ABSTRACT MARTINGALE CONVERGENCE THEOREMS

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Introduction. The study of probability theory in abstract spaces became possible with the introduction of integration theories in such spaces. Thus the idea of the expectation of a random variable which takes its value in a Banach space was studied by Frechet [6] with what amounted to the Bochner integral, and by Mourier [13] with the Pettis integral. Doss [2] studied the problem in a metric space. Kolmogorov [10] generalized the notion of characteristic function. Generalizations of the laws of large numbers and the ergodic theorem appear in Mourier [13] and Fortet-Mourier [5]. In this paper we generalize the concept of martingale and prove various convergence theorems.

Chapter I is devoted to listing various definitions and theorems which we shall have to refer to later. In Chapter II we introduce the idea of the conditional expectation of a Banach space valued random variable. We also prove the existence of the strong conditional expectation for strongly measurable random variables. This part of our work was also done by Moy [14] independently, and without the knowledge of the author. Chapter III is devoted to the definition and study of weak and strong $\mathfrak{X}$-martingales, with emphasis on the latter.

In Chapter IV we prove a series of convergence theorems for $\mathfrak{X}$-Martingales with the help of theorems of Doob [1]. The main theorem says that if $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-Martingale where $\mathfrak{X}$ is a reflexive Banach space, and if $\{\| x_n \|, n \geqq 1\}$ is a uniformly integrable class of functions, then there is a strongly measurable $\mathfrak{X}$-valued function x_∞ such that $\| x_n(\omega) - x_\infty(\omega) \| \to 0$ as $n \to \infty$ with probability 1 and $\{x_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is an $\mathfrak{X}$-martingale. We close by discussing examples where $\mathfrak{X}$ is one of the standard Banach spaces, l^p, $L^p(I)$, and $C(I)$.

CHAPTER I.

PRELIMINARY DEFINITIONS

1. Measurability concepts. A. Let $(\Omega, P, \mathscr{M})$ be a probability space. Thus Ω is an abstract set of points ω, $\mathscr{M}$ is a Borel field of subsets of Ω, and P is a probability measure defined on $\mathscr{M}$. We recall that a Borel field of sets is a class of sets which is closed under countable unions and intersections, and complementation. A probability

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measure P is a completely additive non negative set function defined on a Borel field of sets, such that $P\{\Omega\} = 1$. We will be concerned with functions $x(\cdot)$ defined on Ω, and taking their values in a Banach space $\mathfrak{X}$. The sets of $\mathscr{M}$ will be referred to as the measurable sets.

DEFINITION 1.1. x is a *weak random variable* if it is a weakly measurable function from Ω to $\mathfrak{X}$.

DEFINITION 1.2. x is a *finitely* (*countably*) *valued random variable* if it is constant on each of a finite (countable) number of disjunct measurable sets Λ_j; with $\Omega = \bigcup_j \Lambda_j$.

DEFINITION 1.3. x is a *strong random variable* if it is a strongly measurable function from Ω to $\mathfrak{X}$.

DEFINITION 1.4. x is *almost separably valued* if there is a set Λ in $\mathscr{M}$ such that $P\{\Lambda\} = 0$ and $x(\Omega - \Lambda)$ is separable.

Note. x is strongly measurable if and only if it is weakly measurable and almost separably valued. (Pettis [15] and Hille-Phillips [9] Theorem 3.5.3, p. 72).

B. The measure induced in $\mathfrak{X}$. Suppose x is a function from Ω to $\mathfrak{X}$. We define a class of subsets of $\mathfrak{X}$ in the following way: Let $\mathscr{F}$ be a Borel field of measurable subsets of Ω, $\mathscr{F} \subseteqq \mathscr{M}$. Let $\mathscr{F}_x$ be the class of subsets of $\mathfrak{X}$ with the property that $\mathscr{A} \in \mathscr{F}_x$ if $\mathscr{A} \subseteqq X$ and $\{\omega: x(\omega) \in \mathscr{A}\}$ is an $\mathscr{F}$ set. $\mathscr{F}_x$ is a Borel field.
If $\mathscr{A} \in \mathscr{F}_x$ define $P^x\{\mathscr{A}\} = P\{\omega: x(\omega) \in \mathscr{A}\}$. Clearly P^x is a probability measure on $\mathscr{F}_x$. This gives us a probability triple on $\mathfrak{X}$, $(\mathfrak{X}, P^x, \mathscr{F}_x)$. Now, let $\mathscr{F} = \mathscr{M}$, the class of measurable sets of Ω. In order to assure that $\mathscr{M}_x$ will contain some interesting subsets of $\mathfrak{X}$ we shall have to assume some measurability properties for x, which we now proceed to do.

a. Suppose that x is weakly measurable. Then $f(x)$ is a real measurable function for all $f \in \mathfrak{X}^*$, the real first conjugate space of $\mathfrak{X}$. Thus for every real Borel set B, $\{\omega: f(x(\omega)) \in B\}$ is an $\mathscr{M}$ set. Next $\{\omega: f(x(\omega)) \in B\} = \{\omega: x(\omega) \in f^{-1}(B)\}$. Hence $f^{-1}(B)$ is in $\mathscr{M}_x$ for every f in $\mathfrak{X}^*$ and real Borel set B. Since f is continuous, $f^{-1}(B)$ is open (closed) if B is open (closed).

Further, $\mathscr{M}_x$ contains all the weak neighborhoods of $\mathfrak{X}$ if x is weakly measurable. In fact, let $N(\xi_0; f_1, \cdots, f_n; \varepsilon)$ be a weak neighborhood of $\mathfrak{X}$. Then

$$N(\xi_0; f_1, \cdots, f_n; \varepsilon) = \{\xi: |f_j(\xi) - f_j(\xi_0)| < \varepsilon, \quad j = 1, \cdots, n\}$$
$$= \bigcap_{j=1}^{n} \{\xi: |f_j(\xi) - f_j(\xi_0)| < \varepsilon\} .$$

But the inverse image of each of the sets in the intersection by x is clearly an $\mathscr{M}$ set since $f(x)$ is a real valued measurable function for every linear functional f. Thus $\mathscr{M}_x$ contains all of the weak neighborhoods of $\mathfrak{X}$, and hence the smallest Borel field containing the weak neighborhoods.

Conversely, if $\mathscr{M}_x$ contains all the weak neighborhoods of $\mathfrak{X}$ then x is weakly measurable. To prove this, we must show that $f(x)$ is a real valued measurable function on Ω for every f in $\mathfrak{X}^*$. If f is the zero functional then $f(x(\omega)) = 0$ for all ω, and thus $f(x)$ is clearly measurable. Otherwise f takes on all real values. In this case we show that $\{\omega: f(x(\omega)) \in B\}$ is an $\mathscr{M}$ set for every real Borel set B and linear functional f. If B is the open interval $(a - \varepsilon, a + \varepsilon)$, then $\{\omega: f(x(\omega)) \in B\} = \{\omega: |f(x(\omega)) - a| < \varepsilon\}$. Since f takes on all real values there is an element ξ_0 in $\mathfrak{X}$ such that $f(\xi_0) = a$. Hence $\{\omega: f(x(\omega)) \in B\} = \{\omega: x(\omega) \in N(\xi_0; f; \varepsilon)\}$ which is an $\mathscr{M}$ set by hypothesis for $\mathscr{M}_x$ contains all the weak neighborhoods of $\mathfrak{X}$. Next, every open set in the reals, in fact, in any separable metric space, is a countable union of open spheres. Thus, if B is an open set in the reals $B = \bigcup_n V_n$ where V_n is an open interval for every n. Since $\mathscr{M}$ is closed under countable unions $\{\omega: f(x(\omega)) \in B\} = \bigcup_n \{\omega: f(x(\omega)) \in V_n\}$ is an $\mathscr{M}$ set. Finally, the class of real sets B for which $\{\omega: f(x(\omega)) \in B\}$ is an $\mathscr{M}$ set is a Borel field which contains the open sets, thus it must contain all the real Borel sets, and so x is weakly measurable. Thus the definition of weak measurability may be rephrased as follows:

DEFINITION 1.1.* x is weakly measurable if $\mathscr{M}_x$ contains all the weak neighborhoods of $\mathfrak{X}$, that is, if $\{\omega: x(\omega) \in N\}$ is an $\mathscr{M}$ set for every weak neighborhood N.

b. Suppose that x is strongly measurable. Then there is a sequence x_n of finitely valued functions, and a set Λ in $\mathscr{M}$ such that $P\{\Lambda\} = 0$, $\| x_n(\omega) - x(\omega) \| \to 0$ as $n \to \infty$ for $\omega \in \Omega - \Lambda$. Let g be a real valued continuous function. Then $g(x)$ is a real valued measurable function on Ω. Consequently, $\{\omega: g(x(\omega)) \in B\}$ is an $\mathscr{M}$ set and $g^{-1}(B)$ is an $\mathscr{M}_x$ set for every real Borel set B and real continuous function g. Next let $\mathscr{C}$ be the class of real valued functions g defined on $\mathfrak{X}$ such that $g(x)$ is a real valued measurable function on Ω. Then $\mathscr{C}$ contains the continuous functions and is closed under the limit operation, thus it contains all the Baire functions on $\mathfrak{X}$ to the reals. Now let A be a Borel set in $\mathfrak{X}$. Then there is a real number a and a Baire function g

such that $A = \{\xi: g(\xi) > a\}$. Now $A = g^{-1}(B)$ where $B = (a, \infty)$. Thus A is an $\mathscr{M}_x$ set since $\{\omega: g(x(\omega)) \in B\}$ is an $\mathscr{M}$ set by the measurability of $g(x)$. Therefore if x is strongly measurable, then $\mathscr{M}_x$ contains all the Borel sets of $\mathfrak{X}$, or $\{\omega: x(\omega) \in B\}$ is an $\mathscr{M}$ set for every Borel set B of $\mathfrak{X}$.

C. Independence. Let x and y be (weakly or strongly) measurable random variables on Ω to $\mathfrak{X}$. We can then define a Borel field $\mathscr{M}_{x,y}$ of subsets $\mathfrak{X} \times \mathfrak{X}$ in an analogous way. Consider $\mathscr{M}_x \times \mathscr{M}_y = \{A \times B: A \in \mathscr{M}_x, B \in \mathscr{M}_y\}$. Let $P^{x,y}(A \times B) = P\{\omega: x(\omega) \in A, y(\omega) \in B\}$. This probability is well defined for the set on the right is the intersection of two $\mathscr{M}$ sets and hence is itself an $\mathscr{M}$ set. Let $R_{x,y}$ be the field of finite unions of sets of $\mathscr{M}_x \times \mathscr{M}_y$. Then $P^{x,y}$ can be defined on $R_{x,y}$ to be a probability measure in the obvious way in a unique fashion. Next $P^{x,y}$ can be extended uniquely to $\mathscr{M}_{x,y}$, the smallest Borel field of measurable subsets of $\mathfrak{X} \times \mathfrak{X}$ containing $R_{x,y}$ (Doob [1] Theorem 2.2, p. 605).

DEFINITION 1.5. x and y are said to be independent if $P\{\omega: x(\omega) \in A, y(\omega) \in B\} = P\{\omega: x(\omega) \in A\}P\{\omega: y(\omega) \in B\}$ for A, B subsets of $\mathfrak{X}$ whenever all of the probabilities in the equality are defined; i.e., whenever the above sets are in $\mathscr{M}$. The equality may be rewritten as $P^{x,y}(A \times B) = P^x(A)P^y(B)$.

Notice that this definition can be rephrased to say that the product relationship holds whenever A is in $\mathscr{M}_x$ and B is in $\mathscr{M}_y$, for only then will all of the probabilities in the product be defined. This is the type of definition that has been given by Kolmogorov; e.g., Gnedenko-Kolmogorov ([7], p. 26). The definition used by Doob [1] differs in that it says that the product relationship holds whenever A and B belong to a possibly smaller class of sets, namely the Borel sets. For a full discussion of the connection between the two types of definition the reader is referred to Doob's appendix to the above mentioned book by Gnedenko and Kolmogorov.

THEOREM 1.1. *If x and y are independent, then $f_1(x), \cdots, f_n(x)$ are independent of $g_1(y), \cdots, g_m(y)$ in the sense of Kolmogorov for every finite set of real valued linear functionals $f_1, \cdots, f_n, g_1, \cdots, g_m$ on $\mathfrak{X}$.*

Proof. Let $A_1, \cdots, A_n, B_1, \cdots, B_m$ be real sets such that $\{\omega: f_j(x(\omega)) \in A_j\}$ and $\{\omega: g_k(y(\omega)) \in B_k\}$ are $\mathscr{M}$ sets for $j = 1, \cdots, n$ and $k = 1, \cdots, m$. Then $f_j^{-1}(A_j)$ is in $\mathscr{M}_x$ and $g_k^{-1}(B_k)$ is in $\mathscr{M}_y$. Next, $\bigcap_{j=1}^n f_j^{-1}(A_j) \in \mathscr{M}_x$ and $\bigcap_{k=1}^m g_k^{-1}(B_k) \in \mathscr{M}_y$. Thus

$$P\{\omega: f_1(x(\omega)) \in A_1, \cdots, f_n(x(\omega)) \in A_n, g_1(y(\omega)) \in B_1, \cdots, g_m(y(\omega)) \in B_m\}$$
$$= P\{\omega: x(\omega) \in \bigcap_{j=1}^{n} f_j^{-1}(A_j), y(\omega) \in \bigcap_{k=1}^{m} g_k^{-1}(B_k)\}$$
$$= P\{\omega: x(\omega) \in \bigcap_{j=1}^{n} f_j^{-1}(A_j)\}P\{\omega: y(\omega) \in \bigcap_{k=1}^{m} g_k^{-1}(B_k)\}$$

by the independence of x and y

$$= P\{\omega: f_1(x(\omega)) \in A_1, \cdots, f_n(x(\omega)) \in A_n\}P\{\omega: g_1(y(\omega)) \in B_1, \cdots, g_m(y(\omega)) \in B_m\} \quad \text{Q.E.D.}$$

THEOREM 1.2. *If x and y are weakly measurable and independent, then $f_1(x), \cdots, f_n(x)$ are independent of $g_1(y), \cdots, g_m(y)$ in the sense of Doob for every finite set of real valued linear functionals $f_1, \cdots, f_n, g_1, \cdots, g_m$ on $\mathfrak{X}$.*

Proof. Let A_j and B_k in the above proof be real Borel sets; then $\{\omega: f_j(x(\omega)) \in A_j\}$ and $\{\omega: g_k(y(\omega)) \in B_k\}$ are $\mathscr{M}$ sets for $f_j(x)$ and $g_k(y)$ are real valued measurable functions by the weak measurability of x and y. The rest of the proof goes as above.

THEOREM 1.3. *If x and y are weakly measurable, and such that $f_1(x), \cdots, f_n(x)$ are independent of $g_1(y), \cdots, g_m(y)$ for every finite set of real valued linear functionals $f_1, \cdots, f_n, g_1, \cdots, g_m$ on $\mathfrak{X}$, then x and y are independent relative to the smallest Borel field of $\mathfrak{X}$ sets containing the weak neighborhoods;* i.e.,

$$P\{\omega: x(\omega) \in A, y(\omega) \in B\} = P\{\omega: x(\omega) \in A\}P\{\omega: y(\omega) \in B\}$$

for all A and B in the smallest Borel field containing the weak neighborhoods of $\mathfrak{X}$.

Proof. Let $A = N(\xi_0; f_1, \cdots, f_n; \varepsilon)$ and $B = N(\eta_0; g_1, \cdots, g_m; \delta)$: then

$$\begin{aligned} &P\{\omega: x(\omega) \in A, y(\omega) \in B\} \\ &\quad = P\{\omega: |f_i(x(\omega)) - f_i(\xi_0)| < \varepsilon, \quad i = 1, \cdots, n; \\ &\qquad\qquad |g_j(y(\omega)) - g_j(\eta_0)| < \delta, \quad j = 1, \cdots, m\} \\ &\quad = P\{\omega: |f_i(x(\omega)) - f_i(\xi_0)| < \varepsilon, \quad i = 1, \cdots, n\} \\ &\qquad\qquad P\{\omega: |g_j(y(\omega)) - g_j(\eta_0)| < \delta, \quad j = 1, \cdots, m\} \end{aligned}$$

by the hypothesis, and so

$$P\{\omega: x(\omega) \in A, y(\omega) \in B\} = P\{\omega: x(\omega) \in A\}\cdot P\{\omega: y(\omega) \in B\}$$

when A and B are weak neighborhoods of $\mathfrak{X}$. Now the class of weak neighborhoods is closed under finite intersections and thus the independence multiplicative relationship is preserved if we extend this class to the smallest Borel field containing it (Loève [12] p. 225).

The notion of independence is easily generalized to aggregates of random variables. For a fuller discussion of the measurability concepts mentioned in this section, see Pettis [15] and Hille and Phillips [9].

Note. Let $(\xi, \eta) \in \mathfrak{X} \times \mathfrak{X}$. Define $\|(\xi, \eta)\| = \sqrt{\|\xi\|^2 + \|\eta\|^2}$. By this definition, $\mathfrak{X} \times \mathfrak{X}$ becomes a Banach space. Let f be a real linear functional on $\mathfrak{X} \times \mathfrak{X}$. If $f_1(\xi) = f[(\xi, \theta)]$ and $f_2(\eta) = f[(\theta, \eta)]$, then f_1 and f_2 are real linear functionals on $\mathfrak{X}$, and $f[(\xi, \eta)] = f_1(\xi) + f_2(\eta)$. If x and y are weakly measurable $\mathfrak{X}$-valued functions on Ω, then $f_1(x)$ and $f_2(y)$ are real valued measurable functions on Ω. Thus the weak measurability of x and y implies the weak measurability of (x, y) on Ω to $\mathfrak{X} \times \mathfrak{X}$. Similarly, if x and y are strongly measurable, there exist sequences x_n and y_n of finitely-valued measurable $\mathfrak{X}$-valued functions such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$ with probability 1. But (x_n, y_n) gives a sequence of $\mathfrak{X} \times \mathfrak{X}$ finitely-valued functions, and $\|(x_n, y_n) - (x, y)\| = \sqrt{\|x_n - x\|^2 + \|y_n - y\|^2} \to 0$ with probability 1 as $n \to \infty$. Thus, if x and y are strongly measurable, then so is (x, y).

2. Integrability concepts. Let x be a countably valued function taking the value ξ_j on the measurable set Λ_j. Then x is said to be Bochner integrable if and only if $\|x(\cdot)\|$ is integrable, and by definition

$$(B) \int_\Omega x(\omega) dP = \sum_{j=1}^{\infty} \xi_j P(\Lambda_j) .$$

DEFINITION 2.1. $x(\cdot)$ is integrable in the sense of Bochner if there is a sequence $x_n(\cdot)$ of countably valued random variables converging with probability 1 to $x(\cdot)$, and such that

$$\lim_{m, n \to \infty} \int_\Omega \| x_m(\omega) - x_n(\omega) \| dP = 0 .$$

Then the limit of $(B)\int_\Omega x_n(\omega) dP$ exists and by definition

$$(B)\int_\Omega x(\omega) dP = \lim_{n\to\infty} (B) \int_\Omega x_n(\omega) dP .$$

Since $P\{\Omega\} = 1$, we may again replace the word countably by finitely.

We will later need the following result apparently proved first by Pettis ([15] Theorem 5.2, p. 293), and later by Moy ([14] Theorem 1, pp. 3, 4.)

THEOREM 2.1. *If x is strongly measurable relative to the Borel field*

$\mathscr{F}$ of measurable sets and Bochner integrable and such that $\int_A x(\omega)dP = \theta$ for every set A in $\mathscr{F}$ then $x(\omega) = \theta$ almost everywhere.

CHAPTER II

GENERALIZATIONS OF THE RADON-NIKODYM THEOREM AND ABSTRACT CONDITIONAL EXPECTATIONS

1. It is well known that a real or complex valued completely additive set function which is absolutely continuous on a σ-finite measure space is actually the integral in the usual sense of a finite measurable point function (unique almost everywhere). The existence of this point function is assured by the classical Radon-Nikodym theorem (Halmos [8] p. 128).

Using a theorem due to Dunford and Pettis ([4], p. 339) it is possible to get a definition of conditional expectations for more general random variables such as Dunford and Pettis integrable functions. Since it is too weak for our purposes, we will no longer refer to it in this paper.

2. Strong conditional expectations. If we restrict ourselves to Bochner integrable random variables it is possible to get a sharper version of the conditional expectation.

With this end in mind, let $x(\cdot)\colon \Omega \to \mathfrak{X}$ be finitely valued; in fact, let $x(\omega) = \xi_j$ on Λ_j; $j = 1, \cdots, k$. Then $x(\omega) = \sum_{j=1}^{k} \xi_j \cdot \chi_{\Lambda_j}(\omega)$ where χ_{Λ_j} is the characteristic function of Λ_j.

DEFINITION 2.1. $\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega) = \sum_{j=1}^{k} \xi_j \cdot E\{\chi_{\Lambda_j} / \mathscr{F}\}(\omega)$, where $E\{\chi_{\Lambda_j} \mid \mathscr{F}\}$ is the ordinary conditional expectation (Doob [1]) of χ_{Λ_j} relative to $\mathscr{F}$. $\mathscr{E}^s\{x \mid \mathscr{F}\}$ will be referred to as the strong conditional expectation of x relative to $\mathscr{F}$.

In this section all integrals will be in the sense of Bochner, so we will remove the letter B preceding the integral sign.

LEMMA 2.1. *If x is a measurable finitely valued function on Ω to $\mathfrak{X}$, then $\int_\Lambda x(\omega)dP = \int_\Lambda \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)dP$ for every $\Lambda \in \mathscr{F}$.*

Proof.

$$\int_\Lambda \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)dP = \int_\Lambda \Big(\sum_{j=1}^{k} \xi_j E\{\chi_{\Lambda_j} \mid \mathscr{F}\}(\omega)\Big)dP$$
$$= \sum_{j=1}^{k} \xi_j \int_\Lambda E\{\chi_{\Lambda_j} \mid \mathscr{F}\}(\omega)dP$$

where the integral is in the ordinary sense

$$= \sum_{j=1}^{k} \xi_j P\{A_j \cap A\}$$
$$= \int_A x(\omega) dP . \qquad \text{Q.E.D.}$$

LEMMA 2.2. *If x is a measurable finitely valued function on Ω to $\mathfrak{X}$, then $\| \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega) \| \leqq E\{\|x\| \mid \mathscr{F}\}$ (ω) with probability* 1.

Proof.

$$\| \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega) \| = \left\| \sum_{j=1}^{k} \xi_j E\{\chi_{A_j} \mid \mathscr{F}\}(\omega) \right\|$$
$$\leqq \sum_{j=1}^{k} \| \xi_j \| \cdot E\{\chi_{A_j} \mid \mathscr{F}\}(\omega)$$

for $\chi_{A_j} = 1$ or 0.

$$= E\{\|x\| \mid \mathscr{F}\}(\omega) . \qquad \text{a.e.} \qquad \text{Q.E.D.}$$

LEMMA 2.3. *If $x_1, \cdots, x_k$ are finitely valued measurable functions, and $a_1 \cdots, a_k$ are scalars, then*

$$\mathscr{E}^s\{a_1 x_1 + \cdots + a_k x_k \mid \mathscr{F}\}(\omega) = \sum_{j=1}^{k} a_j \mathscr{E}^s\{x_j \mid \mathscr{F}\}(\omega)$$

with probability 1.

Proof. Let $\{A_m\}$: $m = 1, \cdots, p$ be a decomposition of Ω such that each x_j takes on only one value on each A_m; in fact, let $x_j(\omega) = \varphi_j(A_m)$ for $\omega \in A_m$. Then since $\mathscr{E}^s\{x \mid \mathscr{F}\}$ depends on x and $\mathscr{F}$ and not on the decomposition of Ω, the same representation holds for all the $\mathscr{E}^s\{x_j \mid \mathscr{F}\}$. Hence

$$\mathscr{E}^s\{x_j \mid \mathscr{F}\}(\omega) = \sum_{m=1}^{p} \varphi_j(A_m) E\{\chi_{A_m} \mid \mathscr{F}\}(\omega) .$$

Thus

$$\mathscr{E}^s\{a_1 x_1 + \cdots + a_k x_k \mid \mathscr{F}\}(\omega)$$
$$= \sum_{m=1}^{p} [a_1 \varphi_1(A_m) + \cdots + a_k \varphi_k(A_m)] E\{\chi_{A_m} \mid \mathscr{F}\}(\omega)$$
$$= a_1 \sum_{m=1}^{p} \varphi_1(A_m) E\{\chi_{A_m} \mid \mathscr{F}\}(\omega) + \cdots + a_k \sum_{m=1}^{p} \varphi_k(A_m) E\{\chi_{A_m} \mid \mathscr{F}\}(\omega)$$
$$= \sum_{j=1}^{k} a_j \mathscr{E}^s\{x_j \mid \mathscr{F}\}(\omega) \qquad \text{with probability 1.} \qquad \text{Q.E.D.}$$

THEOREM 2.1. *Let $x(\cdot)$: $\Omega \to \mathfrak{X}$ be integrable in the sense of Bochner and $\mathscr{F}$ a Borel field of measurable Ω sets. Then there exists a func-*

tion $\mathscr{E}^s\{x \mid \mathscr{F}\}(\cdot): \Omega \to \mathfrak{X}$ *which is Bochner integrable, strongly measurable relative to* $\mathscr{F}$, *unique* a.e., *and*

$$\int_\Lambda x(\omega)dP = \int_\Lambda \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)dP \textit{ for all } \Lambda \in \mathscr{F} .$$

Proof. Let x be strongly measurable and integrable in the sense of Bochner. Then there exists a sequence x_n of finitely valued measurable functions such that $x_n(\omega) \to x(\omega)$ with probability 1 as $n \to \infty$; $\int_\Omega \| x_n(\omega) - x_m(\omega) \| dP \to 0$ as $n, m \to \infty$; and $\int_\Omega x_n(\omega)dP \to \int_\Omega x(\omega)dP$. Now $\mathscr{E}^s\{x_n \mid \mathscr{F}\}$ is defined for all x_n by Definition 2.1. Also

$$\int_\Omega \| \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega) - \mathscr{E}^s\{x_m \mid \mathscr{F}\}(\omega) \| dP$$

$$= \int_\Omega \| \mathscr{E}^s\{x_n - x_m \mid \mathscr{F}\}(\omega) \| dP \qquad \text{by Lemma 2.3.}$$

$$\leqq \int_\Omega E\{\| x_n - x_m \| \mid \mathscr{F}\}(\omega)dP \qquad \text{by Lemma 2.2.}$$

$$= \int_\Omega \| x^n(\omega) - x_m(\omega) \| dP \qquad \text{by the definition of ordinary conditional expectations}$$

$\to 0$ by the defining property of the x_n's as $n, m \to \infty$.
Then according to Hille and Phillips ([9] p. 82, Theorem 3.7.7), there exists a function, y, which is Bochner integrable, strongly measurable relative to $\mathscr{F}$, unique a.e., and such that

$$\int_\Omega \| \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega) - y(\omega) \| dP \to 0 \qquad \text{as } n \to \infty . \tag{1}$$

Next,

$$\left\| \int_\Lambda y(\omega)dP - \int_\Lambda x(\omega)dP \right\|$$

$$= \left\| \int_\Lambda y(\omega)dP - \int_\Lambda \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega)dP + \int_\Lambda \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega)dP - \int_\Lambda x(\omega)dP \right\|$$

$$\leqq \int_\Lambda \| y(\omega) - \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega) \| dP + \left\| \int_\Lambda x_n(\omega)dP - \int_\Lambda x(\omega)dP \right\| \qquad \text{by Lemma 2.1.}$$

$\to 0$ as $n \to \infty$ by (1) above and by the definition of $\int_\Lambda x(\omega)dP$. Thus $\int_\Lambda y(\omega)dP = \int_\Lambda x(\omega)dP$ for all $\Lambda \in \mathscr{F}$. We are now justified in calling $y(\cdot)$ the strong conditional expectation of x relative to $\mathscr{F}$ and we use the notation $\mathscr{E}^s\{x \mid \mathscr{F}\}(\cdot)$. Q.E.D.

DEFINITION 2.2. $\mathscr{E}^s\{x \mid \mathscr{F}\}$ is called the strong conditional expectation of x relative to $\mathscr{F}$.

We shall now examine the properties of the strong conditional expectation. In what follows we will be concerned mainly with the strong rather than the weak conditional expectation.

THEOREM 2.2.

1. *If* $x(\omega) = \xi$ *on* Ω *then* $\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega) = \xi$ *with probability* 1.
2. $\mathscr{E}^s\left\{\sum_{j=1}^{n} c_j x_j \mid \mathscr{F}\right\} = \sum_{j=1}^{n} c_j \mathscr{E}^s\{x_j \mid \mathscr{F}\}$ *with probability* 1.
3. $\| \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega) \| \leqq E\{\| x \| \mid \mathscr{F}\}$ *with probability* 1.
4. *If* $\| x_n(\omega) - x(\omega) \| \to 0$ *as* $n \to \infty$ *with probability* 1, *and there is a real random variable* $a(\omega) \geqq 0$ *such that* $\| x_n(\omega) \| \leqq a(\omega)$ *with probability* 1 *and* $E\{a\} < \infty$, *then* $\lim_{n\to\infty} \mathscr{E}^s\{x_n \mid \mathscr{F}\} = \mathscr{E}^s\{x \mid \mathscr{F}\}$ *with probability* 1.

Proof.

(1) The function $x(\omega) = \xi$ has the defining property of $\mathscr{E}^s\{x \mid \mathscr{F}\}$ and is measurable relative to any Borel field $\mathscr{F}$.

(2) $$\int_A \mathscr{E}^s\left\{\sum_{j=1}^{n} c_j x_j \mid \mathscr{F}\right\}(\omega) dP = \int_A \left(\sum_{j=1}^{n} c_j x_j(\omega)\right) dP \qquad \text{by Theorem 2.1.}$$
$$= \int_A \left(\sum_{j=1}^{n} c_j \mathscr{E}^s\{x_j \mid \mathscr{F}\}(\omega)\right) dP \text{ for all } A \in \mathscr{F}.$$

Thus

$$\mathscr{E}^s\left\{\sum_{j=1}^{n} c_j x_j \mid \mathscr{F}\right\} = \sum_{j=1}^{n} c_j \mathscr{E}^s\{x_j \mid \mathscr{F}\} \text{ with probability 1.}$$

(3) Let x_n be as in the proof of Theorem 2.1. and let $A \in \mathscr{F}$.

Now $\| \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega) \| \leqq E\{\| x_n \| \mid \mathscr{F}\}(\omega)$ with probability 1 by Lemma 2.2. Thus

$$\int_A \| \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega) \| \, dP \leqq \int_A E\{\| x_n \| \mid \mathscr{F}\}(\omega) dP .$$

But

$$\int_A \| \mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega) \| \, dP \to \int_A \| \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega) \| \, dP \qquad \text{as } n \to \infty$$

by Theorem 2.1., and

$$\int_A E\{\| x_n \| \mid \mathscr{F}\}(\omega) dP = \int_A \| x_n(\omega) \| \, dP \to \int_A \| x(\omega) \| \, dP$$
$$= \int_A E\{\| x \| \mid \mathscr{F}\}(\omega) dP .$$

Hence

$$\int_\Lambda \|\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)\| dP \leqq \int_\Lambda E\{\|x\| \mid \mathscr{F}\}(\omega)dP \text{ for } \Lambda \in \mathscr{F} ,$$

and thus $\|\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)\| \leqq E\{\|x\| \mid \mathscr{F}\}(\omega)$ with probability 1.

(4) $\|\mathscr{E}^s\{x_n \mid \mathscr{F}\}(\omega) - \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)\|$
$= \|\mathscr{E}^s\{x_n - x \mid \mathscr{F}\}(\omega)\|$ by (2) with probabilility 1.
$\leqq E\{\|x_n - x\| \mid \mathscr{F}\}(\omega)$ by (3) with probability 1.
$\to 0$ as $n \to \infty$ by Doob ([1] p. 23). Q.E.D.

Next it will be convenient to show that every linear transformation distributes over $\mathscr{E}^s$.

THEOREM 2.3. *Let x be Bochner integrable, $\mathscr{F}$ a Borel field of measurable sets, f a linear (bounded) transformation from $\mathfrak{X}$ to another Banach space $\mathfrak{Y}$. Then*

$$f[\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)] = \mathscr{E}^s\{f(x) \mid \mathscr{F}\}(\omega) \textit{ with probability } 1.$$

Proof. Since f is a linear (bounded) transformation, $f(x)$ and $f[\mathscr{E}^s\{x \mid \mathscr{F}\}]$ are Bochner integrable (Hille-Phillips [9] p. 84). Let $\Lambda \in \mathscr{F}$. Then

$$(B)\int_\Lambda f[\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)]dP = f[(B)\int_\Lambda \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)dP]$$

(Hille-Phillips [9] Theorem 3.7.12, p. 83)

$$= f[(B)\int_\Lambda x(\omega)dP]$$

$$= (B)\int_\Lambda f(x(\omega))dP \qquad \text{by the preceding reference}$$

$$= (B)\int_\Lambda \mathscr{E}^s\{f(x) \mid \mathscr{F}\}(\omega)dP .$$

Thus $f[\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)] = \mathscr{E}^s\{f(x) \mid \mathscr{F}\}(\omega)$ with probability 1 by Theorem 2.1. of Chapter I. Q.E.D.

COROLLARY. *Let x be Bochner integrable, $\mathscr{F}$ a Borel field of measurable Ω sets, $f \in \mathfrak{X}^*$, then*

$$f[\mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)] = E\{f(x) \mid \mathscr{F}\}(\omega) \textit{ with probability } 1.$$

A final remark. If $\mathscr{F} \subseteqq \mathscr{S}$, *then*

$$\mathscr{E}^s\{\mathscr{E}^s\{x \mid \mathscr{F}\} \mid \mathscr{S}\} = \mathscr{E}^s\{\mathscr{E}^s\{x \mid \mathscr{S}\} \mid \mathscr{F}\} = \mathscr{E}^s\{x \mid \mathscr{F}\}$$

with probability 1. *For*

$$\int_A \mathscr{E}^s\{\mathscr{E}^s\{x \mid \mathscr{F}\} \mid \mathscr{S}\}(\omega)dP = \int_A \mathscr{E}^s\{x \mid \mathscr{F}\}(\omega)dP \text{ for } \Lambda \in \mathscr{S}$$

$$\begin{aligned}\int_A \mathscr{E}^s\{\mathscr{E}^s\{x \mid \mathscr{S}\} \mid \mathscr{F}\}(\omega)dP &= \int_A \mathscr{E}^s\{x \mid \mathscr{S}\}(\omega)dP \text{ for } \Lambda \in \mathscr{F} \\ &= \int_A x(\omega)dP \text{ for } \Lambda \in \mathscr{S};\ \therefore \text{ also for } \Lambda \in \mathscr{F} \\ &= \int_A E\{x \mid \mathscr{F}\}(\omega)dP \text{ for } \Lambda \in \mathscr{F}. \qquad \text{Q.E.D.}\end{aligned}$$

CHAPTER III.

ABSTRACT MARTINGALES

1. Preliminary definitions.

DEFINITION 1.1. Let T be a linear index set. Let $x_\tau(\cdot)\colon \Omega \to \mathfrak{X}$ be integrable in the sense of Bochner for $\tau \in T$ and $\mathscr{F}_\tau$ be a Borel field of measurable subsets of Ω for $\tau \in T$. Let $\mathscr{F}_\sigma \subset \mathscr{F}_\tau$ if $\sigma < \tau$. Suppose x_τ is strongly measurable relative to $\mathscr{F}_\tau$ or equal almost everywhere to such a function. If $\mathscr{E}^s\{x_\tau \mid \mathscr{F}_\sigma\} = x_\sigma$ with probability 1 when $\sigma < \tau$ then $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is a *strong* $\mathfrak{X}$*-martingale*.

In most of our work we will be concerned with the case in which T is the set of positive integers, and in this case the martingale will be denoted by $\{x_n, \mathscr{F}_n, n \geqq 1\}$ and the martingale equality becomes $\mathscr{E}^s\{x_n \mid \mathscr{F}_m\} = x_m$ with probability 1 for $n > m$.

By using the Dunford-Pettis Theorem alluded to in Chapter II, it is possible to get a definition of weak $\mathfrak{X}$-martingales, but because of a separability assumption in the theorem, they turn out to be strong X-martingales.

2. General properties of strong $\mathfrak{X}$-martingales.

From this point we will denote $(B)\int_A x(\omega)dP$ by $\int_A x(\omega)dP$, $(B)\int_\Omega x(\omega)dP$ by $\mathscr{E}\{x\}$, and $\mathscr{E}^s\{x \mid \mathscr{F}\}$ by $\mathscr{E}\{x \mid \mathscr{F}\}$, and omit the word strong when discussing strong martingales.

THEOREM 2.1. *$\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale if and only if $\int_A x_\tau(\omega)dP = \int_A x_\sigma(\omega)dP$ for $\sigma < \tau$ and A in $\mathscr{F}_\sigma$.*

Proof. If $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale, then $\mathscr{E}\{x_\tau \mid \mathscr{F}_\sigma\} = x_\sigma$ with probability 1. Thus for every A in $\mathscr{F}_\sigma$ we have the equality

$$\int_A x_\sigma(\omega)dP = \int_A \mathscr{E}\{x_\tau \mid \mathscr{F}_\sigma\}(\omega)dP = \int_A x_\tau(\omega)dP ,$$

the last equality following from the definition of conditional expectations. Conversely, if $\int_A x_\tau(\omega)dP = \int_A x_\sigma(\omega)dP$, for A in $\mathscr{F}_\sigma, \sigma < \tau$, then $\int_A \mathscr{E}\{x_\tau \mid \mathscr{F}_\sigma\}(\omega)dP = \int_A x_\sigma(\omega)dP$. Therefore, $\mathscr{E}\{x_\tau \mid \mathscr{F}_\sigma\} = x_\sigma$ with probability 1 by Theorem 2.1 of Chapter I, and hence the process in question is an $\mathfrak{X}$-martingale.

THEOREM 2.2. *If $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale, and f is a linear* (continuous) *transformation from $\mathfrak{X}$ to another Banach space $\mathfrak{Y}$, then $\{f(x_\tau), \mathscr{F}_\tau, \tau \in T\}$ is a $\mathfrak{Y}$-martingale. Thus, in particular, the conclusion is true for every f in $\mathfrak{X}^*$. On the other hand, if $\{f(x_\tau), \mathscr{F}_\tau, \tau \in T\}$ is a real martingale for every f in $\mathfrak{X}^*$, and the x_τ are Bochner integrable, then $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale.*

Proof.

(1) x_τ is strongly measurable relative to $\mathscr{F}_\tau$; thus $f(x_\tau)$ is also strongly measurable relative to $\mathscr{F}_\tau$ by the continuity of f. Next, $\mathscr{E}\{f(x_\tau) \mid \mathscr{F}_\sigma\}(\omega) = f[\mathscr{E}\{x_\tau \mid \mathscr{F}_\sigma\}(\omega)]$ with probability 1 by Theorem 2.3 of Chapter II, where both sides of the equality are in $\mathfrak{Y}$. The expression on the right is equal to $f(x_\sigma(\omega))$ with probability 1 by the definition of $\mathfrak{X}$-martingale. Hence, $\mathscr{E}\{f(x_\tau) \mid \mathscr{F}_\sigma\}(\omega) = f(x_\sigma(\omega))$ with probability 1; thus, $\{f(x_\tau), \mathscr{F}_\tau, \tau \in T\}$ is a $\mathfrak{Y}$-martingale. In particular, this is true for all real linear functionals f, and in this case, the resulting martingale is a real one.

(2) On the other hand, if x_τ is Bochner integrable and strongly measurable relative to $\mathscr{F}_\tau$, then by hypothesis $\mathscr{E}\{f(x_\tau) \mid \mathscr{F}_\sigma\} = f(x_\sigma)$ with probability 1 for every f in $\mathfrak{X}^*$. Then we can write

$$f\Big(\int_A x_\tau(\omega)dP\Big) = \int_A f(x_\tau(\omega))dP = \int_A E\{f(x_\tau) \mid \mathscr{F}_\sigma\}(\omega)dP$$
$$= \int_A f(x_\sigma(\omega))dP = f\Big(\int_A x_\sigma(\omega)dP\Big)$$

for every f in $\mathfrak{X}^*$ and A in $\mathscr{F}_\sigma$. Therefore, $\int_A x_\tau(\omega)dP = \int_A x_\sigma(\omega)dP$ for every A in $\mathscr{F}_\sigma$. Hence $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale by Theorem 2.1. Q.E.D.

Note. By virtue of Hille-Phillips ([9] Theorem 3.7.12, p. 83), the theorem is true for f, a closed additive transformation from $\mathfrak{X}$ to $\mathfrak{Y}$, if we assume that $f(x_\tau)$ is Bochner integrable for every τ in T.

DEFINITION 2.1. Let $\mathfrak{Y}$ be a Banach space. A subset $\mathfrak{K}$ of $\mathfrak{Y}$ is called a positive cone if

(1) $\theta \in \mathfrak{R}$,

(2) $\xi \in \mathfrak{R}$ and a nonnegative imply $a\xi \in \mathfrak{R}$,

(3) if $\xi \in \mathfrak{R}$ and $-\xi \in \mathfrak{R}$, then $\xi = \theta$,

(4) if $\xi \in \mathfrak{R}$ and $\eta \in \mathfrak{R}$, then $\xi + \eta \in \mathfrak{R}$,

(5) $\mathfrak{R}$ is closed. By definition $\xi \geqq \eta$ if and only if $\xi - \eta \in \mathfrak{R}$. The order thus induced is a partial order (Hille-Phillips [9] Theorem 1.11.1, p. 15).

DEFINITION 2.2. Let $\mathfrak{Y}$ be a Banach space with a positive cone. Let T and $\mathscr{F}_\tau$ for $\tau \in T$ be as in Definition 1.1 of this chapter. Let x_τ be a Bochner integrable $\mathfrak{Y}$-valued strongly measurable (relative to $\mathscr{F}_\tau$) function on Ω for $\tau \in T$. Then $\{y_\tau, \mathscr{F}_\tau, \tau \in T\}$ is a $\mathfrak{Y}$-semi-martingale if $\mathscr{E}\{y_\tau \mid \mathscr{F}_\sigma\}(\omega) \geqq y_\sigma(\omega)$ with probability 1 for $\sigma < \tau$.

DEFINITION 2.3. A function g defined on $\mathfrak{X}$ with values in $\mathfrak{Y}$, a Banach space equipped with a positive cone, is said to be sub-additive if $g(\xi + \eta) \leqq g(\xi) + g(\eta)$, positive-homogeneous if $g(a\xi) = ag(\xi)$ for $a \geqq 0$.

THEOREM 2.3. *If x is a Bochner integrable $\mathfrak{X}$-valued function on Ω, $\mathscr{F}$ a Borel field of measurable subsets of Ω, and g a continuous subadditive positive-homogeneous function on $\mathfrak{X}$ to $\mathfrak{Y}$, a Banach space with a positive cone, such that $g(x)$ is Bochner integrable, then $g\left(\int_\Omega x(\omega)dP\right) \leqq \int_\Omega g(x(\omega))dP$ and $g(\mathscr{E}\{x \mid \mathscr{F}\}(\omega)) \leqq \mathscr{E}\{g(x) \mid \mathscr{F}\}(\omega)$ with probability 1. In particular, the conclusion follows for real valued g without the assumption of integrability on $g(x)$.*

Proof. If x and $g(x)$ are Bochner integrable, then by the methods of Hille-Phillips ([9] Corollary, p. 81, and Theorem 3.7.17, p. 83) there exists a sequence of countably valued integrable random variables x_n such that $\| x_n(\omega) - x(\omega) \| \to 0$, $\| g(x_n(\omega)) - g(x(\omega)) \| \to 0$ uniformly with probability 1 as $n \to \infty$, and also $\int_A \| x_n(\omega) - x(\omega) \| \, dP \to 0$ and $\int_A \| g(x_n(\omega)) - g(x(\omega)) \| \, dP \to 0$ as $n \to \infty$ for every measurable set A. Thus $\int_A x_n(\omega)dP \to \int_A x(\omega)dP$ and $\int_A g(x_n(\omega))dP \to \int_A g(x(\omega))dP$ as $n \to \infty$. Furthermore, $\mathscr{E}\{x_n \mid \mathscr{F}\} \to \mathscr{E}\{x \mid \mathscr{F}\}$, $\mathscr{E}\{g(x_n) \mid \mathscr{F}\} \to \mathscr{E}\{g(x) \mid \mathscr{F}\}$ uniformly with probability 1 as $n \to \infty$, and $\int_A \| \mathscr{E}\{x_n \mid \mathscr{F}\} - \mathscr{E}\{x \mid \mathscr{F}\} \| \, dP \to 0$, (Moy [14] p. 7) $\int_A \| \mathscr{E}\{g(x_n) \mid \mathscr{F}\} - \mathscr{E}\{g(x) \mid \mathscr{F}\} \| \, dP \to 0$ as $n \to \infty$ for every measurable set A. Let $x_n(\omega) = \xi_n^j$ for ω in A_n^j, where the A_n^j are disjunct measurable sets such that

$$\sum_{j=1}^{\infty} P\{A_n^j\} = 1 \, .$$

Then

$$\int_\Omega x_n(\omega)dP = \sum_{j=1}^{\infty} \xi_n^j P\{A_n^j\} = \lim_{N\to\infty} \sum_{j=1}^{N} \xi_n^j P\{A_n^j\}.$$

Now

$$g\Big(\sum_{j=1}^{N} \xi_n^j P\{A_n^j\}\Big) \leqq \sum_{j=1}^{N} g(\xi_n^j) P\{A_n^j\}$$

by the subadditivity and positive-homogeneity of g. Further,

$$\int_\Omega g(x_n(\omega))dP = \sum_{j=1}^{\infty} g(\xi_n^j) P\{A_n^j\} = \lim_{N\to\infty} \sum_{j=1}^{N} g(\xi_n^j) P\{A_n^j\} .$$

Hence,

$$\begin{aligned} g\Big(\int_\Omega x_n(\omega)dP\Big) &= g\Big(\lim_{N\to\infty} \sum_{j=1}^{N} \xi_n^j P\{A_n^j\}\Big) = \lim_{N\to\infty} g\Big(\sum_{j=1}^{N} \xi_n^j P\{A_n^j\}\Big) \\ &\leqq \lim_{N\to\infty} \sum_{j=1}^{N} g(\xi_n^j) P\{A_n^j\} = \int_\Omega g(x_n(\omega))dP , \end{aligned}$$

since g is continuous and the positive cone in $\mathfrak{Y}$ is closed. Similarly, $\mathscr{E}\{x_n \mid \mathscr{F}\} = \sum_{j=1}^{\infty} \xi_n^j E\{\chi_{A_n^j} \mid \mathscr{F}\}$ almost everywhere and thus,

$$\begin{aligned} g(\mathscr{E}\{x_n \mid \mathscr{F}\}(\omega)) &= g\Big(\lim_{N\to\infty} \sum_{j=1}^{N} \xi_n^j E\{\chi_{A_n^j} \mid \mathscr{F}\}\Big) \\ &\leqq \lim_{N\to\infty} \sum_{j=1}^{N} g(\xi_n^j) E\{\chi_{A_n^j} \mid \mathscr{F}\} = \mathscr{E}\{g(x_n) \mid \mathscr{F}\}(\omega) \text{ a.e.} \end{aligned}$$

Finally, $g\Big(\int_\Omega x_n(\omega)dP\Big) \to g\Big(\int_\Omega x(\omega)dP\Big)$ and $g(\mathscr{E}\{x_n \mid \mathscr{F}\}) \to g(\mathscr{E}\{x \mid \mathscr{F}\})$ a.e. by the continuity of g and the known convergence of the integrals and conditional expectations in question. Thus,

$$\begin{aligned} g\Big(\int_\Omega x(\omega)dP\Big) &= g\Big(\lim_{n\to\infty} \int_\Omega x_n(\omega)dP\Big) = \lim_{n\to\infty} g\Big(\int_\Omega x_n(\omega)dP\Big) \\ &\leqq \lim_{n\to\infty} \int_\Omega g(x_n(\omega))dP = \int_\Omega g(x(\omega)dP \end{aligned}$$

and

$$\begin{aligned} g(\mathscr{E}\{x \mid \mathscr{F}\}) &= g(\lim_{n\to\infty} \mathscr{E}\{x_n \mid \mathscr{F}\}) \quad \text{a.e.} \\ &= \lim_{n\to\infty} g(\mathscr{E}\{x_n \mid \mathscr{F}\}) \text{ a.e.} \leqq \lim_{n\to\infty} \mathscr{E}\{g(x_n) \mid \mathscr{F}\} \text{ a.e.} \\ &= \mathscr{E}\{g(x) \mid \mathscr{F}\} \text{ a.e.} \end{aligned}$$

If, in particular, g is a real valued subadditive positive-homogeneous continuous function, then there exists a finite nonnegative number M_g, $M_g = \sup\,[g(\xi);\, \|\xi\| \leqq 1]$, such that $|g(\xi)| \leqq M_g(\|\xi\| + 1)$ (Hille-

Phillips [9] Theorem 2.5.2, p. 25). Thus, $|g(x(\omega))| \leqq M_g(\|x(\omega)\| + 1)$, and, since the function on the right is integrable on Ω, it being a finite measure space, $g(x)$ is Lebesgue integrable, and the conclusion of the theorem follows. Q.E.D.

THEOREM 2.4. *Let $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ be an $\mathfrak{X}$-martingale, and let g be a continuous subadditive positive-homogeneous function on $\mathfrak{X}$ to $\mathfrak{Y}$, a Banach space with a positive cone such that $g(x_\tau)$ is Bochner integrable for every τ in T. Then $\{g(x_\tau), \mathscr{F}_\tau, \tau \in T\}$ is a $\mathfrak{Y}$-semi-martingale. In particular, if g is a continuous subadditive positive-homogeneous functional the conclusion is that the resulting process is a real semi-martingale without assuming that $g(x_\tau)$ is integrable. Finally $\{\|x_\tau\|, \mathscr{F}_\tau, \tau \in T\}$ is a real semi-martingale.*

Proof. By Theorem 2.3, $\mathscr{E}\{g(x_\tau) \mid \mathscr{F}_\sigma\}(\omega) \geqq g(\mathscr{E}\{x_\tau \mid \mathscr{F}_\sigma\}(\omega))$ a.e. But the righthand side is equal almost everywhere to $g(x_\sigma(\omega))$ since $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale. Thus, $\mathscr{E}\{g(x_\tau) \mid \mathscr{F}_\sigma\}(\omega) \geqq g(x_\sigma(\omega))$ a.e. for $\sigma < \tau$. Since $g(x_\tau)$ is clearly strongly measurable relative to $\mathscr{F}_\tau$, $\{g(x_\tau), \mathscr{F}_\tau, \tau \in T\}$ is a $\mathfrak{Y}$-semi-martingale. Q.E.D.

Next we consider some examples.

EXAMPLE 2.1. Let z be Bochner integrable and $\{\mathscr{F}_\tau\}$ as before. Let $x_\tau = \mathscr{E}\{z \mid \mathscr{F}_\tau\}$. Then $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale. For let $\Lambda \in \mathscr{F}_\sigma, \sigma < \tau$,

$$\int_\Lambda x_\sigma(\omega)dP = \int_\Lambda \mathscr{E}\{z \mid \mathscr{F}_\sigma\}(\omega)dP = \int_\Lambda z(\omega)dP$$

as a consequence of the definition of $\mathscr{E}\{z \mid \mathscr{F}_\sigma\}$, and

$$\int_\Lambda x_\tau(\omega)dP = \int_\Lambda \mathscr{E}\{z \mid \mathscr{F}_\tau\}(\omega)dP = \int_\Lambda z(\omega)dP\,,$$

the last equality being true for all $\Lambda \in \mathscr{F}_\tau$ and hence for all $\Lambda \in \mathscr{F}_\sigma \subseteqq \mathscr{F}_\tau$. Thus $\int_\Lambda x_\sigma(\omega)dP = \int_\Lambda x_\tau(\omega)dP$ for $\Lambda \in \mathscr{F}_\sigma$. Hence, by Theorem 2.1, $\{x_\tau, \mathscr{F}_\tau, \tau \in T\}$ is an $\mathfrak{X}$-martingale.

Before proceeding to the next example we shall have to prove the following lemma.

LEMMA 2.1. *Let x and y be strongly measurable independent random variables. Let $\mathscr{F}$ be the Borel field of measurable sets generated by x;* i.e., *the smallest Borel field of measurable sets with respect to which x is strongly measurable. Suppose $\mathscr{E}\{y \mid \mathscr{F}\}$ exists, and define $\mathscr{E}\{y \mid x\} = \mathscr{E}\{y \mid \mathscr{F}\}$. Then $\mathscr{E}\{y \mid x\} = \mathscr{E}\{y\}$ with probability* 1.

Proof. If x and y are independent, then $f(x)$ and $f(y)$ are real valued independent random variables by Theorem 1.1 of Chapter I for every f in $\mathfrak{X}^*$. Thus $E\{f(y) \mid \mathscr{F}\} = E\{f(y)\}$ with probability 1. Next, let A be an $\mathscr{F}$ set. Then

$$f\Big(\int_A \mathscr{E}\{y \mid x\}(\omega)dP\Big) = f\Big(\int_A \mathscr{E}\{y \mid \mathscr{F}\}(\omega)dP\Big) = \int_A f(\mathscr{E}\{y \mid \mathscr{F}\}(\omega))dP$$
$$= \int_A E\{f(y) \mid \mathscr{F}\}(\omega)dP = \int_A E\{f(y)\}dP = f\Big(\int_A \mathscr{E}\{y\}dP\Big)$$

by Theorem 3.3 of Chapter II. Thus

$$\int_A \mathscr{E}\{y \mid x\}(\omega)dP = \int_A \mathscr{E}\{y\}dP\,,$$

for every A in $\mathscr{F}$. Hence $\mathscr{E}\{y \mid x\} = \mathscr{E}\{y\}$ with probability 1 by Theorem 2.1 of Chapter I. Q.E.D.

In like manner, it can be shown that if $\{y_n\}$ are mutually independent, then $\mathscr{E}\{y_n \mid \mathscr{F}\} = \mathscr{E}\{y_n\}$ with probability 1 if $\mathscr{F}$ is the smallest Borel field relative to which $y_1, \cdots, y_{n-1}$ are strongly measurable.

EXAMPLE 2.2. Let $\{y_j, j \geqq 1\}$ be mutually independent, $\mathscr{E}\{y_j\} = \theta$ for $j > 1$, $\mathscr{F}_j$ be the smallest Borel field relative to which $y_1, \cdots, y_j$ are all strongly measurable, and $x_n = \sum_{j=1}^n y_j$. Then $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale.

We show that $\mathscr{E}\{x_n \mid \mathscr{F}_{n-1}\} = x_{n-1}$ with probability 1.

Note. $\mathscr{E}\{x_n \mid \mathscr{F}_{n-1}\} = \mathscr{E}\{x_n \mid y_1, \cdots, y_{n-1}\} = \mathscr{E}\{x_n \mid x_1, \cdots, x_{n-1}\}$.

Clearly

$$x_n = \sum_{j=1}^{n} y_j = \sum_{j=1}^{n-1} y_j + y_n = x_{n-1} + y_n\,.$$

Then

$$\begin{aligned}
\mathscr{E}\{x_n \mid \mathscr{F}_{n-1}\} &= \mathscr{E}\{x_{n-1} + y_n \mid \mathscr{F}_{n-1}\} \\
&= \mathscr{E}\{x_{n-1} \mid \mathscr{F}_{n-1}\} + \mathscr{E}\{y_n \mid \mathscr{F}_{n-1}\} \quad \text{by Theorem 2.2 of Chapter II.} \\
&= x_{n-1} + \mathscr{E}\{y_n \mid \mathscr{F}_{n-1}\} \quad \text{with probability 1 for } x_{n-1} \text{ is measurable relative to } \mathscr{F}_{n-1}. \\
&= x_{n-1} + \mathscr{E}\{y_n\} \quad \text{with probability 1 by Lemma 2.1} \\
&= x_{n-1} \text{ for } \mathscr{E}\{y_n\} = \theta \text{ for } n > 1\,.
\end{aligned}$$

Thus $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale.

CHAPTER IV

MARTINGALE CONVERGENCE THEOREMS IN A BANACH SPACE

Let $\mathfrak{X}$ be a Banach space. We will prove various convergence theorems for $\mathfrak{X}$-martingales. Thus we will show that if $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale, then under certain conditions there will exist an $\mathfrak{X}$-valued random variable x such that $x_n \rightarrow x$ with probability 1 in various senses.

THEOREM 1. *Let $\{x_n, \mathscr{F}_n, n \geqq 1\}$ be an $\mathfrak{X}$-martingale, and let $\mathscr{F}_\infty$ be the smallest Borel field of Ω sets such that $\mathscr{F}_\infty \supseteqq \bigcup_{n=1}^{\infty} \mathscr{F}_n$. Let $y_n(\omega) = \| x_n(\omega) \|$. Then*

$$E\{\| x_1 \|\} \leqq E\{\| x_2 \|\} \leqq \cdots \leqq E\{\| x_n \|\} \leqq \cdots$$

(1) If $\text{l.u.b.}_n E\{\| x_n \|\} < \infty$ then $\lim_{n\to\infty} \| x_n \| = y_\infty$ exists with probability 1, and $E\{y_\infty\} < \infty$. In fact, the boundedness condition reduces to $\lim_{n\to\infty} E\{\| x_n \|\} = K < \infty$, and then $E\{y_\infty\} \leqq K$.

(2) a. If the $\| x_n \|$'s are uniformly integrable then

$$\operatorname*{l.u.b.}_n E\{\| x_n \|\} < \infty, \lim_{n\to\infty} E\{| y_\infty - \| x_n \| |\} = 0 ,$$

and the process $\{y_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is a real semi-martingale dominated by a semi-martingale relative to the same fields. (Doob [1] p. 297)

b. If $\text{l.u.b.}_n E\{\| x_n \|\} < \infty$ so that y_∞ exists, and if the process $\{y_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is a real semi-martingale, then $\lim_{n\to\infty} E\{\| x_n \|\} = E\{y_\infty\}$ and the $\| x_n \|$'s are uniformly integrable.

Proof. If $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale, then $\{\| x_n \|, \mathscr{F}_n, n \geqq 1\}$ is a real semi-martingale by Theorem 2.4 of Chapter III, and then $E\{\| x_1 \|\} \leqq \cdots E\{\| x_n \|\} \leqq \cdots$ according to Doob ([1] Theorem 2.1 (ii) p. 311). The other conclusions follow from Theorem 4.1 s of Doob ([1] p. 324–325). Q.E.D.

THEOREM 2. *Let $\{x_n, \mathscr{F}_n, n \geqq 1\}$ be an $\mathfrak{X}$-martingale. Let $\mathfrak{X}$ be reflexive. Suppose $\lim_{n\to\infty} E\{\| x_n \|\} = K < \infty$. Then there is an $\mathfrak{X}$-valued strongly measurable random variable x_∞ such that $x_n \rightarrow x_\infty$ weakly as $n \rightarrow \infty$ with probability* 1.

Proof. Since x_n is strongly measurable, there is a measurable set A_n such that $P\{A_n\} = 0$ and $x_n(\Omega - A_n)$ is separable, for strongly measurable functions are almost separably valued (Hille-Phillips [9] Theorem 3.5.3, p. 72). Let $\mathfrak{Y}_n = x_n(\Omega - A_n)$ and let $\mathfrak{Y}$ be the closed linear manifold spanned by $\bigcup_{n=1}^{\infty} \mathfrak{Y}_n$. Then $\mathfrak{Y}$ is a separable subspace of $\mathfrak{X}$ and $x_n(\omega) \in \mathfrak{Y}$ for almost all ω, for each n. Now $\mathfrak{Y}$ is reflexive since $\mathfrak{X}$ is. (Hille-Phillips [9] Corollary 1 to Theorem 2.10.3, p. 38). Further, since $\mathfrak{Y}$ is separable, then so is $\mathfrak{Y}^{**}$ for $\mathfrak{Y} \cong \mathfrak{Y}^{**}$. But then $\mathfrak{Y}^*$ is separable by Theorem 2.8.4 of Hille-Phillips ([9] p. 34). Now if $f \in \mathfrak{Y}^*$ then $\{f(x_n), \mathscr{F}_n, n \geqq 1\}$ is a real martingale by Theorem 2.2 of Chapter III. Also

$$E\{|f(x_n)|\} \leqq E\{||f|| \, ||x_n||\} = ||f|| \, E\{||x_n||\} \leqq ||f|| \, K$$

because $E\{||x_1||\} \leqq \cdots E\{||x_n||\} \leqq \cdots \leqq K$ by Theorem 1. By virtue of Doob ([1] Theorem 4.1, p. 319) for every $f \in \mathfrak{Y}^*$ there exists a real measurable function z_f, and a measurable set Λ_f such that $P\{\Lambda_f\} = 0$ and $|f(x_n(\omega)) - z_f(\omega)| \to 0$ as $n \to \infty$ for $\omega \in \Omega - \Lambda_f$. By the separability of $\mathfrak{Y}^*$ there is a countable dense subset $\{f_j\}$ of $\mathfrak{Y}^*$. Thus for every f_j there is a Λ_{f_j} and z_{f_j} as we have seen, such that $P\{\Lambda_{f_j}\} = 0$ and $|f_j(x_n(\omega)) - z_{f_j}(\omega)| \to 0$ as $n \to \infty$ for $\omega \in \Omega - \Lambda_{f_j}$. Let $\Lambda_1 = \bigcup_{j=1}^{\infty} \Lambda_{f_j}$. Then

$$P\{\Lambda_1\} = P\left\{\bigcup_{j=1}^{\infty} \Lambda_{f_j}\right\} \leqq \sum_{j=1}^{\infty} P\{\Lambda_{f_j}\} = 0 .$$

By Theorem 1 there is a measurable set M such that $P\{M\} = 0$ and such that $||x_n(\omega)||$ is a convergent sequence for $\omega \in \Omega - M$. Let $\Lambda = \Lambda_1 \cup M$. Then $P\{\Lambda\} = 0$. Next, let $\omega \in \Omega - \Lambda$. Then $\omega \in \Omega - M$ so that $||x_n(\omega)||$ is a convergent sequence. Thus there is a constant C such that $||x_n(\omega)|| \leqq C$ for all n.

Define $Q_n(f) = f(x_n(\omega))$ for $f \in \mathfrak{Y}^*$. The Q_n's form an equi-continuous sequence of functions on $\mathfrak{Y}^*$, for, given $\epsilon > 0$, $\exists \delta = \varepsilon/C$ such that for every n, $||f - g|| < \delta$ implies

$$|Q_n(f) - Q_n(g)| = |f(x_n(\omega)) - g(x_n(\omega))| \leqq ||f - g|| \, ||x_n(\omega)|| < \varepsilon/C \cdot C = \varepsilon .$$

Furthermore, since $\omega \in \Omega - \Lambda_{f_j}$ for every j,

$$|Q_n(f_j) - Q_m(f_j)| = |f_j(x_n(\omega)) - f_j(x_m(\omega))| \to 0 \text{ as } n, m \to \infty .$$

But, an equicontinuous sequence of functions converging on a dense set of a metric space converges on the whole space. Thus for every $f \in \mathfrak{Y}^*$, $|Q_n(f) - Q_m(f)| \to 0$ as $n, m \to \infty$; i.e., $|f(x_n(\omega)) - f(x_m(\omega))| \to 0$ as $n, m \to \infty$ for every $\omega \in \Omega - \Lambda$.

Therefore $f(x_n(\omega))$ is a convergent sequence for all $\omega \in \Omega - \Lambda$ and $f \in \mathfrak{Y}^*$. The reflexiveness of $\mathfrak{X}$ and $\mathfrak{Y}$ implies that $\mathfrak{X}$ and $\mathfrak{Y}$ are weakly complete. Thus there is an x_∞ (strongly measurable) such that for every $f \in \mathfrak{Y}^*$ and $\omega \in \Omega - \Lambda$ we have $|f(x_n(\omega)) - f(x_\infty(\omega))| \to 0$ as $n \to \infty$; i.e. x_n converges to x_∞ weakly with probability 1. Q.E.D.

Note. x_∞ is strongly measurable since it is the weak limit of strongly measurable functions (Hille-Phillips [9] Theorem 3.5.4, p. 74). Theorem 2 may be restated as follows:

THEOREM 2*. *Let* $\{x_n, \mathscr{F}_n, n \geqq 1\}$ *be an* $\mathfrak{X}$*-martingale. Let* $\mathfrak{X}$ *be weakly complete and suppose that* $\mathfrak{X}^*$ *is separable, and* $\lim_{n\to\infty} E\{\|x_n\|\} = K < \infty$. *Then there is an* $\mathfrak{X}$*-valued strongly measurable random variable* x_∞ *such that* x_n *converges to* x_∞ *weakly with probability* 1.

COROLLARY 1. *Let* $\{x_n, \mathscr{F}_n, n \geqq 1\}$ *be an* $\mathfrak{X}$*-martingale. Suppose* $\mathfrak{X}$ *is a Hilbert space, and that* $\lim_{n\to\infty} E\{\|x_n\|\} = K < \infty$. *Then there exists a strongly measurable* $\mathfrak{X}$*-valued random variable* x_∞ *such that* $x_n \to x_\infty$ *weakly with probability* 1.

Proof. Since $\mathfrak{X}$ is a Hilbert space, it is reflexive and weakly complete. Hence all of the hypotheses of Theorem 2 are satisfied, and so the above conclusion follows. Q.E.D.

By making a stronger assumption on the $\|x_n\|$'s we will show that the last result may be sharpened to give strong convergence with probability 1.

THEOREM 3. *Let* $\{x_n, \mathscr{F}_n, n \geqq 1\}$ *be an* $\mathfrak{X}$*-martingale; let* $\mathfrak{X}$ *be reflexive. If the* $\|x_n\|$*'s are uniformly integrable, then there is a strongly measurable* $\mathfrak{X}$*-valued random variable* x_∞ *such that* $\|x_n(\omega) - x_\infty(\omega)\| \to 0$ *as* $n \to \infty$ *with probability* 1, *and in fact* $\{x_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ *is an* $\mathfrak{X}$*-martingale.*

Proof. As in the proof of Theorem 2, there is a separable sub-space $\mathfrak{Y}$ of $\mathfrak{X}$, and for each n, $x_n(\omega) \in \mathfrak{Y}$ for almost all ω. Also $\mathfrak{Y}$ is reflexive, so therefore $\mathfrak{Y}^{**}$ is separable, which implies that $\mathfrak{Y}^*$ is separable. Now $E\{\|x_1\|\} \leqq E\{\|x_2\|\} \leqq \cdots \leqq E\{\|x_n\|\} \leqq \cdots$ since $\{\|x_n\|, \mathscr{F}_n, n \geqq 1\}$ is a semi-martingale. Therefore $\lim_{n\to\infty} E\{\|x_n\|\} = K \leqq \infty$, while $\lim_{n\to\infty} E\{|f(x_n)|\} \leqq \lim_{n\to\infty} \|f\| E\{\|x_n\|\} = \|f\| K$. But the uniform integrability of the $\|x_n\|$'s makes $K < \infty$ (Doob [1] Theorem 4.1, p. 319). Theorem 1 tells us that there is a y_∞ such that $|\|x_n\| - y_\infty| \to 0$ as $n \to \infty$ with probability 1, and such that $\{y_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is a real semi-martingale, where $y_n(\omega) = \|x_n(\omega)\|$ and $y_\infty(\omega) = \lim_{n\to\infty} \|x_n(\omega)\|$. In fact, $E\{|y_\infty - \|x_n\||\} \to 0$ as $n \to \infty$. By Theorem 2, there is a strongly measurable $\mathfrak{X}$-valued random variable x_∞ such that $|f(x_n(\omega)) - f(x_\infty(\omega))| \to 0$ as $n \to \infty$ with probability 1 for every $f \in \mathfrak{Y}^*$. Furthermore, if the $\|x_n\|$'s are uniformly integrable, then so are the $f(x_n)$'s for every $f \in \mathfrak{Y}^*$ because, first of all,

$$\{\omega\colon |f(x_n(\omega))| > M\} \subseteqq \left\{\omega\colon \|x_n(\omega)\| > \frac{M}{\|f\|}\right\}$$

if $\|f\| < 0$. (If $\|f\| = 0$, then trivially the $f(x_n)$'s are uniformly integrable.) Thus

$$\int_{\{\omega: |f(x_n(\omega))| > M\}} |f(x_n(\omega))| \, dP \leqq \int_{\{\omega: \|x_n(\omega)\| > M/\|f\|\}} |f(x_n(\omega))| \, dP$$
$$\leqq \|f\| \int_{\{\omega: \|x_n(\omega)\| > M/\|f\|\}} \|x_n(\omega)\| \, dP \to 0$$

uniformly in n as $M \to \infty$.

By the uniform integrability of the $\|x_n\|$'s, thus proving the uniform integrability of the $f(x_n)$'s for every $f \in \mathfrak{Y}^*$. Hence $\{f(x_n), \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is a real martingale for every $f \in \mathfrak{Y}^*$ by Doob ([1] Theorem 4.1, p. 319).

Next, x_∞ is strongly measurable (in fact, relative to $\mathscr{F}_\infty$) by Theorem 2. Furthermore, $E\{\|x_\infty\|\} < \infty$, for, $x_n \to x_\infty$ weakly with probability 1. Hence $\|x_\infty(\omega)\| \leqq \lim_{n\to\infty} \inf \|x_n(\omega)\|$ for almost all ω. But the right hand side equals $y_\infty(\omega)$ with probability 1 by Theorem 1. Thus $\|x_\infty(\omega)\| \leqq y_\infty(\omega)$ a.e. Since y_∞ is integrable, so is $\|x_\infty\|$; hence, by Theorem 3.7.4 of Hille-Phillips ([9] p. 80), x_∞ is Bochner integrable. Thus, by Theorem 2.2 of Chapter III, $\{x_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is an $\mathfrak{X}$-martingale. Then $\{\|x_n\|, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is a semi-martingale by Theorem 3.4 of Chapter III. But so is $\{\|x_1\|, \cdots, \|x_n\|, \cdots, y_\infty\}$ relative to $\mathscr{F}_1, \cdots, \mathscr{F}_n, \cdots, \mathscr{F}_\infty$.

We now show that $\|x_\infty\| = y_\infty$ with probability 1. We have already shown $E\{\|x_\infty\|\} \leqq E\{y_\infty\}$. But $E\{\|x_n\|\} \leqq E\{\|x_\infty\|\}$ since $\{\|x_n\|, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is a semi-martingale, and since $E\{\|x_n\|\} \to E\{y_\infty\}$ by Theorem 1, we have $E\{y_\infty\} \leqq E\{\|x_\infty\|\}$. Hence, $E\{\|x_\infty\|\} = E\{y_\infty\}$. But $\|x_\infty(\omega)\| \leqq y_\infty(\omega)$ for almost all ω. Therefore by Theorem B of Halmos ([8] p. 104), $\|x_\infty(\omega)\| = y_\infty(\omega)$ for almost all ω, and $\|x_n(\omega)\| \to \|x_\infty(\omega)\|$ with probability 1, even as $x_n \to x_\infty$ weakly with probability 1. Next, let $\xi \in \mathfrak{Y}$. Then $\{x_n - \xi, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale, for

$$\mathscr{E}\{x_n - \xi \mid \mathscr{F}_m\} = \mathscr{E}\{x_n \mid \mathscr{F}_m\} - \mathscr{E}\{\xi \mid \mathscr{F}_m\}$$ with probability 1 by Theorem 2.2 of Chapter II

$$= x_m - \xi$$ with probability 1, since $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale, and by Theorem 2.2 of Chapter II.

Now by what we have already proved in this theorem, since the $\|x_n - \xi\|$'s are clearly uniformly integrable, there is a u_∞ such that $f(x_n - \xi) \to f(u_\infty)$ with probability 1 for every $f \in \mathfrak{Y}^*$ and $\|x_n(\omega) - \xi\| \to u_\infty(\omega)$ with probability 1. But $f[x_n(\omega) - \xi] = f(x_n(\omega)) - f(\xi) \to f(x_\infty(\omega)) - f(\xi) = f[x_\infty(\omega) - \xi]$ as $n \to \infty$ with probability 1. Thus $u_\infty(\omega) = x_\infty(\omega) - \xi$ with probability 1. Hence $\|x_n(\omega) - \xi\| \to \|x_\infty(\omega) - \xi\|$ with probability 1. Let $\{\xi_j\}$ be a denumerable dense set in $\mathfrak{Y}$. Then there is a Λ_j such that $P\{\Lambda_j\} = 0$ and $\|x_n(\omega) - \xi_j\| \to \|x_\infty(\omega) - \xi_j\|$ for

$\omega \in \Omega - \Lambda_j$. Let $\Lambda = \bigcup_{j=1}^{\infty} \Lambda_j$. Then $P\{\Lambda\} = 0$. Let $\omega \in \Omega - \Lambda$, and define $R_n(\xi) = \| x_n(\omega) - \xi \|$ for $\xi \in \mathfrak{Y}$. The R_n's form an equicontinuous sequence of functions on $\mathfrak{Y}$, for given $\varepsilon > 0$, $\exists \delta = \varepsilon$, such that for every n, $\| \xi - \eta \| < \delta = \varepsilon$ implies $| R_n(\xi) - R_n(\eta) | = | \| x_n(\omega) - \xi \| - \| x_n(\omega) - \eta \| | \leqq \| \xi - \eta \| < \varepsilon$. Furthermore, since $\omega \in \Omega - \Lambda_j$ for every j,

$$| R_n(\xi_j) - \| x_\infty(\omega) - \xi_j \| | = | \| x_n(\omega) - \xi_j \| - \| x_\infty(\omega) - \xi_j \| | \to 0 \text{ as } n \to \infty .$$

But, as an equicontinuous sequence of functions converging on a dense set of a metric space converges on the whole space, thus for every $\xi \in \mathfrak{Y}$, $| R_n(\xi) - \| x_\infty(\omega) - \xi \| | = | \| x_n(\omega) - \xi \| - \| x_\infty(\omega) - \xi \| | \to 0$ as $n \to \infty$ for every $\omega \in \Omega - \Lambda$. Now, for $\omega \notin \Lambda$, let $\xi = x_\infty(\omega)$. Then $\| x_n(\omega) - x_\infty(\omega) \| \to \| x_\infty(\omega) - x_\infty(\omega) \| = 0$. Thus there is a measurable set Λ such that $P\{\Lambda\} = 0$ and such that for $\omega \in \Omega - \Lambda$, $\| x_n(\omega) - x_\infty(\omega) \| \to 0$ as $n \to \infty$. Q.E.D.

COROLLARY 2. *If $\mathfrak{X}$ is a Hilbert space, or l^p, or L^p, $1 < p < \infty$, and $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale in which the $\| x_n \|$'s are uniformly integrable, then there is an x_∞ such that $\{x_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is an $\mathfrak{X}$-martingale, and $\| x_n(\omega) - x_\infty(\omega) \| \to 0$ as $n \to \infty$ with probability 1.*

Proof. All of the above named Banach spaces are reflexive, and thus the result follows from Theorem 3.

REMARK. Let $\mathfrak{X}$ be a Banach space with a partial order induced by a positive cone. Suppose $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-semi-martingale. Then, as in Doob ([1] p. 297), x_n can be represented in the form

$$x_n = x'_n + \sum_{j=1}^{n} \Delta_j ,$$

where $\Delta_1 = \theta$; $\Delta_j = \mathscr{E}\{x_j \mid \mathscr{F}_{j-1}\} - x_{j-1} \geqq \theta, j > 1$; and $\{x'_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale. Thus convergence problems for $\mathfrak{X}$-semi-martingales can be reduced to convergence of $\mathfrak{X}$-martingales if reasonable conditions can be found for the convergence of the monotone sequence $y_n = \sum_{j=1}^{n} \Delta_j$.

THEOREM 4. *Let $\{x_n, \mathscr{F}_n, n \leqq -1\}$ be an $\mathfrak{X}$-martingale in which $\mathfrak{X}$ is reflexive, and let $\mathscr{F}_{-\infty} = \bigcap_{-\infty}^{-1} \mathscr{F}_n$. Then $x_{-\infty}$ exists, such that $\| x_n(\omega) - x_{-\infty}(\omega) \| \to 0$ as $n \to -\infty$ with probability 1, and $\{x_n, \mathscr{F}_n, -\infty \leqq n \leqq -1\}$ is an $\mathfrak{X}$-martingale.*

Proof. $\{\| x_n \|, \mathscr{F}_n, n \leqq -1\}$ is a real semi-martingale; thus by Doob ([1] Theorem 4.25, p. 329) $\lim_{n \to -\infty} \| x_n(\omega) \| = y_{-\infty}$ exists with probability 1, and $-\infty \leqq y_{-\infty} < \infty$, while $\{\| x_n \|, \mathscr{F}_n, -\infty \leqq n \leqq -1\}$ is a semi-martingale. By Theorem 4.2 of Doob ([1] p. 328) $\lim_{n \to -\infty} f(x_n)$ exists for almost all ω and every $f \in \mathfrak{X}^*$. Using the methods of Theorem 2, we can show that there is an $x_{-\infty}$ such that $f(x_n(\omega)) \to f(x_{-\infty}(\omega))$ as $n \to -\infty$ for almost all ω and all f. Using the methods of Theorem 3, we show that $\{x_n, \mathscr{F}_n, -\infty \leqq n \leqq -1\}$ is an $\mathfrak{X}$-martingale, and that $\|x_{-\infty}\| = y_{-\infty}$ and $\|x_n(\omega) - x_{-\infty}(\omega)\| \to 0$ as $n \to -\infty$ with probability 1. Q.E.D.

THEOREM 5. *Let z be a strongly measurable random variable, $\mathfrak{X}$ reflexive, with $E\{\|z\|\} < \infty$; let $\cdots \mathscr{F}_{-n} \subseteqq \cdots \subseteqq \mathscr{F}_0 \subseteqq \cdots \subseteqq \cdots \subseteqq \mathscr{F}_n \subseteqq \cdots$ be Borel fields of measurable Ω sets. Let $\mathscr{F}_{-\infty} = \bigcap_{n=-\infty}^{\infty} \mathscr{F}_n$, be the smallest Borel field of Ω sets with $\mathscr{F}_\infty \supseteqq \bigcup_{n=-\infty}^{\infty} \mathscr{F}_n$. Then $\lim_{n\to-\infty} \mathscr{E}\{z \mid \mathscr{F}_n\} = \mathscr{E}\{z \mid \mathscr{F}_{-\infty}\}$, and $\lim_{n\to\infty} \mathscr{E}\{z \mid \mathscr{F}_n\} = \mathscr{E}\{z \mid \mathscr{F}_\infty\}$ with probability* 1.

Proof. Let $x_n = \mathscr{E}\{z \mid \mathscr{F}_n\}$, $-\infty \leqq n \leqq \infty$. Then $\{x_n, \mathscr{F}_n, -\infty \leqq n \leqq \infty\}$ is an $\mathfrak{X}$-martingale by Example 2.1 of Chapter III. Thus by Theorem 4, $\lim_{n\to-\infty} \mathscr{E}\{z \mid \mathscr{F}_n\} = \mathscr{E}\{z \mid \mathscr{F}_{-\infty}\}$. Next, $\{\|x_n\|, \mathscr{F}_n, -\infty \leqq n \leqq \infty\}$ is a real semi-martingale, with a last term in which all the random variables are nonnegative. Thus by Theorem 3.1 of Doob ([1] p. 311) the $\|x_n\|$'s are uniformly integrable. Hence by Theorem 3, there is a y such that $\|x_n(\omega) - y(\omega)\| \to 0$ as $n \to \infty$ for almost all ω and $\{x_n, 1 \leqq n < \infty, y\}$ is an $\mathfrak{X}$-martingale. We finally must show that $x_\infty(\omega) = y(\omega)$ with probability 1. But this is true for both x_∞ and y are equal almost everywhere to functions measurable relative to $\mathscr{F}_\infty$. Also $\int_A x_\infty(\omega) dP = \int_A \mathscr{E}\{z \mid \mathscr{F}_\infty\}(\omega) dP = \int_A z(\omega) dP$ for $A \in \mathscr{F}_\infty$ and $\int_A y(\omega) dP = \int_A x_n(\omega) dP = \int_A \mathscr{E}\{z \mid \mathscr{F}_n\}(\omega)\, dP = \int_A z(\omega) dP$ for every $A \in \mathscr{F}_n$ and thus for every $A \in \bigcup_n \mathscr{F}_n$. Hence $\int_A y(\omega) dP = \int_A x_\infty(\omega) dP$ for every $A \in \bigcup_n \mathscr{F}_n$; thus, $\int_A f(y(\omega)) dP = \int_A f(x_\infty(\omega)) dP$ for every $A \in \bigcup_n \mathscr{F}_n$ and $f \in \mathfrak{X}^*$. But these integrals define completely additive set functions of $\mathscr{F}_\infty$ sets which are identical on the fields $\bigcup_n \mathscr{F}_n$ and therefore identical on $\mathscr{F}_\infty$ (Doob [1] Theorem 2.1, p. 605). Thus $\int_A y(\omega) dP = \int_A x_\infty dP$ for every $A \in \mathscr{F}_\infty$. Hence $y(\omega) = x_\infty(\omega)$ with probability 1 and $\lim_{n\to\infty} \mathscr{E}\{z \mid \mathscr{F}_n\} = \mathscr{E}\{z \mid \mathscr{F}_\infty\}$ with probability 1. Q.E.D.

COROLLARY 3. *Let z be a strongly measurable random variable, with $E\{\|z\|\} < \infty$ and let $y_1, y_2, \cdots$ be strongly measurable. Let $\mathscr{S}_n$ be the smallest Borel field with respect to which $y_n, y_{n+1} \cdots$ are strongly measurable. Then $\lim_{n\to\infty} \mathscr{E}\{z \mid \mathscr{S}_n\} = \mathscr{E}\{z \mid \bigcap_{n=1}^{\infty} \mathscr{S}_n\}$, $\lim_{n\to\infty} \mathscr{E}\{z \mid \mathscr{H}_n\} = \mathscr{E}\{z \mid \mathscr{H}_\infty\}$ where $\mathscr{H}_n$ is the smallest Borel field relative to which $y_1, y_2, \cdots, y_n$ are strongly measurable, $\mathscr{H}_\infty$ the smallest Borel field containing $\bigcup_{n=1}^{\infty} \mathscr{H}_n$.*

Proof. In Theorem 5, let $\mathscr{S}_n = \mathscr{F}_{-n}$ and $\mathscr{H}_n = \mathscr{F}_n$. Q.E.D.

Using this corollary it is possible to get a proof of the Banach space version of the strong law of large numbers. In fact, such a proof is virtually along the lines outlined in Doob ([1] p. 341). Mourier [13] has proved an ergodic theorem, more general than this one, by a more direct approach.

EXAMPLE 1. Let $\mathfrak{X} = l^p, 1 < p < \infty$ (real l^p). Then

$$x_n(\omega) = (\xi_1^{(n)}(\omega), \cdots, \xi_j^{(n)}(\omega), \cdots) \text{ where } \sum_{j=1}^{\infty} |\xi_j^{(n)}(\omega)|^p < \infty ,$$

and

$$\| x_n(\omega) \| = \left\{\sum_{j=1}^{\infty} |\xi_j^{(n)}(\omega)|^p\right\}^{1/p} .$$

If x_n is Bochner integrable, then its integral satisfies the equation

$$\int_\Omega x_n(\omega) dP = \left\{\int_\Omega \xi_1^{(n)}(\omega) dP, \cdots, \int_\Omega \xi_j^{(n)}(\omega) dP, \cdots\right\}$$

where the components are ordinary Lebesgue integrals; thus the components of x_n are real-valued Lebesgue integrable functions.

The martingale equality becomes

$$\left\{\int_\Lambda \xi_1^{(n)}(\omega) dP, \cdots, \int_\Lambda \xi_j^{(n)}(\omega) dP, \cdots\right\}$$
$$= \left\{\int_\Lambda \xi_1^{(m)}(\omega) dP, \cdots, \int_\Lambda \xi_j^{(m)}(\omega) dP, \cdots\right\}, m < n, \Lambda \in \mathscr{F}_m \subset \mathscr{F}_n$$

or, alternatively,

$$\int_\Lambda \xi_j^{(n)}(\omega) dP = \int_\Lambda \xi_j^{(m)}(\omega) dP, m < n, \Lambda \in \mathscr{F}_m \subset \mathscr{F}_n \text{ for } j = 1, 2, \cdots .$$

Thus for every j, $\{\xi_j^{(n)}, \mathscr{F}_n, n \geqq 1\}$ is a real martingale, which can also be seen by noticing that the mapping from an l^p vector to any of its components is a linear functional. Then if

$$E\{\| x_n \|\} = \int_\Omega \left\{\sum_{j=1}^{\infty} |\xi_j^{(n)}(\omega)|^p\right\}^{1/p} dP \leqq K < \infty ,$$

by Theorem 2 there is an $x(\omega) = \{\xi_1(\omega), \cdots, \xi_j(\omega), \cdots\} \in l^p$ such that for every $\eta = (\eta_1, \cdots, \eta_j, \cdots) \in l^q$, $1/p + 1/q = 1$, $\sum_{j=1}^{\infty} \eta_j \xi_j^{(n)}(\omega) \to \sum_{j=1}^{\infty} \eta_j \xi_j(\omega)$ as $n \to \infty$ for almost all ω. Note that the boundedness assumption on the $E\{\| x \|\}$'s implies boundedness for $E\{| \xi_j^{(n)} |\}$'s for every j; thus we could get convergence in each component by the ordinary martingale convergence theorems.

Finally, if the $\|x_n\|$'s are uniformly integrable, that is, if

$$\int_{A_K}\Big[\sum_{j=1}^{\infty}|\xi_j^n(\omega)|^p\Big]^{1/p}dP \to 0$$

uniformly in n as $K \to \infty$, $A_K = \{\omega: [\sum_{j=1}^{\infty}|\xi_j^n(\omega)|^p]^{1/p} > K\}$. We can get by the ordinary martingale convergence theorem that $\int_A \xi_j(\omega)dP = \int_A \xi_j^n(\omega)dP$ for $A \in \mathscr{F}_n$, $n \geqq 1$ for every j.

However, we get more by Theorem 3, namely, $\sum_{j=1}^{\infty}|\xi_j^n(\omega) - \xi_j(\omega)|^p \to 0$ with probability 1 as $n \to \infty$, and also, of course $\sum_{j=1}^{\infty}|\xi_j^n(\omega)|^p \to \sum_{j=1}^{\infty}|\xi_j(\omega)|^p$ with probability 1 as $n \to \infty$.

EXAMPLE 2. Let $\mathfrak{X} = L^p(I)$, where I is the closed unit interval with Lebesgue measure, $p > 1$. Then $x_n(\omega) = g_n(t, \omega)$ where $\int_\Omega |g_n(t,\omega)|^p dt < \infty$. Now if $x_n(\omega)$ is strongly measurable relative to $\mathscr{F}_n$, there is a representation $g_n(t, \omega)$ which is measurable over $\Omega \times I$ such that $g_n(\cdot, \omega) = x_n(\omega)$ in $L^p(I)$ a.e. in Ω, and any two representations of $x_n(\cdot)$ differ over $\Omega \times I$ on at most a set of measure zero. (Dunford-Pettis [4] Theorem 1.3.2, p. 336).

If $x_n(\cdot)$ is Bochner integrable, then besides being strongly measurable, $\int_\Omega \|x_n(\omega)\| dP < \infty$.

Thus

$$\int_\Omega\Big\{\int_I |g_n(t, \omega)|^p dt\Big\}^{1/p} dP = \int_\Omega \|x_n(\omega)\| dP < \infty .$$

Hence

$$\int_\Omega\Big\{\int_I |g_n(t, \omega)|\, dt\Big\}dP \leqq \int_\Omega\Big\{\int_I |g_n(t, \omega)|^p dt\Big\}^{1/p} dP < \infty$$

by the Hölder Inequality. Therefore, by the Fubini Theorem,

$$\int_\Omega\int_I g_n(t, \omega)dt\, dP = \int_I\int_\Omega g_n(t, \omega)dP\, dt ,$$

and

$$\int_I\Big\{\int_\Omega x_n(\omega)dP\Big\}(t)dt = \int_\Omega\int_I x_n(\omega)(t)dt\, dP$$

$$= \int_\Omega\int_I g_n(t, \omega)dt\, dP = \int_I\int_\Omega g_n(t, \omega)dP\, dt .$$

Hence

$$\Big\{\int_\Omega x_n(\omega)dP\Big\}(t) = \int_\Omega g_n(t, \omega)dP$$

for almost all t.

If $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an L^p-martingale, then $\int_\Lambda x_n(\omega)dP = \int_\Lambda x_m(\omega)dP$ for $\Lambda \in \mathscr{F}_m$, $m < n$, i.e., $\int_\Lambda g_n(t, \omega)dP = \int_\Lambda g_m(t, \omega)dP$ for almost all t, and $\Lambda \in \mathscr{F}_m$, $m < n$. Hence, for almost all $t \in I$ (Lebesgue measure) if $\mathscr{F}_n$ is generated by countably many sets, $\{g_n(t, \cdot), \mathscr{F}_n, n \geqq 1\}$ is a real martingale.

Next, if

$$E\{\| x_n \|\} = \int_\Omega \left[\int_I | g_n(t, \omega)|^p dt\right]^{1/p} dP \leqq K < \infty ,$$

there is an $x(\omega) = g(t, \omega) \in L^p(I)$, $\int_I | g(t, \omega) |^p dt < \infty$ by Theorem 2 such that $\int_I h(t)g_n(t, \omega)dt \to \int_I h(t)g(t, \omega)dt$ as $n \to \infty$ with probability 1 for every $h \in L^q(I)$, $1/p + 1/q = 1$

Furthermore, by Theorem 3, if the $\| x_n \|$'s are uniformly integrable, then $\int_I | g_n(t, \omega) |^p dt \to \int_I | g(t, \omega) |^p dt$ as $n \to \infty$ with probability 1, and even better, $\int_I | g_n(t, \omega) - g(t, \omega) |^p dt \to 0$ as $n \to \infty$ with probability 1.

The uniform integrability condition says that

$$\int_{\Lambda_N} \left[\int_I | g_n(t, \omega) |^p dt\right]^{1/p} dP \to 0$$

uniformly in n as $N \to \infty$,

$$\left\{\omega: \left[\int_I | g_n(t, \omega) |^p dt\right]^{1/p} > N\right\} = \Lambda_N .$$

This implies uniform integrability of the random variables in the real martingales $\{g_n(t, \cdot), \mathscr{F}_n, n \geqq 1\}$. Thus for almost all t, we can apply the ordinary Doob martingale theorems, and thus get convergence theorems in each coordinate.

The functions $g_n(t, \omega)$ as functions of t might, as a further illustration, be sample functions of a sequence of measurable stochastic processes (Doob [1] p. 60) with the property of being absolutely integrable over $\Omega \times I$.

EXAMPLE 3. We have seen in Example 2.2 of Chapter III that if $\{y_j, j \geqq 1\}$ are mutually independent, as $\mathfrak{X}$-valued random variables, with $\mathscr{E}\{y_j\} = \theta$ for $j > 1$, and $\mathscr{F}_j$ is the smallest Borel field relative to which $y_1, \cdots, y_j$ are all strongly measurable, and if $x_n = \sum_{j=1}^n y_j$, then $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-martingale. Theorem 2 tells us that if $\lim_{n\to\infty} E\{\|x_n\|\} = K < \infty$, then $\sum_{j=1}^\infty f(y_j(\omega))$ converges with probability 1. If, further, the $\| x_n \|$'s are uniformly integrable, then by Theorem 3, $\sum_{j=1}^\infty y_j(\omega)$ converges with probability 1.

Examples 1 and 2 above illustrate an important point. It is clear from them that an l^p-martingale is really a countable collection of one-dimensional martingales, while an L^p-martingale is a non-denumerable collection of ordinary real martingales. Thus, it is possible to prove convergence theorems for l^p or L^p by first proving convergence in each coordinate, using the Doob theorems on convergence of ordinary martingales. One could prove the convergence theorem for abstract Hilbert space by first proving the theorem for l^2 in each coordinate and then using the fact that there is a one-to-one linear norm preserving transformation between l^2 and abstract Hilbert space. In fact, one could prove convergence theorems for any $\mathfrak{X}$-martingale in which $\mathfrak{X}$ is a function space or a coordinate space by first proving martingale convergence theorems in each coordinate.

Let $\{\xi_t, t \in I = [0, 1]\}$ be a separable Brownian motion process (Doob [1] p. 52, p. 392). Then there is a measurable set $\Omega_0 \subset \Omega$, such that $P\{\Omega - \Omega_0\} = 0$, and such that for $\omega \in \Omega_0$, $\xi_t(\omega)$ is a continuous function of $t \in I$. Let $x(\omega) = \xi_t(\omega) = g(t, \omega)$. Then $x(\cdot)\colon \Omega \to C(I)$, the continuous function space on the unit interval, and $\| x(\omega) \| = \sup_{t\in I} | g(t, \omega) |$.

We next show that $x(\cdot)$ is strongly measurable. Let $f \in \mathfrak{X}^* = C(I)^*$. Then there is a function of bounded variation F such that $f(x(\omega)) = \int_I g(t, \omega) dF(t)$

$$= \lim_{\max|t_j - t_{j-1}| \to 0} \sum_{j=1}^{n} g(u_j, \omega)[F(t_j) - F(t_{j-1})]$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$ and $t_{j-1} < u_j < t_j$. But each sum is clearly measurable in ω, so the limit must be too. Thus $x(\cdot)$ is weakly measurable, but since $C(I)$ is separable, this is equivalent to strong measurability of x.

To show that $x(\cdot)$ is Bochner integrable, we need only show that $E\{\| x \|\} < \infty$, for $x(\cdot)$ is strongly measurable. To this end, let $\xi_0 = 0$ with probability 1, and let $h(\omega) = \| x(\omega) \| = \sup_{t\in I} | g(t, \omega) |$.

Then

$$P\{\omega\colon h(\omega) \geqq n\} \leqq \frac{\sigma}{n}\sqrt{\frac{2}{\pi}} e^{-n^2/2\sigma^2} .$$

(Doob [1] p. 392) Thus

$$\sum_{n=1}^{\infty} P\{\omega\colon h(\omega) \geqq n\} \leqq \sigma\sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n^2/2\sigma^2} < \infty .$$

Hence, $E\{\| x \|\} < \infty$, and $x(\cdot)$ the sample function of a separable Brownian motion process is Bochner integrable.

Let $\mathscr{F}_1$ be the Borel field of Ω sets generated by $\xi_0, \xi_{1/2}, \xi_1$; $\mathscr{F}_2$ the

Borel field generated by $\xi_0, \xi_{1/4}, \xi_{1/2}, \xi_{3/4}, \xi_1$, and in general $\mathscr{F}_n$ the Borel field generated by $\xi_0, \xi_{1/2^n}, \cdots, \xi_{2^n-1/2^n}, \xi_1$. Then $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_n \subset \cdots$ Let $f_n(t)(\omega) = E\{\xi_t \mid \mathscr{F}_n\}(\omega)$. Lévy ([11]) p. 18) has shown that $f_n(t)(\omega)$ is a polygonal line function of t for almost all ω, and that $|f_n(t)(\omega) - \xi_t(\omega)| \to 0$ as $n \to \infty$ uniformly in t for almost all ω. If we let $y_n(\omega) = f_n(t)(\omega) \in C(I)$ for $\omega \in \Omega$, then $\{y_n, \mathscr{F}_n, n \geqq 1\}$ is a $C(I)$-martingale. Lévy's result does not as yet come out of our work because $C(I)$ is not reflexive.

The validity of the Martingale Convergence Theorem for non-reflexive spaces is not known to the author. In fact, various, attempts in proving it have failed. If it were established, then further interesting examples like the last one for important non-reflexive spaces, e.g., L^1 or l^1, could be given.

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