the open sets of $\mathscr{S}(X)$. It is easy to see that if X is compact Hausdorff, so is $\mathscr{S}(X)$.

THEOREM 1. *Let S be a locally compact partially ordered local semigroup with unit u, and let U_0 be a non-degenerate open connected set about u with U_0^6 defined. Then S contains a non-degenerate compact connected linearly ordered local sub-semigroup L with $u \in L \subset U_0$.*

Received December 7, 1959. This work was supported by the United States Air Force through the Air Force Office of Scientific Research, Air Research and Development Command, under Contract AF 18 (603)-89

Proof. Let U_1 be an open set containing u with U_1^* compact and $U_1^* \subset U_0$. Define $\leqq$ on $U_1^* \times U_1^*$ by: $a \leqq b$ if and only if $a = bc$ for some $c \in U_1^*$. From the compactness of U_1^* it is easily seen that Graph $(\leqq)$ is closed in $U_1^* \times U_1^*$. We show first that $\leqq$ is transitive on some neighborhood of u. Let U_2 be an open set about u with $U_2^2 \subset U_1$. We claim there is an open set U containing u, $U \subset U_2$, such that if $a, b \in U^*$ with $a = bc$ for some $c \in U_1^*$, then $c \in U_2$. If this is false, then for any open set U with $u \in U \subset U_2$, there are elements a and b of U^* with $a = bc$ for some $c \in U_1^* \backslash U_2$. Hence there are nets a_α and b_α converging to u with $a_\alpha = b_\alpha \cdot c_\alpha$ where $c_\alpha \in U_1^* \backslash U_2$. It follows that c_α must also converge to u, a contradiction. Since $U_2^2 \subset U_1$ it follows that $\leqq$ is transitive on U^*. Also the restriction of $\leqq$ on $U^* \times U^*$ is closed and hence U^* is locally convex [6]. We show next that there exists an open set V_1 with $u \in V_1 \subset U$ such that $e^2 = e \in V_1$ implies $eU_0e \neq e$. Suppose the contrary; we can then find a net of idempotents $e_\alpha \rightarrow u$ with $e_\alpha U_0 e_\alpha = e_\alpha$. Let $x \in U_0$; then $e_\alpha = e_\alpha x e_\alpha$ converges to $uxu = x$, so that $x = u$ and U_0 is degenerate, a contradiction. Let V be a convex open set with $u \in V \subset V^* \subset (V^*)^2 \subset V_1$. Then $e^2 = e \in V$ implies $eU_0e \neq e$.

Let $\mathscr{C}$ denote the collection of all closed chains C in U^* with $u \in C$, $C \cap S \backslash V \neq \square$, and $(C \cap V)^2 \subset C$. Note that $\mathscr{C} \neq \square$, for if $a \in F(V)$, then the elements u and a constitute an element of $\mathscr{C}$.

(i) *$\mathscr{C}$ is closed in $\mathscr{S}(U^*)$*. We will show that $\mathscr{C}$ is an intersection of closed set. Since the collection of all closed chains which contain u and meet $S \backslash V$ is closed [4], it remains to show that the collection of closed chains C satisfying $(C \cap V)^2 \subset C$ is closed. Suppose A is a closed chain with $(A \cap V)^2 \not\subset A$; then there are elements a and b of $A \cap V$ with $ab \in S \backslash A$. Hence there exist open sets U_a, U_b, and W containing a, b, and A respectively, with $U_a \cdot U_b \cap W = \square$. Now $N(W, U_a) \cap N(W, U_b)$ is an open set about A, and contains no chain C with $(C \cap V)^2 \subset C$. This establishes (i).

As in [4], we define $L(x) = \{y \mid y \leqq x\}$, $M(x) = \{y \mid x \leqq y\}$, and $(x, y) = \{z \mid x < z < y\}$. Let δ be an open cover of U^*, and define a subset M_δ of $\mathscr{S}(U^*)$ by: $C \in M_\delta$ if and only if C is a closed chain in U^*, and for any x and y in C with $x < y$ and $(x, y) \cap C = \square$, there exists $D \in \delta$ such that D^* meets both $L(x) \cap C$.

(ii) *$M_\delta \cap \mathscr{C} \neq \square$ for any open cover δ of U^**. Let δ be an open cover of U^*, and let $\mathscr{D}$ be the collection of all closed chains C with $u \in C \subset U$, $C \in M_\delta$, and $(V \cap C)^2 \subset C$. Let τ be a maximal tower in $\mathscr{D}$, and let $T = U\tau$. Then T^* is a closed chain, $u \in T^* \subset U^*$, and $(V \cap T^*)^2 \subset T^*$. As in [4], $T^* \in M_\delta$, and it remains to show that $T^* \in \mathscr{C}$, i.e., that $T^* \cap S \backslash V \neq \square$. Suppose $T^* \subset V$; (note then that $T = T^*$) then since $(T \cap V)^2 \subset T$, T is a compact chain and a semigroup. Let $e = \inf T$. Since $e^2 \leqq e$ and $e^2 \in T$ we have $e^2 = e$. We show next that e is a zero for T. Let $y \in T$, then $ey \in T$ and $ey \leqq e$, so $ey = e$ and e is a left zero for T. Hence the minimal ideal K of T consists of left zeros for T [1]. Let $f \in K$; then $e \leqq f$ so there exists $c \in U_1^*$ with $e = fc$. Therefore $f = fe = e$, and e is the unique left zero, and hence a zero for T. Let $W \in \delta$ with $e \in W$. If $eU_0e \cap W \cap V$ contains an idempotent $g \neq e$, then $T \cup g$ is a semigroup: for if $x \in T$ then $xg = x(eg) = eg = g$ and $gx = (ge)x = g(ex) = ge = g$. Also $T \cup g$ is a chain, so by the maximality of τ, $T = T \cup g$, a contradiction.

Hence we may assume that $eU_0e \cap W \cap V$ has a unique idempotent e. Since $\leqq$ is antisymmetric, the maximal subgroup of S containing e is e. Also eU_0e is a local semigroup with unit e, $eU_0e \neq e$, and e is not isolated in eU_0e which is the continuous image of U_0 and hence connected. Hence [5; 122] there is a non-degenerate one parameter local semigroup A with $e \in A \subset eU_0e \cap W \cap V$; let $a \in A$ with $a \neq e$ and $a^2 \in A$. Define $a^0 = e$ and let $B_k = \bigcup_{n=0}^{k} a^n[a, e]$, $B_\infty = \bigcup_{n=0}^{\infty} a^n[a, e]$ where $[a, e]$ denotes the sub-arc of A from a to e. We assume temporarily that all products involved in forming B_k and B_∞ are defined. Each of the sets $a^n[a, e]$ is a compact connected chain (hence an arc) with minimal element a^{n+1} and maximal element a^n. Hence B_k is a compact connected chain from a^{k+1} to e. Also B_∞ is a connected chain, hence B_∞^* is a closed connected chain. Using the easily established commutativity of B_k and B_∞^* it follows that for $x \in T$ and $b \in B_k$ (or B_∞^*) then $xb = x(eb) = (xe)b = eb = b$, and similarly $bx = b$. Hence $[(T \cup B_k^2) \cap V]^2 \subset T \cup (B_k^2 \cap V)^2$ and similarly with B_k replaced by B_∞^*. We distinguish two cases:

Case 1: For some $k \geqq 0$, $a^{k+1} \in V$ and $a^{k+2} \notin V$. Then since V is convex, $a^0, a, \cdots, a^{k+1}$ are in V and all products involved in forming B_k are defined, so that $B_k \subset V$ and $B_{k+1} \not\subset V$. We show first that $B_k^2 \cap V \subset B_k$. Let $z \in B_k^2 \cap V$; then $z = xy$ with $x, y \in B_k$, so $x = a^n x'$ and $y = a^m y'$ with x' and y' in $[a, e]$. Hence $xy = a^{m+n}x'y'$. If $x'y' \in A$, then since $z \in V$ it follows that $m + n \leqq k$. If $x'y' \notin A$, then $x'y' = at$ for some $t \in A$, so $xy = a^{m+n+1}t$ and $m + n + 1 \leqq k$. In either case, then, $z \in B_k$. Note that $(T \cup B_k)^2 \in M_\delta$ since B_k^2 is a connected chain. Also $[(T \cup B_k^2) \cap V]^2 \subset T \cup (B_k^2 \cap V)^2 \subset T \cup B_k^2$, so that $T \cup B_k^2 \in \mathscr{D}$. This contradicts the maximality of τ.

Case 2: $a^k \in V$ for each $k \geqq 0$. Using the convexity of V we see that all products involved in forming B_∞ are defined, and $B_\infty = B_\infty^2 \subset V$, hence $B_\infty^* = B_\infty^{*2}$. Since B_∞^* is a connected chain, it follows that $T \cup B_\infty^* \in M_\delta$. Also $[(T \cup B_\infty^*) \cap V]^2 \subset T \cup B_\infty^*$, so that $T \cup B_\infty^* \in \mathscr{D}$, a contradiction to the maximality of τ. The proof of (ii) is now complete.

(iii) *$M_\delta \cap \mathscr{C}$ is closed for each finite open cover δ of U^*.*

This proof is similar to that in [4], and is omitted.

For any finite open cover δ of U^*, let $P_\delta = M_\delta \cap \mathscr{C}$. The collection of sets $\{P_\delta\}$ is a descending family, so $\bigcap P_\delta \neq \square$. If $C \in \bigcap P_\delta$, then as shown in [4], C is an arc. Clearly C is a local semigroup, and the proof is complete.

In what follows, a *standard thread* is a compact connected semigroup irreducibly connected between a zero and a unit. The structure of standard threads is known [5; 130]. The example in [4] shows that a compact connected semigroup with zero and unit need not contain a standard thread joining the zero to the unit. The problem of finding standard threads joining zero to unit has an affirmative solution in case either

(1) S is compact, connected, and one-dimensional [3], or

(2) S is compact, connected, and each element is idempotent [4]. A third solution is given by the following corollary.

COROLLARY 1. *If S is a non-degenerate compact connected partially ordered semigroup with unit u, then the minimal ideal K consists of left zeros for S, K consists of the set of minimal elements, and some elements of K can be joined by a standard thread to the unit.*

Proof. Note that Graph $(\leqq)$ is closed since S is compact. Let G be a compact group in S, with unit e. Since $x^2 \leqq x$ for each $x \in S$, then for $x \in G$ we have $e \geqq x \geqq x^2 \geqq \cdots$, and $\{x^n\}$ clusters at an idempotent, which must be e. We conclude that $x = e$, and hence that each compact group in S is trivial. From this fact it is clear that K is proper, for otherwise $K = S$ would be a compact group [1]. From the fact that $aS = bS$ implies $a = b$ we conclude that each minimal right ideal is a single element, hence each element of K is a left zero for S [1]. Since a minimal element x of S is characterized by the equality $xS = x$, it is clear that K consists of the set of minimal elements of S, and hence that $S\backslash K$ is convex. In the proof of the Theorem, we take $S = U_0 = U_1 = U_2 = U$, and $V = S\backslash K$. Hence there is a compact connected linearly ordered local semigroup L containing u, with $L \cap S\backslash V \neq \square$. Since the elements of K are minimal it follows that L is a semigroup, and hence a standard thread.

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