ON COMPLETE AND INDEPENDENT SETS OF
OPERATIONS IN FINITE ALGEBRAS

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In [4] Post obtained a variety of results about truth functions in 2-valued sentential calculus. He studied sets of truth functions which could be used as primitive notions for various systems of 2-valued logics. In particular, he was interested in complete sets of truth functions, i.e., sets having the property that every truth function with an arbitrary finite number of arguments is definable in terms of the truth functions belonging to the set. Among other results Post established a computable criterion for a set of truth functions to be complete. Using this criterion he showed that there is a finite upper bound for the number of elements in any complete and independent set of primitive notions for the 2-valued sentential calculus (and that actually the number 4 is the least upper bound). Alfred Tarski has asked to what extent these results can be extended to n-valued sentential calculus, for any finite n. It will be seen from this note that those results of Post concerning complete sets of truth functions can actually be extended. On the other hand it has been shown recently by A. Ehrenfeucht that the result concerning arbitrary sets of functions cannot be extended.

Both the results of Post and those of this note can be formulated in terms of truth functions of the 2-valued (n-valued) sentential calculus or in terms of finitary operations in arbitrary 2 element (n element) algebras. We choose the second alternative since the many-valued sentential calculi have a rather restricted significance in logic and mathematics.

Thus we shall concern ourselves with finitary operations under which a given set A with n elements is closed. For simplicity we restrict our attention to the case when A is the set N of all natural numbers less than n. This restriction implies no loss of generality, since all the results can be extended by isomorphism to any finite set with n elements. For convenience we will identify N with n, as is frequently done in modern set theory.

For any given natural number k, let n^k be the set of all k-termed sequences $x = \langle x_0, x_1, \cdots, x_{k-1} \rangle$ with terms in n. Denote by $F_{n,k}$ the set of all k-ary operations on and to elements of n, i.e., of all function

on n^k to n.   Let $F_n = \bigcup_{k<\infty} F_{n,k}$, i.e., F_n is the set of all finitary opera-
tions on and to elements of n.

For any subset X of F_n we will denote by $\bar{X}$ the smallest set Y
which includes X and satisfies the following conditions:

(i) if $f \in Y$ and h is obtained from f by exchanging or identifying
two arguments, then $h \in Y$;

(ii) if $f \in Y$, $g \in Y$, and h is obtained from f by replacing an
argument by g, then $h \in Y$.   A function f is said to be *generated* by
a set X if $f \in \bar{X}$.   The set X is called *closed* if $\bar{X} = X$, it is called
complete if $\bar{X} = F_n$, it is called *independent* if there is no proper subset
X' of X for which $\bar{X}' = \bar{X}$.   A set Y is called a *basis* for a set X if
$Y \subseteq X$ and $\bar{Y} = \bar{X}$.   A function $f \in F_{n,k}$ is called *reducible* or reducible
to first order if its values depend on at most one argument; i.e., if
there is an $i < k$ and an $h \in F_{n,1}$ such that for every sequence $x \in n^k$
we have $f(x) = h(x_i)$.   Hence f is not reducible if and only if for every
$q < k$ there are $x, y \in n^k$ with $x_q = y_q$ and $f(x) \neq f(y)$.   We will denote
by $\mathscr{R}(f)$ the range of f.   We single out two functions in F_n. $\bigvee_n$ is
the function of two arguments defined by the formula:

$$x \bigvee_n y = \max(x, y) .$$

$\sim_n$ is the function of one argument defined by:

$$\sim_n x = x + 1 (\mathrm{mod}\ n) .$$

In the following few lemmas we will establish some properties of
the notions just defined:

LEMMA 1.   *If $f \in F_{n,k}$, $n \geq 3$, f not reducible and $\mathscr{R}(f) = n$, then
$\{f\} \cup F_{n,1}$ generates a function $g \in F_{n,2}$, g not reducible and $\mathscr{R}(g) = n$.*

Proof.   We first establish that there exist $q < k$ and $u, v \in n^k$ such
that $f(u) \neq f(v)$ and $u_i = v_i$ for all $i \neq q$, $i < n$.   Since $\mathscr{R}(f)$ contains
more than one element there exist $a, b \in n^k$ such that $f(a) \neq f(b)$.   There
are $k + 1$ sequences $c^{(0)}, c^{(1)}, \cdots, c^{(k)} \in n^k$ with $c^{(0)} = a$, $c^{(k)} = b$, and such
that for any $i < k$, $c^{(i+1)}$ is obtained from $c^{(i)}$ by replacing $c_i^{(i)}$ by $c_i^{(k)}$.
Hence $c^{(i)}$ and $c_i^{(i+1)}$ differ only in the ith coordinate, moreover
$c_i^{(j)} = a_i$ for $i \geq j$ and $c_i^{(j)} = b_i$ for $i < j$.   Since $f(a) = f(c^{(0)}) =
f(c_0^{(0)}, c_1^{(0)}, \cdots, c_{k-1}^{(0)})$, $f(b) = f(c^{(k)})$ and $f(a) \neq f(b)$ it cannot be the case
that $f(c^{(i)}) = f(c^{(i+1)})$ for all $i < k$.   Therefore there is some $q < k$ such
that $f(c^{(q)}) \neq f(c^{(q+1)})$ and $c_i^{(q)} = c_i^{(q+1)}$ for all $i \neq q$, $i < k$.   Take $c^{(q)}$ for
u and $c^{(q+1)}$ for v.

Since $\mathscr{R}(f) = n$, we can choose n sequences $y^{(0)}, y^{(1)}, \cdots, y^{(n-1)} \in n^k$
such that $f(y^{(0)}) = f(u)$, $f(y^{(1)}) = f(v)$, and each value in n is taken on
by f for some $y^{(i)}$.   There also exist $w, z \in n^k$ for which $w_q = z_q$ and

$f(w) \neq f(z)$, since if this were not the case f would depend only on its qth argument and thus be reducible.

There are two possible cases: (i) there exist $w, z \in n^k$, with $w_q = z_q$, and $f(w) \neq f(z)$, $f(w) \neq f(u)$, $f(w) \neq f(v)$; (ii) for every $w, z \in n^k$ with $w_q = z_q$ and $f(w) \neq f(z)$ neither $f(w)$ nor $f(z)$ is different from both $f(u)$ and $f(v)$.

If case (i) holds there exist $w, z \in n^k$ for which $w_q = z_q$, and $f(w) \neq f(z)$, $f(w) \neq f(u)$, $f(w) \neq f(v)$. With no loss of generality we may assume $f(y^{(2)}) = f(w)$. We define k functions $h_0, h_1, \cdots, h_{k-1} \in F_{n,1}$ separately for h_q and for h_j, $j \neq q$,

$$h_q(x) = \begin{cases} u_q & x = 0 \\ v_q & x = 1 \\ z_q & x = 2 \\ y_q^{(x)} & x > 2 \end{cases} \qquad h_j(x) = \begin{cases} u_j & x = 0 \\ z_j & x = 1 \\ w_j & x = 2 \\ y_j^{(x)} & x > 2. \end{cases}$$

We define $g \in F_{n,2}$ as follows:

(1) $g(x, y) = f(h_0(x), h_1(x), \cdots, h_{q-1}(x), h_q(y), h_{q+1}(x), \cdots, h_{k-1}(x))$.

Notice that y appears only in the qth coordinate of f. $\mathscr{R}(g) = n$ since $g(0, 0) = f(u) = f(y^{(0)})$, $g(0, 1) = f(v) = f(y^{(1)})$, $g(2, 2) = f(w) = f(y^{(2)})$, and $g(i, i) = f(y^{(i)})$ for $i > 2$. Moreover g is not reducible, since $g(0, 0) \neq g(0, 1)$ and $g(1, 2) \neq g(2, 2)$.

If case (ii) holds we take for h_q the identity function in $F_{n,1}$ and using any $w, z \in n^k$ for which $w_q = z_q$ and $f(w) \neq f(z)$ we choose $k - 1$ functions $h_j \in F_{n,1}$ for $j \neq q$, satisfying

$$h_j(x) = \begin{cases} u_j & x = 0 \\ w_j & x = 1 \\ z_j & x = 2 \end{cases}$$

and then define g by (1). Now for any $s \in n^k$ with $s_q = y_q^{(i)}$, $2 < i \leq n$, condition (ii) guarantees that $f(s) = f(y^{(i)})$ since $f(y^{(i)}) \neq f(u)$ and $f(y^{(i)}) \neq f(v)$. Therefore $g(0, u_q) = f(u) = f(y^{(0)})$, $g(1, v_q) = f(v) = f(y^{(1)})$, $g(m, y_q^{(i)}) = f(y^{(i)})$ for $m < n$ and $2 \leq i < n$. Hence $\mathscr{R}(g) = n$. The function g is not reducible, since $g(0, u_q) \neq g(0, v_q)$ and $g(1, w_q) \neq g(2, w_q)$.

LEMMA 2. *If $f \in F_{n,2}$ is not reducible and $\mathscr{R}(f)$ has p elements $p \geq 3$, then there exist $i, j, k, l < n$ such that among the function values $f(i, k), f(i, l), f(j, k), f(j, l)$ at least three distinct values are represented.*

Proof. There are two possibilities;
   (i) in the table of f all value in $\mathscr{R}(f)$ are taken on across some row, i.e., there is an $i < n$ such that the set of all values $f(i, j)$ with

$j < n$ coincides with $\mathscr{R}(f)$;

(ii)   in no row are all values in $\mathscr{R}(f)$ taken on.

If (i) holds, since f is not reducible there must be $j, l < n$ for which $f(i, l) \neq f(j, l)$. Since $p \geqq 3$ and all values in $\mathscr{R}(f)$ are taken on in the ith row there must be a $k < n$ such that $f(i, k)$ is distinct from both $f(i, l)$ and $f(j, l)$.

If (ii) holds, since f is not reducible there is some non-constant row, i.e., some i, j', j'' such that $f(i, j') \neq f(i, j'')$. However, by assumption there is a $w \in \mathscr{R}(f)$ which does not appear in this row, i.e., for all $x < p$, $f(i, x) \neq w$. Since $w \in \mathscr{R}(f)$ there are $j, l < n$ for which $f(j, l) = w$. Hence $f(i, l) \neq f(j, l)$. Since w does not appear in the ith row and the ith row is not constant there is some $k < n$ such that the value $f(i, k)$ is different from both $f(i, l)$ and $f(j, l)$.

LEMMA 3.   *If $f \in F_{n,2}$ is not reducible and $\mathscr{R}(f)$ has exactly p elements, $3 \leqq p \leqq n$, then there exist two functions $h_1, h_2 \in F_{n,1}$ with $\mathscr{R}(h_1)$, $\mathscr{R}(h_2)$ each consisting of at most $p - 1$ elements, and such that for every $x \in \mathscr{R}(f)$ we have $f(h_1(x), h_2(x)) = x$.*

Proof.   By Lemma 2 we can find $i, j, k, l < n$ such that $f(i, k)$, $f(i, l)$, $f(j, k)$, $f(j, l)$ represent at least three distinct values. Assume $f(i, k) = u$, $f(i, l) = v$, $f(j, k) = w$ are all different. Functions $h_1, h_2 \in F_{n,1}$ can be found such that

$$
\begin{aligned}
h_1(u) &= i & \qquad h_2(u) &= k \\
h_1(v) &= i & \qquad h_2(v) &= l \\
h_1(w) &= j & \qquad h_2(w) &= k
\end{aligned}
$$

and

$$
h_1(x) = i \, , \quad h_2(x) = k \quad \text{for} \quad x \notin \mathscr{R}(f)
$$
$$
f(h_1(x), h_2(x)) = x \quad \text{for} \quad x \in \mathscr{R}(f) \sim \{u, v, w\} \, .
$$

Clearly, h_1 and h_2 satisfy the requirements of the Lemma. The proof in the other cases is analogous.

LEMMA 4.   *If $f \in F_{n,2}$ and $2 \leqq p \leqq n$, and there exist $i, j, k < n$ such that for all $y < p$*

$$
f(i, y) = y \quad \text{and} \quad f(j, y) = k
$$

then f together with the functions in $F_{n,1}$ generate a function $g \in F_{n,2}$ such that $g(x, y) = x \bigvee_n y$ for $x, y < p$.

Proof.   We shall prove, by induction on p, a slightly weakened form of the theorem, replacing the condition $i, j, k < n$ by $i, j, k < p$. The

theorem as stated then follows, since $\{f\} \cup F_{n,1}$ generates a function satisfying the strengthened hypothesis.

For $p = 2$, since $i, j < p$ the function f must agree on $\{0, 1\}$ with one of the following four tables:

(i)	0	1
0	0	1
1	1	1

(ii)	0	1
0	0	1
1	0	0

(iii)	0	1
0	0	0
1	0	1

(iv)	0	1
0	1	1
1	0	1

In case (i) f itself may be taken for g. In the other cases using any homomorphism $h \in F_{n,1}$ which exchanges 0 and 1, g may be constructed as (ii) $h(f(x, h(y)))$, (iii) $h(f(h(x), h(y)))$, (iv) $f(h(x), y)$.

Assume the theorem is true for $p - 1$. From $\{f\} \cup F_{n,1}$ we can construct a function satisfying the induction hypothesis for $p - 1$. Thus we can generate a function g' such that

$$g'(x, y) = x \bigvee_n y \quad \text{for} \quad x, y < p - 1 \ .$$

Now choose functions $h_1, h_2 \in F_{n,1}$ such that

$$h_1(x) = \begin{cases} i & \text{if} \quad x < p - 1 \\ j & \text{if} \quad x = p - 1 \end{cases} \qquad h_2(x) = \begin{cases} p - 1 & \text{if} \quad x = k \\ k & \text{if} \quad x = p - 1 \\ x & \text{otherwise} \end{cases}$$

and construct $f' \in F_{n,2}$ as follows:

$$f'(x, y) = h_2(f(h_1(x), h_2(y))) \ .$$

It can be seen that

$$f'(x, y) = \begin{cases} y & \text{if} \quad x < p - 1 \quad \text{and} \quad y < p \\ p - 1 & \text{if} \quad x = p - 1 \quad \text{and} \quad y < p \ . \end{cases}$$

Now we define $g \in F_{n,2}$:

$$g(x, y) = f'(f'(x, y), g'(x, y)) \ .$$

For $x, y < p - 1$, $g(x, y) = g'(x, y) = x \bigvee_n y$; for $x = p - 1$, $y < p$, $g(x, y) = f'(p - 1, g'(p - 1, y)) = p - 1$; for $x < p - 1$ and $y = p - 1$, $g(x, y) = f'(p - 1, g'(x, p - 1)) = p - 1$. Therefore g agrees with $\bigvee_n$ for $x, y < p$, which estalishes the Lemma.

LEMMA 5. *If $f \in F_{n,2}$ is not reducible and $\mathscr{R}(f) = p$, $3 \leq p \leq n$, then f together with the functions in $F_{n,1}$ generate a function $g \in F_{n,2}$ such that $g(x, y) = x \bigvee_n y$ for all $x, y < p$.*

Proof. The proof is by induction on p. For $p = 3$, using the i, j, k, l of Lemma 3 and appropriate homomorphisms from $F_{n,1}$ we can

generate a function $h \in F_{n,2}$ such that the values of h on $\{0, 1\}$ agree
with one of the two tables:

(i)

	0	1
0	0	1
1	2	2

(ii)

	0	1
0	0	1
1	2	0

.

We then choose five functions $h_1, h_2, h_3, h_4, h_5 \in F_{n,1}$ such that

$$h_1(0) = 0 \qquad h_2(0) = 0 \qquad h_3(0) = 0 \qquad h_4(0) = 0 \qquad h_5(0) = 0$$
$$h_1(1) = 0 \qquad h_2(1) = 1 \qquad h_3(1) = 0 \qquad h_4(1) = 1 \qquad h_5(1) = 2$$
$$h_1(2) = 1 \qquad h_2(2) = 1 \qquad h_3(2) = 0 \qquad h_4(2) = 0 \qquad h_5(2) = 2 .$$

In case (i) g can be constructed as

$$g(x, y) = h(h(x, h_1(y)), h(x, h_2(y))) .$$

To construct g in case (ii) we define $g', g'' \in F_{n,2}$

$$g'(x, y) = h(h(h_3(x), h_1(y)), h_2(h(h_1(x), h_4(y))))$$
$$g''(x, y) = h_4(g'(x, h_5(y)))$$

and then

$$g(x, y) = g'(g''(x, y), g'(x, y)) .$$

Assuming the theorem true for $p - 1$, we prove it for p, $3 < p \leq n$.
First we construct a function f'' satisfying the induction hypothesis for
$p - 1$. To do this we apply Lemma 2, taking an i, j, k, l such that
$f(i, k), f(i, l), f(j, k), f(j, l)$ represent at least 3 distinct values, u, v, w.
Since $p > 3$ there is a value $z \in \mathscr{R}(f) \sim \{u, v, w\}$. Let $h \in F_{n,1}$ such
that

$$h(x) = \begin{cases} u & x = z \\ x & \text{otherwise.} \end{cases}$$

Then the function $h(f(x, y))$ is not reducible and has $p - 1$ elements in
its range. The application of an appropriate isomorphism from $F_{n,2}$ will
produce a function $f'' \in F_{n,2}$ with $\mathscr{R}(f'') = p - 1$ and f'' not reducible.
Then by the induction assumption we can generate a function $g'' \in F_{n,2}$
such that

$$g''(x, y) = x \vee_n y \quad \text{for} \quad x, y < p - 1 .$$

Next by Lemma 3 there exist $h_1, h_2 \in F_{n,1}$ with $\mathscr{R}(h_1), \mathscr{R}(h_2)$ each
consisting of at most $p - 1$ elements and such that

$$f(h_1(x), h_2(x)) = x$$

for $x < p$.

Let $h_3, h_4 \in F_{n,1}$ be isomorphisms such that

$$h_3(x) < p - 1 \quad \text{for} \quad x \in \mathscr{R}(h_1)$$
$$h_4(x) < p - 1 \quad \text{for} \quad x \in \mathscr{R}(h_2) ,$$

and define $f' \in F_{n,2}$, $h_5, h_6 \in F_{n,1}$:

$$h_5(x) = h_3(h_1(x)) , \quad x < n$$
$$h_6(x) = h_4(h_2(x)) , \quad x < n$$
$$f'(x, y) = f(h_3^{-1}(x), h_4^{-1}(y)) , \quad x, y < n .$$

Then

$$f'(h_5(x), h_6(x)) = x \quad \text{for} \quad x < p$$

and $\mathscr{R}(h_5), \mathscr{R}(h_6) \subseteqq p - 1$.

We can now define a function $g' \in F_{n,2}$ as follows:

$$g'(x, y) = f'(g''(x, h_1(y)), g''(x, h_2(y))) .$$

Then

$$g'(0, y) = f'(h_5(y), h_6(y)) = y \quad \text{for} \quad y < p$$

and

$$g'(p - 2, y) = f'(p - 2, p - 2) \quad \text{for} \quad y < p .$$

Therefore by Lemma 4 we can generate a function $g \in F_{n,2}$ such that $g(x, y) = x \bigvee_n y$ for $x, y < p$.

LEMMA 6. *If $f \in F_n$, $n \geqq 3$, f is not reducible, and $\mathscr{R}(f) = n$ then $F_{n,1} \cup \{f\}$ is complete.*

Proof. By definition $\sim_n \in F_{n,1}$. By Lemma 1 there is a $g \in \overline{F_{n,1} \cup \{f\}} \cap F_{n,2}$ such that $\mathscr{R}(g) = n$ and g is not reducible. Using Lemma 5 with $p = n$ we see that $\bigvee_n \in \overline{F_{n,1} \cup \{f\}}$. It is known[1] that the set $\{\bigvee_n, \sim_n\}$ is complete. Clearly, if $X \subseteqq \overline{Y}$ and X is complete then Y is complete. Therefore $F_{n,1} \cup \{f\}$ is complete.

In [4] Post established a necessary and sufficient condition for a set $X \subseteqq F_2$ to be complete. In order to extend this result to $n > 2$ we use his method. This consists in constructing a finite family $\mathscr{M}_n$ of proper closed subsets of F_n satisfying the condition that every proper closed subset of F_n is included in some set of the family. The existence of such a finite family of maximal sets is an important property of the lattice of all closed subsets of F_n.

By our definition $F_{n,1}$ is closed. Moreover $F_{n,1}$ is finite, containing

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[1] Post [3].

exactly n^n elements; therefore the family of all closed subsets of $F_{n,1}$ is finite. For each closed $S \subseteq F_{n,1}$, $n \geq 3$, we define a set $M(S)$ as follows:

   (i) if $S = F_{n,1}$ then $M(S)$ is the set of all functions $f \in F_n$ such that either f is reducible or $\mathscr{R}(f)$ is a proper subset of n;

   (ii) if $S \neq F_{n,1}$, then $M(S)$ is the set of all functions $f \in F_n$ satisfying the following condition: if in f we replace zero or more arguments by functions in S and then identify all arguments, the resulting function is in S.

Finally, we take as $\mathscr{M}_n$, $n \leq 3$, the family of all sets $M(S)$ where S is any closed subset of $F_{n,1}$.[2] That this family $\mathscr{M}_n$ actually has the property mentioned above is seen from the following.

LEMMA 7. *Let S is a closed subset of $F_{n,1}$, $(n \geq 3)$. Then*
   (i) *$M(S)$ is closed.*
   (ii) *$M(S)$ is a proper subset of F_n.*
   (iii) *$M(S) \cap F_{n,1} = S$.*
   (iv) *If Y is a proper closed subset of F_n with $Y \cap F_{n,1} = S$ then $Y \subseteq M(S)$.*

Proof. We establish Lemma 7 first for the case $S \neq F_{n,1}$. $M(S)$ is closed, since the defining property for $M(S)$ is preserved under exchange or identification of variables and also under substitution. Since S is a proper subset of $F_{n,1}$, $M(S)$ does not contain all functions of $F_{n,1}$. Therefore $M(S)$ is a proper subset of F_n. $S \subseteq M(S) \cap F_{n,1} \subseteq S$, hence $M(S) \cap F_{n,1} = S$. The fourth property can be verified directly from the definition of $M(S)$: Let Y be any proper closed subset of F_n with $Y \cap F_{n,1} = S$, $f \in Y$, and h a function obtained by replacing zero or more arguments of f by functions in S and then identifying all arguments. Since $S \subseteq Y$ and Y is closed, $h \in Y$. But $Y \cap F_{n,1} \subseteq S$ and $h \in F_{n,1}$, so $h \in S$. Thus every function $f \in Y$ is in $M(S)$. Therefore $Y \subseteq M(S)$.

We turn now to the case $S = F_{n,1}$. That $M(F_{n,1})$ is closed follows from its definition: Both reducibility and range different from n are preserved under exchange or identification of variables. Let $f, g \in M(F_{n,1})$, h a function obtained by replacing an argument of f by the function g. If $\mathscr{R}(g) \neq n$ then $\mathscr{R}(h) \neq n$ and $h \in M(F_{n,1})$. If g is reducible, either h is reducible or $h = f$, so $h \in M(F_{n,1})$. Clearly $M(F_{n,1})$ is a proper subset of F_n since there exist functions in F_n with full range n which are not reducible. $M(F_{n,1}) \cap F_{n,1} = F_{n,1}$, since every function in $F_{n,1}$ is reducible. The proof of the fourth property follows from Lemma 6:

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[2] The corresponding family for $n = 2$ contains nine elements since $F_{2,1}$ has exactly nine closed subsets. In [4] Post defined these nine sets individually: C_4, R_2, R_3, R_9, C_2, C_3, D_3, A_1, L_1. Our definition of $M(S)$ is directly applicable to the eight proper closed subsets of $F_{2,1}$. However, it is of interest to note that in the case $S = F_{n,1}$ the structure of $M(S)$ is essentially different for $n > 2$.

Let Y be any proper closed subset of F_n with $Y \cap F_{n,1} = F_{n,1}$. Clearly, Y cannot be complete. By Lemma 6, since $F_{n,1} \subsetneqq Y$, Y cannot contain any function f, with $\mathscr{R}(f) = n$, which is not reducible. Hence $Y \subseteq M(F_{n,1})$. This completes the proof.

Thus we see that the family $\mathscr{M}_n$ of all sets $M(S)$ where S is any closed subset of $F_{n,1}$ consists of finitely many proper closed subsets of F_n. Moreover, if X is any proper closed subset of F_n by property (iv) of Lemma 7 $X \subseteq M(X \cap F_{n,1}) \in \mathscr{M}_n$, since $X \cap F_{n,1}$ is a closed subset of $F_{n,1}$.

We now state the main result of the note:

THEOREM 1. *A necessary and sufficient condition for a set $X \subseteq F_n$, $n \geq 3$, to be complete is that for every closed subset S of $F_{1,n}$ there is an $f \in X$ such that $f \notin M(S)$.*[3]

The proof of this theorem follows directly from Lemma 7. If there were any closed subset S of $F_{1,n}$ such that $X \subseteq M(S)$ then $\bar{X} \subseteq M(S)$ since $M(S)$ is closed and hence X would not be complete. On the other hand if for every closed subset S of $F_{1,n}$ there is an $f \in X \sim M(S)$ then $\bar{X} \not\subseteq M(S)$ for any closed subset S of $F_{1,n}$. By Lemma 7, $\bar{X}$ cannot be a proper subset of F_n. Therefore X must be complete.

COROLLARY 1. *A set $X \subseteq F_n$ is complete if and only if $F_{n,1} \subseteq \bar{X}$, $n \geq 3$, and there is an $f \in X$ such that $\mathscr{R}(f) = n$, and f is not reducible.*

Proof. If X is complete then $F_{n,1} \subseteq \bar{X}$; and by Theorem 1, $X \not\subseteq M(F_{n,1})$. Therefore there is an $f \in X \sim M(F_{n,1})$; i.e., $f \in X$, $\mathscr{R}(f) = n$, and f not reducible. On the other hand if $F_{n,1} \subseteq \bar{X}$, then X is not included in $M(S)$ for any proper closed subset S of $F_{n,1}$. If, in addition, there is an $f \in X$ with $\mathscr{R}(f) = n$ and f not reducible, then $X \not\subseteq M(F_{n,1})$. Therefore by Theorem 1, X must be complete.

We now state two further results, Theorems 2 and 3,[4] which follow easily from Theorem 1;

THEOREM 2. *There exist finite decision procedures to determine*

[3] An analogous result for $n = 2$ with the family $\mathscr{M}_n$ replaced by the set $\{D_3, C_2, C_3, A_1, L_1\}$ was obtained by Post in [4]. It may be noted that our theorem can be sharpened to include this result by adding the restriction: S contains the identity function and at least one other element.

[4] Yablonskii in [5] states without proof Theorem 2, which he attributes to A. V. Kuznecov. He also attributes to Kuznecov another result which he states (again without proof) as follows: Every complete set (included in F_n) contains a finite complete subset (i.e., a finite basis). In this form the result is rather obvious and follows directly from the results of Post [3]; compare the first part of the proof of Corollary 2. The subsequent remarks of Yablonskii make it likely, however, that Kuznecov obtained a stronger result established here as Corollary 2.

whether or not any finite subset of F_n, $n \geq 3$, is complete and if complete whether independent.[5]

This theorem depends essentially on the computable character of our definition of the sets $M(S)$. This means that for any $f \in F_n$ and any closed subset S of $F_{n,1}$ we can tell by a finite procedure whether or not $f \in M(S)$. Therefore if X is a finite set, Theorem 1 provides a finite method for determining whether or not X is complete. This part of the proof can also be based on Corollary 1. The only thing to be shown is that all functions belonging to $F_{n,1} \cap \bar{X}$ can be obtained by means of a well determined finite procedure.

If X is complete then X is independent if and only if no proper subset of X is complete. For a finite complete X, therefore, we can determine in finitely many steps whether or not X is independent.

THEOREM 3. *For any natural number n, $n \geq 2$, there is a natural number p such that every complete and independent subset of F_n has at most p elements.*[6]

For $n = 2$ this theorem was proved by Post in [4]. For $n \geq 3$ it can be derived directly from Theorem 1. Let p be the number of elements in the family $\mathscr{M}_n$ of all sets $M(S)$ for S any proper closed subset of $F_{n,1}$. By Theorem 1, any set which contains an $f \notin M(S)$ for each $M(S)$ in $\mathscr{M}_n$ is complete. Thus any complete set with more than p elements would contain a proper subset which is complete.

COROLLARY 2. *For any number n, $n \geq 2$, there is a natural number p such that every complete set included in F_n has a finite basis with at most p element.*

If $X \subseteq F_n$ is complete, then $\mathbf{V}_n, \sim_n \in \bar{X}$. Hence $\{\mathbf{V}_n, \sim_n\}$ can be generated by a finite subset Y of X. Since $\{\mathbf{V}_n, \sim_n\}$ is complete Y must be complete. Let Z be any complete independent subset of Y. Z is a finite basis of X and by Theorem 3, Z has at most p elements.

By modifying the proof of this result (and in fact making the argument independent of Theorem 1) A. Tarski has obtained the following generalization of Theorem 3.

THEOREM 4. *For any closed set $X \subseteq F_n$ which has a finite basis there is a natural number q such that every independent basis of X has at most q elements.*

The method of proof is similar to the proof of Theorem 3. We replace the family $\mathscr{M}_n$ by a finite family $\mathscr{L}_X = \{L_0, L_1, \cdots, L_{q-1}\}$ of closed proper subsets of X with the property that for any closed proper

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[5] By Post's results in [4], this theorem is also valid for $n = 2$ since the conditions defining the sets D_3, C_2, C_3, A_1 and L_1 are finitely computable.

[6] For $n = 3$, Yablonskiĭ in [5] found $p = 18$.

subset Y of X there is a set L_i, $i < q$, in $\mathscr{L}_x$ such that $Y \subseteq L_i$. $\mathscr{L}_x$ is constructed as follows. Let B be any finite basis of X. Since B is finite there is a natural number k such that $B \subseteq \bigcup_{i<k} F_{i,n}$. For each A satisfying

$$A \subseteq X \cap \bigcup_{i<k} F_{n,i} \,, \quad A \neq X \,, \quad \text{and} \quad \bar{A} \cap \bigcup_{i<k} F_{n,i} = A$$

we define the set $L(A)$ in the same manner as the sets $M(S)$ were defined for $S \neq F_{n,1}$. The proof that the sets $L(A)$ are closed proper sets with the required property in the lattice all closed subsets of X is entirely analogous to the proof of Lemma 7.

For complete sets the upper bound q of Theorem 4 is much larger than the p of Theorem 3.

For $n = 2$, Post in [4] computed an upper bound $p = 5$ in Theorem 3, and then showed that actually 4 was the least upper bound. He also proved that every closed subset of F_2 has a finite basis. Therefore, two further questions arise for $n \geq 3$:

(1) does every closed system of functions in F_n have a finite basis;

(2) (proposed by A. Tarski) is there any finite procedure to determine the least upper bound for the number of elements in any independent basis of the complete system F_n.

The solutions of these two problems have been communicated to me by A. Ehrenfeucht. He has shown that the solution of problem (2) is positive, while that of problem (1) is negative. Ehrenfeucht exhibits a very simple closed subset of F_n, $n \geq 3$, which has no finite basis.

(*Added in proof.*) It has been communicated to me by Professor C. C. Chang that Lemma 5 was obtained by Jerzy Slupecki in "A criterion of fullness of many-valued systems of propositional logic", Comptes Rendues des séances de la Société des Sciences et des lettres de Varsovie 33, 1939, Classe III, pp 102–109. Slupecki proves the following extension of Lemma 5: *If $F_{n,1} \subseteq X$ then X is complete if and only if there is an $f \in X$, $f \in F_{n,2}$, f not reducible and $\mathscr{R}(f) = n$.* Note that Lemma 6 and Corollary 1 extend this result by using Lemmas 1 and 7 to remove the condition $f \in F_{n,2}$, which is necessary for the main results of this note.

BIBLIOGRAPHY

1. Jean W. Butler, *On operations in finite algebras*, Bull. Amer. Math. Soc., **62** (1956), Abstract 552.
2. A. Ehrenfeucht, review of [**5**], Journal of Symbolic Logic 20 (1955), 175.
3. E. L. Post, *Introduction to a general theory of elementary propositions*, American Journal of Mathematics, **43** (1921), 163-185.
4. E. L. Post, *The two-valued interative systems of mathematical logic*, Princeton Annals of Math. Studies, **5** (1941).
5. S. V. Yablonskiĭ (S. V. Áblonskij), *On functional completeness in the three-valued calculus*, Doklady Akademii Nauk SSSR, **95** (1954), 1153-1155.

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