RELATION OF A DIRECT LIMIT GROUP TO ASSOCIATED VECTOR GROUPS

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A set M with a binary, transitive relation $<$ is said to be directed if for each pair a, b in M, there is a c in M such that $a < c, b < c$. Let $\{G_a\}_{a \in M}$ be a collection of groups indexed by a directed set $M = \{a, b, \cdots; <\}$, and for each $a < b$ in M let h_b^a be a homomorphism of G_a into G_b. The homomorphisms are assumed to satisfy the relations

(i) $h_c^b h_b^a = h_c^a$ if $a < b < c$

and

(ii) if $a < a$, then h_a^a is the identity.

We call such a system a direct system of groups and define a direct limit group of this system in the following manner. Two elements $g_a \in G_a$ and $\bar{g}_b \in g_b$ are said to be equivalent if there is a $c > a, b$ such that $h_c^a(g_a) = h_c^b(\bar{g}_b)$. Let g_a^* denote the collection of elements which are equivalent to g_a. Now given any two equivalence classes g_a^* and $\bar{g}_b^*$, there exists a c and elements $g_c, \bar{g}_c$ in G_c such that $g_a^* = g_c^*$ and $\bar{g}_b^* = \bar{g}_c^*$. We define $g_a^* \cdot \bar{g}_b^* = (g_c \bar{g}_c)^*$. This multiplication is a well defined binary operation on the set, G^*, of equivalence classes. And it may be shown that G^* is a (multiplicative) group, which we define to be the direct limit group of the given system.

Let $G = \prod G_a$ be the restricted direct product of the given groups G_a, and consider the groups G_a as subgroups of G. An element in G of the form $g_a^{-1} h_b^a(g_a)$ is called a relation. Let H be the subgroup generated by the relations of G. Note that the inverse of a relation is a relation. By a "last" element of M we mean an element b such that $a < b$ for all a in M. If M contains no last element, it is immediate that given $a_1, a_2, \cdots, a_k$ in M, there exists a $b \in M$ with the property $a_i < b, a_i \neq b$ for $i = 1, 2, \cdots, k$.

LEMMA 1. *If M contains no last element, the commutator group K of G is contained in H.*

Proof. Let $x = g_{a_1} g_{a_2} \cdots g_{a_k}$ and $y = \bar{g}_{b_1} \bar{g}_{b_2} \cdots \bar{g}_{b_j}$ be arbitrary elements of G, where $a_m = a_n$ or $b_m = b_n$ implies that $m = n$. First choose a with the property that $a_i < a, a_i \neq a$, and $b_i \neq a$ for all i. Then choose b such that $b_i < b, b_i \neq b, a_i \neq b$, and $a \neq b$. We have

$$xyx^{-1}y^{-1} = \prod_{i=1}^{k} g_{a_i} \prod_{i=1}^{j} \bar{g}_{b_i} \prod_{i=k}^{1} g_{a_i}^{-1} \prod_{i=j}^{1} \bar{g}_{b_i}^{-1}$$

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$$= \prod_{i=1}^{k} g_{a_i} \prod_{i=1}^{k} h_a^{a_i}(g_{a_i}^{-1}) \prod_{i=1}^{j} \bar{g}_{b_i} \prod_{i=1}^{j} h_b^{b_i}(\bar{g}_{b_i}^{-1})$$
$$= \prod_{i=k}^{1} g_{a_i}^{-1} \prod_{i=k}^{1} h_a^{a_i}(g_{a_i}) \prod_{i=j}^{1} \bar{g}_{b_i}^{-1} \prod_{i=j}^{1} h_b^{b_i}(\bar{g}_{b_i})$$
$$= \prod_{i=1}^{k} g_{a_i} h_a^{a_i}(g_{a_i}^{-1}) \prod_{i=1}^{i} \bar{g}_{b_i} h_b^{b_i}(\bar{g}_{b_i}^{-1}) \prod_{i=k}^{1} g_{a_i}^{-1} h_a^{a_i}(g_{a_i}) \prod_{i=j}^{1} \bar{g}_{b_i}^{-1}(\bar{g}_{b_i}) .$$

Thus $xyx^{-1}y^{-1} \in H$. Since H is a group, the lemma follows.

COROLLARY 1. *If M contains no last element, then H is a normal subgroup of G.*

The following example shows that H may not be normal in G if M contains a last element.

EXAMPLE. Suppose that M is $\{1, 2; \leqq\}$. Let G_2 be the symmetric group on the set $\{1, 2, 3\}$, and let G_1 be the subgroup of G_2 of those elements fixing 3. Define h_2^1 to be the identity isomorphism of G_1 into G_2. Then H, a cyclic group of order 2, is generated by $((1, 2), (1, 2))$. It is, therefore, not normal in G.

LEMMA 2. *If g_a in G_a is in H, then there exists a b such that $h_b^a(g_a) \in K_b$, the commutator group of G_b.*

Proof. In general, if x_a in G_a is the product $x_{a_1}x_{a_2} \cdots x_{a_n}$ where $x_{a_i} \in G_{a_i}$ and if $b > a, a_i$ for $i = 1, 2, \cdots, n$, then $h_b^a(x_a)$ can be written as the product of the elements $h_b^{a_1}(x_{a_1}), h_b^{a_2}(x_{a_2}), \cdots, h_b^{a_n}(x_{a_n})$ in some order. This fact is easily proved by induction on n. If $n > 1$, by the induction hypothesis we may as well assume that the factors x_{a_i} are nontrivial. Thus two of the factors must be contained in a single group G_{a_i}. And the product $x_{a_1}x_{a_2} \cdots x_{a_n}$ can be contracted to a product of the same form with one less factor by taking one of the new factors to be the product of two of the old and letting the other factors remain unchanged (except, possibly, for the order in which they appear).

Since g_a is in H, it can be written in the form $\prod_{i=1}^{k} g_{a_i}^{-1} h_{b_i}^{a_i}(g_{a_i})$. Choose b such that $b > a, b_i$ for $i = 1, 2, \cdots, k$. Then

$$h_b^a(g_a)K_b = \prod_{i=1}^{k} h_b^{a_i}(g_{a_i}^{-1}) h_b^{b_i} h_{b_i}^{a_i}(g_{a_i})K_b$$
$$= \prod_{i=1}^{k} h_b^{a_i}(g_{a_i}^{-1}) h_b^{a_i}(g_{a_i})K_b = 1_b K_b = K_b ,$$

which proves the lemma.

THEOREM 1. *If H is a normal subgroup of G, then G/H is a homomorphism image of G^*, where the kernel of the homomorphism is contained in the commutator subgroup, K^**

REMARKS. The theorem is well known [1] in case the groups G_a are abelian. In this case H is necessarily normal and $K^* = 1$ is the identity. Thus $G^* \cong G/H$, and we have two equivalent definitions for the direct limit.

Proof of theorem. Let f be the mapping of G^* into G/H, defined by: $g_a^* \to g_a H$. In order to show that f is single-valued, let $g_a^* = \bar{g}_b^*$. There exists $c > a, b$ such that $h_c^a(g_a) = h_c^b(\bar{g}_b)$. Thus $g_a^{-1}\bar{g}_b = g_a^{-1}h_c^a(g_a)h_c^b(\bar{g}_b)^{-1}\bar{g}_b$. Since $h_c^b(\bar{g}_b)^{-1}\bar{g}_b = \bar{g}_b h_c^b(\bar{g}_b^{-1})$, we have $g_a^{-1}\bar{g}_b \in H$, which implies that f is independent of the representative of g_a^*. The multiplicative property of f is immediate. We next show that f is onto. Let $gH \in G/H$ and let $g = g_{a_1}g_{a_2}\cdots g_{a_k}$, where the a_i's are distinct. Choose b such that $a_i < b$ for $i = 1, 2, \cdots, k$. If $a_i = b$ for some i, we may as well assume that $i = k$ since the g_{a_i}'s commute. For each i, $g_{a_i}^{-1}h_b^{a_i}(g_{a_i}) \in H$. Thus

$$\bar{g} = \prod_{i=k}^{1} g_{a_i}^{-1}h_b^{a_i}(g_{a_i}) = \prod_{i=k}^{1} g_{a_i}^{-1} \prod_{i=k}^{1} h_b^{a_i}(g_{a_i})$$

is in H, which implies that $g\bar{g}H = gH$. But $g\bar{g} = \prod_{i=k}^{1} h_b^{a_i}(g_{a_i})$ is in G_b. Hence $f((g\bar{g})^*) = gH$, and f is onto. Since $g_a \in K_a$, the commutator group of G_a, implies that $g_a^* \in K^*$, it follows from Lemma 2 that the kernel of f is contained in K^*.

THEOREM 2. *If M contains no last element, then $G^*/K^* \cong G/H$.*

Proof. By Corollary 1, H is normal in G. Thus by Theorem 1, we need only show that the kernel of f is the whole commutator group, K^*. However, if g_a^* is a commutator of G^*, then there exist a b and a commutator $\bar{g}_b \in K_b$ such that $g_a^* = \bar{g}_b^*$. Since $K_b \subseteqq K$, by Lemma 1 $\bar{g}_b \in H$. Thus $f(g_a^*) = H$, and the theorem folllows.

The limit group G^* is abelian if and only if for every a in M and for every commutator g_a of the group G_a, there exists a $b > a$ (depending on g_a) such that $h_b^a(g_a) = 1_b$. Also, under this condition the commutator subgroups, K_a, of the groups G_a are contained in H, and H is normal in G since the conjugate of a generator of H transformed by a general element of G

$$\begin{aligned}\{x_c\}_{c\in M}g_a^{-1}h_b^a(g_a)\{x_c\}_{c\in M}^{-1} &= x_a x_b g_a^{-1}h_b^a(g_a)x_a^{-1}x_b^{-1}\\ &= x_a g_a^{-1}x_a^{-1}g_a \cdot g_a^{-1}h_b^a(g_a)\cdot h_b^a(g_a)^{-1}x_b h_b^a(g_a)x_b^{-1}\end{aligned}$$

remains in H.

COROLLARY 2. *If the limit group G^* is abelian, then $G^* \cong G/H$. Moreover, the converse holds if M contains no last element.*

A directed set $M = \{a, b, \cdots; <\}$ is said to be completely directed if for every a in M all but a finite number of b's in M satisfy the relation $a < b$. In particular, the positive integers are completely directed by $<$.

Letting $G' = \prod' G_a$ be the complete direct product of the given groups G_a, we have

LEMMA 3. *If M is completely directed and has no last element, then G^* is contained (in the sense of isomorphism) in the factor group G'/G.*

Proof. Define a mapping h of G^* into G'/G by: $g_a^* \to \{x_b\}_{b \in M} G$, where $x_a = g_a$ and $a < b$ implies that $x_b = h_b^a(g_a)$. The coordinate x_b may be chosen as an arbitrary element of G_b if b fails to satisfy $a \leqq b$. It may be shown that h is a homomorphism with trivial kernel, which proves the lemma.

Letting F be the inverse image of $h(G^*)$ under the natural homomorphism of G' onto G'/G, we observe

COROLLARY 3. *Let M satisfy the conditions of Lemma 3, and let G^* be abelian. Then in the chain*

$$G' \supseteqq F \supseteqq G \supseteqq H \supseteqq 1$$

we have $F/G \cong G/H \cong G^*$.

REFERENCE

1. S. Lefschetz, *Algebraic Topology*, New York, American Mathematical Society Colloquium Publications, 1942.

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