SOME ZERO SUM TWO-PERSON GAMES WITH MOVES IN THE UNIT INTERVAL

MARTIN FOX

Introduction. Consider the following zero sum two person game. The players alternately choose points $t_i \in [0, 1]$ for $i = 1, 2, \cdots, n$, the choice being made by player I if i is odd and by player II if i is even. After the ith move the player who is to make the $(i+1)$st move observes the value of $\phi_i(t_1, t_2, \cdots, t_i)$ where ϕ_i is some function on the i-dimensional closed unit cube to some set A_i. The payoff is $f(t_1, t_2, \cdots, t_n)$ where f is a continuous, real-valued function.

If all the ϕ_i are constant we have the case of no information. Ville [1] showed that in this case such a game has a value. At the other extreme, if the ϕ_i are all one-to-one we have the case of perfect information so the game has a value.

The purpose of the present paper is to show that, in general, games of the form introduced in the first paragraph do not have values and to consider two cases in which they do. The counter-examples to be presented will be compared with Ville's classical example of a game on the unit square which has no value.

It is shown in § 2 that the games considered always have values when $n = 2$.

An example of a game with no value is presented in § 3. In this example $n = 3$ and the ϕ_i take only a finite number of values.

In § 4 it is shown that the additional hypothesis of continuity of the ϕ_i is not sufficient to guarantee existence of a value. In that example $n = 4$. The case $n = 3$ with continuous ϕ_i remains unsolved.

Section 5 deals with a special case for which n is arbitrary and yet the game has a value. In this case the ϕ_i each take only a finite number of values and each is constant on sets which are finite unions of i-dimensional generalized intervals.

1. Preliminary remarks. In this section the notation to be used in this paper will be introduced. This will be facilitated by the introduction of the normal forms of the games under consideration.

A pure strategy for player I is a vector $x = (x_1, x_2, \cdots, x_{[(n+1)/2]})$ where $x_1 \in [0, 1]$ and the x_i for $i = 2, 3, \cdots, [(n+1)/2]$ are functions on A_{2i-2} to $[0, 1]$. If moves $t_1, t_2, \cdots, t_{2i-2}$ have been made, then the ith move made by player I (the $(2i-1)$st move in the game) will be $x_i(\phi_{2i-2}(t_1, t_2, \cdots, t_{2i-2}))$. His first move will be x_1.

Received November 16, 1959. Excerpt from dissertation for the degree of Doctor of Philosophy, University of California, 1959. Partial support was given by Office of Ordnance Research, U.S. Army under Contract DA-04-200-ORD-171, Task Order 3.

A pure strategy for player II is a vector $y = (y_1, y_2, \cdots, y_{[n/2]})$ where each y_i is a function on A_{2i-1} to $[0, 1]$. If moves $t_1, t_2, \cdots, t_{2i-1}$ have been made, then the ith move made by player II (the $(2i)$th move in the game) will be $y_i(\phi_{2i-1}(t_1, t_2, \cdots, t_{2i-1}))$.

When player I uses the pure strategy x and player II uses the pure strategy y let $t_i(x, y)$ be the ith move made in the game. The t_i are defined recursively as follows:

$$\begin{aligned} t_1(x, y) &= x_1 ; \\ t_{2i}(x, y) &= y_i(\phi_{2i-1}(t_1(x, y), t_2(x, y), \cdots, t_{2i-1}(x, y))) \\ &\qquad\qquad \text{for } i = 1, 2, \cdots, [n/2] ; \\ t_{2i-1}(x, y) &= x_i(\phi_{2i-2}(t_1(x, y), t_2(x, y), \cdots, t_{2i-2}(x, y))) \\ &\qquad\qquad \text{for } i = 2, 3, \cdots, [(n+1)/2] . \end{aligned}$$

The payoff function is given by $M(x, y) = f(t_1(x, y), t_2(x, y), \cdots, t_n(x, y))$. The payoff as a function of mixed strategies will also be denoted by M.

In our case, since the moves are points in $[0, 1]$, the strategy spaces X and Y are products, usually infinite dimensional, each coordinate space being $[0, 1]$. Hence, the choice of a strategy by player I is equivalent to the choice of a distribution function F on X. It will be convenient to let the space P of mixed strategies for player I be the family of all distribution functions on X which assign probability 1 to a finite subset of X. The same will be done for Q, the space of mixed strategies for player II.

If H is a distribution function on the real line and S is any subset of the real line which is Borel measurable, we will let HS be the probability assigned to S by H.

For $F \in P$ we let $F_{i,\alpha}$ denote the marginal distribution function of the coordinate of player I's strategy which corresponds to his ith move when $\phi_{2i-2} = \alpha$. Similar notation will be used for $G \in Q$.

2. The case $n = 2$. In this section it will be shown that any game $\mathscr{G}$ of the type given in the introduction for which $n = 2$ has a value. It is not even necessary to assume that ϕ_1 is a measurable function.

For any $\alpha \in A_1$ let $\mathscr{G}(\alpha) = (\phi_1^{-1}(\alpha), [0, 1], M_\alpha)$ where M_α is f restricted to $\phi_1^{-1}(\alpha) \times [0, 1]$. It follows by the proof used for Ville's minimax theorem that each $\mathscr{G}(\alpha)$ has a value $v(\alpha)$. Let

$$v = \sup_{\alpha \in A_1} v(\alpha) .$$

Fix $\varepsilon > 0$ and let α^* be such that $v(\alpha^*) > v - \varepsilon$. For each $\alpha \in A_1$ let $F^{(\alpha)}$ and $G^{(\alpha)}$ be ε-good strategies for players I and II, respectively, in $\mathscr{G}(\alpha)$. The distribution function $F^{(\alpha)}$ assigns probability 1 to a finite subset of $\phi_1^{-1}(\alpha)$. Since $F^{(\alpha^*)}$ is a distribution function on $[0, 1]$ which is the strategy space for player I in $\mathscr{G}$, it can also be used as a strategy in $\mathscr{G}$. Let y be any pure strategy for player II in $\mathscr{G}$. Since $y_1(\alpha^*) \in [0, 1]$, it follows that $y_1(\alpha^*)$ is a pure strategy for player II in $\mathscr{G}(\alpha^*)$. Hence,

$$\begin{aligned} M(F^{(\alpha^*)}, y) &= \int_{\phi_1^{-1}(\alpha^*)} f(t, y_1(\alpha^*))F^{(\alpha^*)}(dt) \\ &= \int M_{\alpha^*}(t, y_1(\alpha^*))F^{(\alpha^*)}(dt) \\ &= M_{\alpha^*}(F^{(\alpha^*)}, y_1(\alpha^*)) \\ &> v(\alpha^*) - \varepsilon > v - 2\varepsilon\ . \end{aligned}$$

Let G be any strategy for player II in $\mathscr{G}$ such that $G_{1,\alpha} = G^{(\alpha)}$ for all $\alpha \in A_1$. Let x be any pure strategy for player I in $\mathscr{G}$. For some $\alpha \in A_1$ it must be true that $x \in \phi_1^{-1}(\alpha)$ so that x is also a pure strategy for player I in $\mathscr{G}(\alpha)$. Then,

$$\begin{aligned} M(x, G) &= \int f(x, t)G_{1,\alpha}(dt) \\ &= \int M_\alpha(x, t)G^{(\alpha)}(dt) \\ &= M_\alpha(x, G^{(\alpha)}) \\ &< v(\alpha) + \varepsilon \leqq v + \varepsilon\ . \end{aligned}$$

From the two inequalities obtained above it follows that the value of $\mathscr{G}$ is v.

3. A counter-example for $n = 3$.

In this section the counter-example for $n = 3$ will be given. The functions ϕ_i $(i = 1, 2)$ each take only a finite number of values. The similarity of this example to Ville's example will be discussed.

For this example let

$$\phi_1(t_1) \equiv 0\ ;$$

$$\phi_2(t_1, t_2) = \begin{cases} -1 \text{ if } t_1 = 0 \text{ or } 0 < \min(t_2, 1 - t_2) \leqq t_1\ ; \\ t_2 \text{ if } t_2 = 0 \text{ or } 1 \text{ and } t_1 \neq 0\ ; \\ 2 \text{ if } 0 < t_1 < t_2 \leqq \dfrac{1}{2} \\ 3 \text{ if } 0 < t_1 < 1 - t_2 < \dfrac{1}{2} \end{cases}$$

$$f(t_1, t_2, t_3) = -|\, t_3 - t_2\,|.$$

Let F be any strategy for player I. Fix $\varepsilon > 0$ and let $\delta \in (0, \varepsilon)$ be sufficiently small so that $F_1(0, \delta) < \varepsilon$. Let $G\{\delta\} = G\{1 - \delta\} = 1/2$. Then,

$$M(F, G) \leqq -\frac{1}{2}(F_1[\delta, 1] + F_1\{0\})\left[\int |t_3 - \delta| F_{2,-1}(dt_3) + \int |t_3 - (1-\delta)| F_{2,-1}(dt_3)\right]$$
$$< -\frac{1}{2}(1-\varepsilon)\left[\left(\frac{1}{2} - \delta\right) + \left(1 - \delta - \frac{1}{2}\right)\right] < -\frac{1}{2} + \frac{3}{2}\varepsilon$$

so that

$$\sup_F \inf_G M(F, G) \leqq -\frac{1}{2} .$$

Let G be any strategy for player II. Fix $\varepsilon > 0$ and let $x_1 \in (0, 1/2)$ be sufficiently small so that $G(0, x_1] + G[1 - x_1, 1) < \varepsilon$.
Let

$$x_2(\alpha) = \begin{cases} \frac{1}{2} \text{ if } \alpha = -1 ; \\ \alpha \text{ if } \alpha = 0 \text{ or } 1 ; \\ \frac{1}{4} \text{ if } \alpha = 2 ; \\ \frac{3}{4} \text{ if } \alpha = 3 . \end{cases}$$

Let $x = (x_1, x_2)$ so that x is a pure strategy for player I. Then,

$$M(G, x) \geqq -\int_{(0,x_1]}\left(\frac{1}{2} - t_2\right)G(dt_2) - \int_{[1-x_1,1)}\left(t_2 - \frac{1}{2}\right)G(dt_2)$$
$$-\int_{[0,1/2]}\left|\frac{1}{4} - t_2\right|G(dt_2) - \int_{(1/2,1]}\left|\frac{3}{4} - t_2\right|G(dt_2)$$
$$> -\varepsilon - \frac{1}{4}$$

so that

$$\inf_G \sup_F M(F, G) \geqq -\frac{1}{4}$$

and the game has no value.

In Ville's example the payoff function is such as to force each player to attempt to choose a point closer to 1 than does his opponent without actually choosing 1. It is impossible for either player to guarantee he will achieve this with any preassigned positive probability no matter what pure strategy his opponent may use. In the example just presented a similar situation arises on the first two moves. In Ville's example the competition to choose a point close to the endpoint is a direct competition over payoff. In the present example this competition is over the information player I will receive, which, of course, helps determine the payoff. If on his first move player I chooses a point closer to 0 (but not 0) than the choice of his opponent is to both 0 and 1, then he will obtain more accurate information about the location of his opponent's choice than would be the case otherwise. Player II is prevented from choosing an endpoint since to do so would be to give his opponent perfect information.

4. A counter-example with continuous ϕ_i. In this section a counter-example will be presented in which the functions ϕ_i are all continuous. In this example $n = 4$. Again a comparison will be made with Ville's example.

Let

$$\phi_1(t_1) \equiv 0 ;$$

$$\phi_2(t_1, t_2) = t_1(1 - t_1)t_2 ;$$

$$\phi_3(t_1, t_2, t_3) = \begin{cases} 0 \text{ if } \min(t_1, 1 - t_1) \leqq t_2 \leqq \max(t_1, 1 - t_1) ; \\ t_2(1 - t_2)(t_1 - t_2)\left|t_1 - \frac{1}{2}\right| \text{ if } t_2 < t_1 < \frac{1}{2} \\ \qquad\qquad \text{or } \frac{1}{2} < t_1 < t_2 ; \\ t_2(1 - t_2)[t_1 - (1 - t_2)]\left|t_1 - \frac{1}{2}\right| \text{ if } \frac{1}{2} \leqq t_1 < 1 - t_2 \\ \qquad\qquad \text{or } 1 - t_2 < t_1 \leqq \frac{1}{2} ; \end{cases}$$

$$f(t_1, t_2, t_3, t_4) = |\, t_1 - t_4 \,| - 10 \,|\, t_2 - t_3 \,| .$$

Assume $t_2 \neq 0$ or 1. Then, $\phi_3(t_1, t_2, t_3) > 0$ for $\min(t_2, 1 - t_2) < t_1 < 1/2$ while $\phi_3(t_1, t_2, t_3) < 0$ for $1/2 < t_1 < \max(t_2, 1 - t_2)$. On the other hand, $\phi_3(t_1, t_2, t_3) = 0$ otherwise.

Let F be any strategy for player I. Fix $\varepsilon > 0$ and let $\delta \in (0, \varepsilon)$ be sufficiently small so that $F_1(0, \delta] + F_1[1 - \delta, 1) < \varepsilon$. Let

$$y_2(\alpha) = \begin{cases} \frac{1}{2} & \text{if } \alpha = 0 ; \\ \frac{1}{4} & \text{if } \alpha > 0 ; \\ \frac{3}{4} & \text{if } \alpha < 0 . \end{cases}$$

Let G assign probability 1/2 to each of the pure strategies (δ, y_2) and $(1 - \delta, y_2)$. Then,

$$M(F, G) \leqq \int_{[0,\delta]} \left(\frac{1}{2} - t_1\right) F_1(dt_1) + \int_{[1-\delta,1]} \left(t_1 - \frac{1}{2}\right) F_1(dt_1)$$
$$+ \int_{(\delta,1/2)} \left| t_1 - \frac{1}{4} \right| F_1(dt_1) + \int_{(1/2,1-\delta)} \left| t_1 - \frac{3}{4} \right| F_1(dt_1)$$
$$- 10[F_1\{0\} + F_1\{1\}] \left[\frac{1}{2}\int | \delta - t_3 | F_{2,0}(dt_3) + \frac{1}{2}\int | 1 - \delta - t_3 | F_{2,0}(dt_3)\right]$$
$$< \frac{1}{2}[F_1\{0\} + F_1\{1\}] + \frac{1}{2}\varepsilon + \frac{1}{4}[1 - \varepsilon - F_1\{0\} - F_1\{1\}]$$
$$-5[F_1\{0\} + F_1\{1\}]\left[\left(\frac{1}{2} - \delta\right) + \left(1 - \delta - \frac{1}{2}\right)\right]$$
$$= \frac{1}{4} + \frac{1}{4}\varepsilon - [F_1\{0\} + F_1\{1\}]\left[5(1 - 2\delta) - \frac{1}{4}\right]$$
$$< \frac{1}{4} + 11\varepsilon$$

so that $\sup_F \inf_G M(F, G) \leqq 1/4$.

Let G be any strategy for player II. Fix $\varepsilon > 0$ and let $\delta \in (0, \varepsilon) \cap (0, 1/2)$ be sufficiently small so that $G_{1,0}(0, \delta) + G_{1,0}(1 - \delta, 1) < \varepsilon$. Let $x_2(\alpha) = \alpha/[\delta(1 - \delta)]$ and let F assign probability 1/2 to each of the pure strategies (δ, x_2) and $(1 - \delta, x_2)$. When player I uses the strategy F the value of the nonpositive term in f will always be zero. Thus,

$$M(F, G) \geqq \left[1 - G_{1,0}(0, \delta) - G_{1,0}(1 - \delta, 1)\right]$$
$$\times \left[\frac{1}{2}\int | \delta - t_4 | G_{2,0}(dt_4) + \frac{1}{2}\int | 1 - \delta - t_4 | G_{2,0}(dt_4)\right]$$
$$> \frac{1}{2}(1 - \varepsilon)\left[\left(\frac{1}{2} - \delta\right) + \left(1 - \delta - \frac{1}{2}\right)\right]$$
$$> \frac{1}{2} - \frac{3}{2}\varepsilon$$

so that $\inf_G \sup_F M(F, G) \geqq 1/2$ and the game has no value.

Here again the primary competition between the players is to make their first moves as close to the endpoints as possible without actually choosing the endpoints. If player I is successful in choosing a point t_1 at least as close to one of the endpoints as is player II's choice, then player II will have less information about t_1 than would be the case otherwise. Player I is prevented from choosing an endpoint by the fact that if he does so he will get no information about his opponent's first move so that he cannot guarantee that he can keep the negative term close to zero. Player II is prevented from choosing an endpoint by the fact that when he does so the function ϕ_3 will take the value zero no matter what his opponent does so that he will have no information about player I's first move.

5. The case of information sets which are unions of generalized intervals. The case to be considered here is that in which each ϕ_i takes only a finite number of values and each is constant only on sets which are finite unions of i-dimensional generalized intervals. This is the only case considered in this paper in which n remains arbitrary.

Let the values of ϕ_i be $1, 2, \cdots, m_i$. Let $P_j\phi_i^{-1}(k)$ be the projection on the jth coordinate of $\phi_i^{-1}(k)$ where $j = 1, 2, \cdots, i$. The interval $[0, 1]$ can be subdivided into disjoint sets $B_{j1}, B_{j2}, \cdots, B_{jl_j}$ such that for each B_{jl} there exist $i_1, i_2, \cdots, i_r$ and $k_1, k_2, \cdots, k_u$, all integers, such that $t \in B_{jl}$ if, and only if, $t \in P_j\phi_i^{-1}(k)$ whenever $i \in \{i_1, i_2, \cdots, i_r\}$ and $k \in \{k_1, k_2, \cdots, k_u\}$ while $t \notin P_j\phi_i^{-1}(k)$ otherwise. Suppose j is even so that player II makes the jth move. Let $y = (y_1, y_2, \cdots, y_{[n/2]})$ and $y' = (y'_1, y'_2, \cdots, y'^{[n/2]})$ be any strategies for player II such that $y_i = y'_i$ for $i \neq j/2$ and if $y_{j/2}(k) \in B_{jl}$, then $y'_{j/2}(k) \in B_{jl}$. For any pure strategy x for player I we have $t_i(x, y) = t_i(x, y')$ for $i = 1, 2, \cdots, j-1$ since for these values of i player II's moves are unchanged. If $t_j(x, y) \in B_{jl}$, then $t_j(x, y') \in B_{jl}$. Hence,

$$\phi_j(t_1(x, y), t_2(x, y), \cdots, t_j(x, y)) = \phi_j(t_1(x, y'), t_2(x, y'), \cdots, t_j(x, y'))$$

so that $t_{j+1}(x, y) = t_{j+1}(x, y')$. Suppose that $t_i(x, y) = t_i(x, y')$ for $i = j+1, j+2, \cdots, i_0$. Then, $\phi_{i_0}(t_1(x, y), t_2(x, y), \cdots, t_{i_0}(x, y)) = \phi_{i_0}(t_1(x, y'), t_2(x, y'), \cdots, t_{i_0}(x, y'))$ so that $t_{i_0+1}(x, y) = t_{i_0+1}(x, y')$. Thus, $t_i(x, y) = t_i(x, y')$ for all $i \neq j$.

For each $j = 1, 2, \cdots, n-1$ fix $\delta_j > 0$ and select points $t_{j1}, t_{j2}, \cdots, t_{jv_j}$ such that for any $t_j \in B_{jl}$ there exists $t_{jv} \in B_{jl}$ such that for any $t_1, t_2, \cdots, t_{j-1}, t_{j+1}, \cdots, t_n$ we have

$$\begin{aligned} |f(t_1, t_2, \cdots, t_{j-1}, t_j, t_{j+1}, \cdots, t_n) \\ - f(t_1, t_2, \cdots, t_{j-1}, t_{jv}, t_{j+1}, \cdots, t_n)| < \delta_j . \end{aligned}$$

Select the t_{jv} in such a way that as $\delta_j \downarrow$ the set of all the t_{jv} increases monotonically.

Let the game $\mathscr{G}(\delta_1, \delta_2, \cdots, \delta_i) = (X(\delta_1, \delta_2, \cdots, \delta_i), Y(\delta_1, \delta_2, \cdots, \delta_i), M_{\delta_1, \delta_2, \cdots, \delta_i})$ be our original game with the jth move for $j = 1, 2, \cdots, i$ restricted to $t_{j1}, t_{j2}, \cdots, t_{jv_j}$. In $\mathscr{G}(\delta_1, \delta_2, \cdots, \delta_{n-1})$ the player who makes the $(n-1)$st move has only a finite number of strategies so that $\mathscr{G}(\delta_1, \delta_2, \cdots, \delta_{n-1})$ has a value (see Wald [2]).

Suppose $\mathscr{G}(\delta_1, \delta_2, \cdots, \delta_{i-1}, \delta_i)$ has a value for all $\delta_i > 0$. It follows, by a proof similar to Ville's, that $\mathscr{G}(\delta_1, \delta_2, \cdots, \delta_{i-1})$ has a value. Thus, by induction, $\mathscr{G}$ will also have a value.

Acknowledgment. The author wishes to express his gratitude to Professor David Blackwell for suggesting the problem treated here and for his continued interest in its solution.

References

1. J. Ville, *Sur la théorie générale des jeux où intervient l'habilité des joueurs,* note in *Traité du Calcul des Probabilites et de ses Applications,* by E. Borel, Tome IV, Fascicule II, Applications aux jeux de hasard, (1938), 105–113.
2. A. Wald, *Generalization of a theorem by von Neumann concerning zero sum two person games,* Ann. Math. **46** (1945), 281–286.

Michigan State University