# INVOLUTIONS ON LOCALLY COMPACT RINGS

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By a proper involution $*$ on a ring $R$ we mean a mapping $x \to x^*$ defined on $R$ with the following properties:

(i) $(x+y)^* = x^* + y^*$,

(ii) $(xy)^* = y^*x^*$,

(iii) $(x^*)^* = x$ and

(iv) $xx^* = 0$ if and only if $x = 0$. If (iv) is not assumed, the mapping is simply termed an involution. If $F$ is a field with an involution $\sharp$ and $R$ is an algebra over $F$, we say that an involution on $R$ is an algebra involution if in addition to (i)-(iv) above the following holds:

(v) $(\alpha x)^* = \alpha^\sharp x^*$ for all $x \in R$ and $\alpha \in F$.

We are concerned principally with involutions on two types of locally compact semi-simple rings, namely those which are compact or connected. The main result is that involutions on such rings are automatically continuous. As a byproduct we determine the form of any proper involution on a total matric ring $R$ over a division ring. If in addition $R$ is topological and the division ring admits only continuous involutions, then we note that $R$ has only continuous involutions.

LEMMA *Let $D$ be a division ring with center $Z$. Let $R$ be a total matric ring over $D$. Any ring involution $*$ on $R$ induces an involution $\sharp$ on $Z$, and $*$ is an algebra involution on $R$ with respect to the involution $\sharp$ on $Z$.*

Direct calculation shows that the center of $R$ consists of the totality of elements of the form $\alpha I$ where $\alpha \in Z$ and $I$ is the identity of $R$. Suppose $x$ is in the center of $R$ and $y \in R$, then $x^*y = (y^*x)^* = (xy^*)^* = yx^*$, so $x^*$ is in the center of $R$. Since $I^* = I$ is immediate, it follows that for any $\alpha \in Z$, there is a $\beta \in Z$ such that $(\alpha I)^* = \beta I$. Denote $\beta$ by $\alpha^\sharp$. It is clear that $\sharp$ is an involution on $Z$. Moreover, if $\alpha \in Z$ and $x \in R$, $(\alpha x)^* = [(\alpha I)x]^* = x^*\alpha^\sharp I = \alpha^\sharp x$, so $*$ is an algebra involution on $R$ with respect to the involution $\sharp$ on $Z$.

THEOREM 2. *Let $R$ be a total matric ring over $D$, where $D$ is a division ring with center $Z$. Let $*$ be a proper ring involution on $R$, and let $\sharp$ be the induced involution on $Z$. Then there exist a set of matrix units $\{g_{ij}\}$ in $R$ such that $g_{ii}^* = g_{ii}$ and a set of non-zero elements $\gamma_i$ of $Z$ such that $\gamma_i^\sharp = \gamma_i$ such that the involution $*$ has the following form: If $x = \sum \alpha_{ij} e_{ij}$, with $\alpha_{ij} \in D$, then $x^* = \sum \gamma_j^{-1} \alpha_{ij} \gamma_i e_{ji}$.*

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Let $e_{ij}$, $i, j = 1, \cdots, n$ be a set of matric units for $R$. The right ideal $e_{11}R$ is minimal, so by a theorem of Rickart [7] there is a unique idempotent $u_1 \in e_{11}R$ such that $u_1^* = u_1 \neq 0$. Let $L_1 = Ru_1$, and $L_k = Re_{1k} = Re_{kk}$, $k = 2, \cdots, n$. The $L_k$ are minimal left ideals so by the Rickart theorem there are unique idempotents $u_k \in L_k$ such that $u_k^* = u_k \neq 0$, $k = 1, \cdots, n$.

We denote by $[A, B, \cdots, C]$ the smallest left ideal containing $A, B, \cdots, C$. The linear independence of $u_1$ and the $e_{1k}$, $k = 2, \cdots, n$ implies that $L_k \not\subset [L_1, \cdots, L_{k-1}]$ for $1 < k \leqq n$. It is readily verified that $R = [L_1, \cdots, L_n]$.

Let $g_1 = u_1$ and suppose that $g_1, \cdots, g_{k-1}$ have been defined so that $g_j = g_j^2 = g_j^* \neq 0$, $g_j \in [L_1, \cdots, L_j]$ and $g_i g_j = 0$ for $i \neq j$, $i, j = 1, 2, \cdots, k-1$. We next show that $g_k$ may be defined with the corresponding properties.

Let $v = u_k - \sum_{j=1}^{k-1} u_k g_j$. Since $L_k \not\subset [L_1, \cdots, L_{k-1}]$, $u_k \notin [L_1, \cdots, L_{k-1}]$ and thus $v \neq 0$. Since $L_k = Ru_k$ is a minimal left ideal $u_k R u_k$ is a division ring with unit $u_k$. The propriety of the involution then yields $vv^* \neq 0$. Since $vv^* \in u_k R u_k$, there is an element $s \in u_k R u_k$ such that $s(vv^*) = (vv^*)s = u_k$. If we apply the involution to the prior relation $(vv^*)s^* = s^*(vv^*) = u_k$, and the uniqueness of inverses in a division ring yields $s = s^*$.

It is claimed that $g_k = v^* s v$ has the desired properties. Since $v g_k v^* = vv^* s vv^* = u_k vv^* = vv^* \neq 0$, it follows that $g_k \neq 0$. Clearly $g_k = g_k^*$ and $g_k^2 = v^* s vv^* s v = v^* u_k s v = v^* s v = g_k$. If $i = 1, \cdots, k-1$, $g_i v^* = g_i(u_k - \sum_{j=1}^{k-1} g_j u_k) = 0$ by the inductive hypothesis, thus $g_i g_k = g_i v^* s v = 0$. By applying the involution we obtain $g_k g_i = 0$. The induction is thus complete and we may suppose that $g_1, \cdots, g_n$ have been defined.

Clearly $[g_1] = [L_1]$. Suppose that for $1 < k \leqq n$, $[g_1, \cdots, g_{k-1}] = [L_1, \cdots, L_{k-1}]$. The defining property for $g_k$ yields $[g_1, \cdots, g_k] \subset [L_1, \cdots, L_k] = [[g_1, \cdots, g_{k-1}], L_k]$. Thus $g_k = x_1 g_1 + \cdots + x_{k-1} g_{k-1} + x_k e_{1k}$. Right multiplication of the last relation by $g_k$ shows that $x_k e_{1k} \neq 0$. Since $L_k$ is a minimal left ideal, there is a $z \in R$ such that $z x_k e_{1k} = e_{1k}$. This may be expressed as $z[g_k - x_1 g_1 \cdots - x_{k-1} g_{k-1}] = e_{1k}$. Thus $L_k \subset [g_1, \cdots, g_k]$ and hence $[g_1, \cdots, g_k] = [L_1, \cdots, L_k]$ for $k = 1, \cdots, n$. In particular $R = [g_1, \cdots, g_n]$.

The spaces $Rg_k$ must be irreducible over $R$, otherwise we would have $R$ decomposed into sums of irreducible $R$-spaces of different lengths. Thus the ideals $Rg_k$ are minimal. Furthermore if we denote the unit element of $R$ by $e$, we have $e = y_1 g_1 + \cdots + y_n g_n$. Right multiplication by $g_j$ shows that $g_j = y_j g_j$ and thus $e = g_1 + \cdots + g_n$.

The form of an idempotent in $e_{11}R$ and $Re_{kk}$, $k = 2, \cdots, n$, together with the fact that $\lambda e_{ij} = e_{ij}\lambda$ yields $\lambda u_k = u_k \lambda = u_k \lambda u_k$, $k = 1, \cdots, n$ for any $\lambda \in D$. The inductive method of defining $g_k$ then permits one to deduce that $\lambda g_k = g_k\lambda = g_k\lambda g_k$. For suppose that $\lambda g_j = g_j\lambda$ for $j = 1, \cdots, k-1$. From the way in which $v$ and $v^*$ were defined $\lambda v = v\lambda$ and $\lambda v^* = v^*\lambda$. Since $\lambda g_k = v^*\lambda s v = v^* u_k \lambda s u_k v$, and $g_k\lambda = v^* s\lambda v = v^* u_k s\lambda u_k v$, it is sufficient if we show that $u_k\lambda s u_k = u_k s\lambda u_k$ for all $\lambda \in D$. But $(u_k s\lambda u_k)(vv^*) = svv^*\lambda = u_k\lambda = \lambda u_k = \lambda svv^* = u_k\lambda s u_k(vv^*)$. Since $u_k R u_k$ is a division ring, $u_k s\lambda u_k = u_k\lambda s u_k$ as desired. Hence $\lambda g_k = g_k\lambda$ for all $\lambda \in D$ and $k = 1, \cdots, n$.

Since $(0) \neq Rg_iR$ is a two sided ideal of $R$, $Rg_iRg_k = Rg_k \neq (0)$, and thus $g_iRg_k \neq (0)$. Suppose $i < k$, and $g_irg_k \neq 0$. Then, by the propriety of the involution, $0 \neq (g_irg_k)(g_irg_k)^* = g_irg_kr^*g_i$. Since the left ideal $Rg_i$ is minimal, $g_iRg_i$ is a division ring, and there exists $t \in R$ such that $(g_itg_i)(g_irg_kr^*g_i) = g_i$. If we take adjoints of the expressions in the preceding equation, we see that $g_itg_i = g_it^*g_i$. Let $g_{ik} = g_itg_irg_k$ and $g_{ki} = g_kr^*g_i$. Then $g_{ik}g_{ki} = g_i$, and consequently $(g_{ik}g_{ki})(g_{ik}g_{ki}) = g_i$, so $0 \neq g_{ki}g_{ik} \in g_kRg_k$, which is a division ring. Also $g_{ki}g_{ik}$ is idempotent so $g_{ki}g_{ik} = g_k$. Finally if we define $g_{ii} = g_i$, we obtain a set of matrix units $\{g_{ij}\}$ for $R$ such that $g_{ii}^* = g_{ii}$. The form of the involution * on $R$ is then an immediate consequence of a theorem of Jacobson and Rickart [2].

We are now in a position in which we may discuss the continuity of involutions.

THEOREM 3. *Let $D$ be a topological division ring such that any involution on $D$ is continuous. If $R$ is a total matric ring over $D$, then any proper ring involution on $R$ is continuous.*

The result is immediate by virtue of the representation of the involution given in Theorem 2, together with the fact that convergence in $R$, when it is regarded as a finite dimensional vector space, involves [1] convergence of the coefficients of the representation in terms of a given basis.

We turn now to locally compact semi-simple rings which are either connected or compact. The first item needed concerns their topological algebraic structure.

LEMMA. (a) *A compact semi-simple ring is the topological direct sum of total matric algebras over finite fields.*

(b) *A locally compact connected semi-simple ring is the topological direct sum of a finite number of total matric rings over locally compact division rings.*

Statement (a) is immediate from Theorem 16 of Kaplansky [4]. In the second statement, the semi-simplicity allows the use of Theorem 2 of Kaplansky [5], which shows that the ring is the direct sum of a semi-simple algebra over the reals with a unit and a totally disconnected ring. Since the decomposition is the Peirce decomposition relative to the algebra unit, it is easily seen that one has a topological direct sum. The connectedness then forces the second summand to be zero. The conclusion of the lemmas then follows from Theorem 10 of [5].

It might further be noted that the division rings involved must be connected. Consequently, since the only connected locally compact division rings are the reals, the complexes and the quaternions [3], [6], these are the only rings involved in the conclusion of (*b*).

LEMMA 5. *If * is a proper involution on a direct sum of total matric rings over division rings, then each matric ring is invariant under *. Thus * restricted to an individual matric ring is a proper involution on that ring.*

Let $R$ be the direct sum of rings $R_j$. Let $e^\circ$ be the unit of a summand $R^\circ$. Say $e^\circ = e_1 + \cdots + e_n$ is the decomposition of $e^\circ$ in terms of the vector units of $R^\circ$. The right ideal $e_iR = e_iR^\circ$ is a minimal right ideal of $R$. Hence, by the theorem of Rickart used previously, there exists a unique idempotent $f_k$ in $e_iR$ such that $0 \neq f_i = f_i^*$. Thus $e_i = f_ie_i$ and $e_i^* = e_i^*f_i$. Consequently if $x \in R^\circ$, $x = e_1x + \cdots + e_nx = f_1e_1x + \cdots + f_ne_nx$, and $x^* = x^*e_1^*f_1 + \cdots + x^*e_n^*f_n$ is in $R^\circ$.

We are now in a position to establish the continuity of proper involutions on the class of semisimple rings under discussion.

THEOREM 6. *If $R$ is a semi-simple locally compact ring which is either compact or connected then any proper involution * on $R$ is continuous.*

In view of Lemmas 4 and 5, it is sufficient to prove the continuity of * on an individual matric ring. Thus the proof is complete for the compact ring. For the connected ring, all we need note is that the only involutions on the reals, complexes and quaternions are automatically continuous. Hence Theorem 3 applies and the proof is complete.

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