AN EMBEDDING OF RIEMANN SURFACES OF GENUS ONE

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The C^k embedding of a Riemann surface S will mean here the construction of a C^k surface S' in 3-space which is conformally equivalent to S, if angles on the surface S' are measured in the natural way.[1] The result to be obtained is:

THEOREM. *Any compact Riemann surface of genus one can be C^∞ embedded in 3-space.*

As is well known, any Riemann surface of genus one is conformally equivalent to a parallelogram in the plane with opposite sides identified. The method used here utilizes surfaces which are approximately isometric to the canonical surfaces determined by parallelograms. The parallelogram for a given conformal class may be picked in a standard way. We may take the vertices at the points $0, 2\pi, \omega, \omega + 2\pi$ in the complex plane. Then the parallelogram is determined by a single complex number ω. For any surface S conformally equivalent to this parallelogram with opposite sides identified, ω will be called a modulus of S, and the parallelogram a fundamental parallelogram of S. ω is not completely determined. A complete set of inequivalent canonical surfaces corresponds to the values of $\omega = \theta + i\lambda$ in the region

$$(1) \qquad -\pi < \theta \leqq \pi, \quad \theta^2 + \lambda^2 > 4\pi^2$$

or

$$0 \leqq \theta \leqq \pi, \quad \theta^2 + \lambda^2 = 4\pi^2 .$$

For each value of ω in this region a surface is needed.

A torus has a pure imaginary modulus which is easily computed. More generally, any surface with a plane of symmetry has pure imaginary modulus. Thus there are many ways in which one can construct a family of surfaces whose moduli fill the line $\theta = 0, \lambda \geqq 2\pi$.

For finding surfaces with $\theta \neq 0$, we may note first that under a reflection of space a surface with modulus $\theta + i\lambda$ is transformed into one with modulus $-\theta + i\lambda$. This means that if surfaces whose moduli represent all points of the region

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Received February 23, 1960. This research was performed while the first author was supported by the United States Air Force under Contract No. AF 49(638)-42, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

[1] In our considerations compact surfaces of 3-space will be considered oriented by the outward pointing normal.

$$(1') \qquad 0 < \theta \leqq \pi, \quad \theta + \lambda^2 \geqq 4\pi^2$$

are available, then every point of (1) with $\theta < 0$ will be found among the moduli of the reflected surfaces.

One type of surface whose modulus can be computed is a canal surface, the envelope of a one-parameter family of spheres. In many cases it becomes difficult, however, to determine when the surface enveloped by a given family of spheres is really a good surface of genus one, with no undesired self-intersections. One two parameter family of canal surfaces has been given [2] which yields all values of ω in (1) with $|\theta| < \theta_0$, where $\theta_0 > 0$. More complicated families yield these moduli and also those for which $\lambda > \lambda_0$. In all the families which have been investigated by the authors, the surfaces for which ω is close to the vertex $\pi + (\pi + i\pi\sqrt{3})$ of (1) have self-intersections. Perhaps there is a region of values of ω near this point which cannot be realized by canal surfaces. However, by using the methods of Nash [4] as extended by Kuiper [3] it should not be hard to show that there exists an analytic surface whose modulus is arbitrarily close to any given modulus.

The method used here to prove the existence of embeddings for all moduli is to construct C^∞ surfaces which are approximations to singular surfaces. The singular surfaces used are composed of polygonal faces joined along edges and at vertices, and have the property that although points on different faces are distinguished, the faces all lie in the same plane in space, and may partly or wholly overlap. For each value of ω in a region including (1′) a singular surface is constructed. It is isometric be the canonical surface of modulus ω.

The singular surfaces are idealizations of figures which may be physically constructed by folding paper parallelograms and joining the edges. The physical model approaches the ideal surfaces as the thickness of the paper approaches zero. From this it follows that the singular surface can be approximated by a true surface. Such an approximation is given by the central plane of the paper.

1. The deformation lemma.[2] Suppose that we have a C^1 mapping of a surface S onto a surface S' such that all first derivatives of the mapping do not simultaneously vanish at any point of S. Then the dilation quotient D of the mapping is a function of position on S defined as follows: The image of an infinitesimal circle about the point P of S is an ellipse of definite eccentricity. $D(P)$ is the ratio of major to minor diameters in this ellipse. If S and S' are Riemann surfaces and the mapping is given in a neighborhood of P in terms of local uniformizers z, z' by $z' = f(z)$, then

[2] The methods used in this section are inspired by the work of Teichmuller [5] and Ahlfors [1].

$$D(P) = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|}\bigg|_{z=z(P)} \tag{2}$$

if the mapping preserves orientation. For a conformal mapping $D = 1$. In general, $1 \leqq D \leqq \infty$.

Now let S and S' be surfaces of genus one, and let the mapping be one-to-one. If the moduli of S, S' are ω, ω' respectively, then the induced mapping of the fundamental parallelograms of S and S' has the form

$$z' \equiv f(z) \quad (\mathrm{mod}\ 2\pi, \omega') ,$$

where f is a differentiable function of z and $\bar z$. The boundary values of $f(z)$ are related by the condition that equivalent values of z go into equivalent values of z'. We can choose the fundamental parallelogram of S' so that its sides are homotopic to the images under $S \to S'$ of the corresponding sides of the fundamental parallelogram of S. Then

$$\begin{aligned} f(z + 2\pi) &= f(z) + 2\pi , \\ f(z + \omega) &= f(z) + \omega' . \end{aligned} \tag{3}$$

After these preliminaries we may prove the following

LEMMA. *Let S, S' be Riemann surfaces of genus* 1. *Suppose there is a one-to-one mapping of S onto S' which is piecewise C^1, is conformal except on a region R of the fundamental parallelogram of S of area α, and has $D \leqq D_0 < \infty$. Then if $\omega = \theta + i\lambda$ is the modulus of S, S' has a modulus ω' such that*

$$|\omega - \omega'| < \sqrt{\eta(\eta + 2\lambda)} \tag{4}$$

where

$$\eta = \frac{\alpha}{2\pi D_0}(D_0 - 1)^2 .$$

Proof. Let $z = x + iy, z' = x' + iy'$. If P is the fundamental parallelogram of S and C its boundary, we have, by using (3),

$$\int_C f(z)dz = 2\pi(\omega - \omega') .$$

By Green's theorem,

$$\int_C f(z)dz = 2i\iint_P f_{\bar z}dxdy .$$

Since $f_{\bar z} = 0$ outside R,

$$-\pi i(\omega - \omega') = \iint_R f_{\bar z} dxdy \,.$$

Now apply Schwarz' inequality:

$$\pi^2 |\omega - \omega'|^2 < \iint_R dxdx \iint_R |f_{\bar z}|^2 dxdy \,. \tag{5}$$

To estimate the last integral, we use (2) to get

$$|f_{\bar z}|^2 = \frac{(D-1)^2}{4D}\{|f_z|^2 - |f_{\bar z}|^2\} = \frac{(D-1)^2}{4D} \frac{\partial(x', y')}{\partial(x, y)} \,.$$

Hence

$$\iint_R |f_{\bar z}|^2 dxdy < \iint_P \frac{(D_0-1)^2}{4D_0} \frac{\partial(x', y')}{\partial(x, y)} dxdy = \frac{(D_0-1)^2}{4D_0} \cdot 2\pi\lambda' \,,$$

where $\lambda' = \operatorname{Im} \omega'$. Inserting this in (5),

$$|\omega - \omega'|^2 \leqq \frac{\alpha}{\pi^2} \iint_R |f_{\bar z}|^2 dxdy \leqq \eta\lambda' \,.$$

To get an inequality without λ' on the right, note that

$$(\lambda' - \lambda)^2 \leqq |\omega - \omega'|^2 \leqq \eta\lambda'$$

implies $\lambda' < \eta + 2\lambda$.

Thus

$$|\omega - \omega'| \leqq \sqrt{\eta\lambda'} < \sqrt{\eta(\eta + 2\lambda)} \,.$$

Observe that $|\omega - \omega'|$ is small, according to the lemma, in two cases: (1) if D_0 is close to 1, and (2) for fixed D_0, if α is small. Both cases will occur in the applications of the lemma.

2. The singular surfaces. The construction of a singular surface will be described in terms of operations on a paper parallelogram of modulus $\omega = \theta + i\lambda$. If ideal paper of zero thickness is used, the resulting surface is isometric to the canonical surface of modulus ω.

The first operation is to bend the parallelogram into a cylinder of radius 1 which fits together along the sides of length $|\omega|$. Glueing the cylinder together along the line where these sides meet gives the proper identification of the vertical sides of the fundamental parallelogram. The cylinder must be folded up so that the ends come into coincidence in the proper way.

If $\theta = 0$, the points at the ends of each generator must be identified. To do this, first flatten the cylinder by folding along two opposite generators. Then fold on the center line perpendicular to the generators, so that the two ends of the cylinder come together. Along the line where the ends meet, there are the edges of four layers of paper. If these are numbered in order of position, the proper identification of points is accomplished by joining sheet 1 to sheet 4 and sheet 2 to sheet 3.

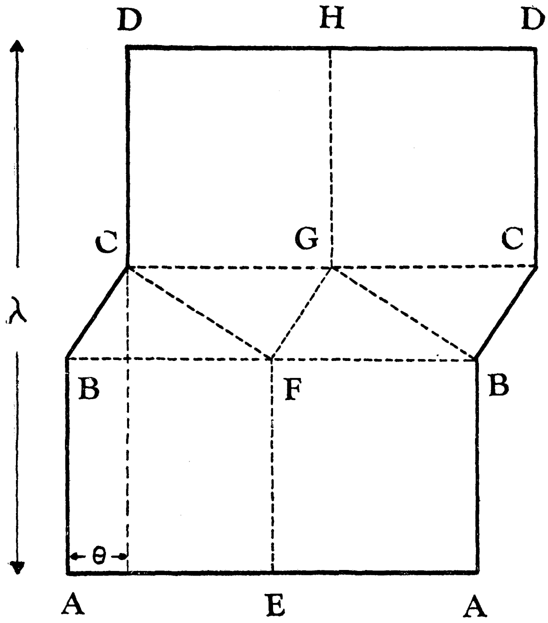

Figure IA

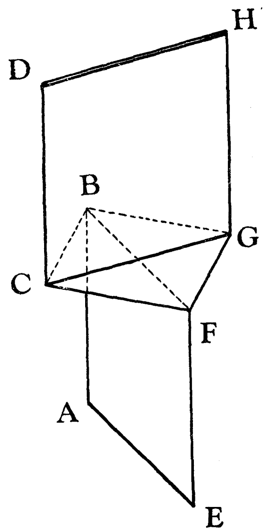

Figure IB

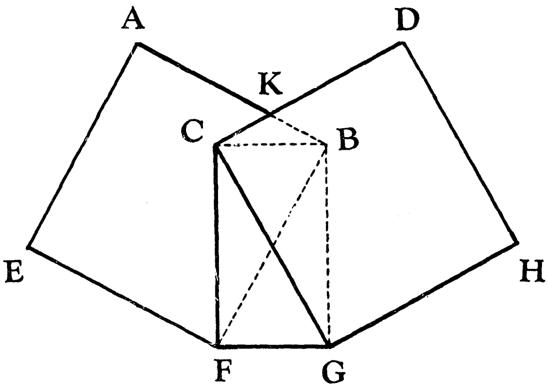

Figure IC

If $\theta \neq 0$, the point at the bottom of a generator must be identified with the point at the top of another generator, separated from the first by the angle θ. The cylinder will be folded flat in such a way that at each end it is folded along two opposite generators, but the generators used at the top are displaced from those used at the bottom by the angle θ. Then the two ends can be connected as they were for $\theta = 0$.

One such method of folding the cylinder is illustrated in figure 1. In 1(a) the surface of the cylinder is shown, developed on the plane. AB, EF and CD, GH are pairs of opposite generators, with CD to the right of AB by the distance θ. CGC and BFB are perpendicular to the generators. $AB = CD$, and the angles BCF, CFG, etc. are right angles. This determines the construction of 1(a). The cylinder may now be folded into a polygonal surface as shown in 1(b). Each line in 1(a) becomes one or more edges in 1(b). The part of the cylinder outside the lines CGC and BFB are flattened into the rectangles $DHGC$, $BFAE$. The part between these lines is flattened into the rectangle $BGFC$.

Next 1(b) is flattened out to give 1(c) by bending 1(b) along the lines CG, BF until all faces lie in the same plane. In figure 1(c) there

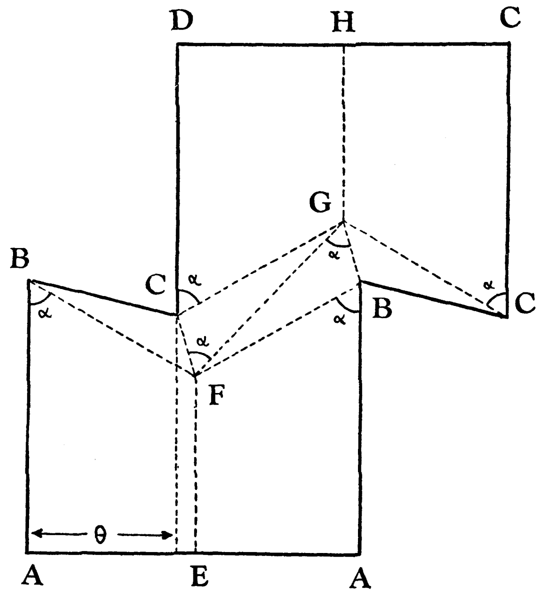

Figure 2 A

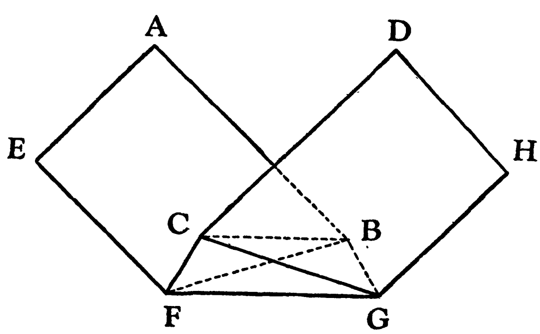

Figure 2 B

are six sheets, joined in appropriate ways along their edges. The flattened circles DHD and AEA are now in a suitable position and may be identified as when $\theta = 0$, after folding 1(c) along its vertical center line. This gives a singular surface consisting of twelve polygonal faces lying in the same plane. It is a Riemann surface isometric to the original parallelogram with its opposite sides identified.

A restriction must be placed on ω for this construction to work. λ must be sufficiently large that the lines AE, DH lie outside the rectangle $CBGF$ in 1(c). Also the lines CD and BA of 1(c) must extend at least as far as their intersection K. The first condition implies the second if $\theta \leqq \pi/2$, which is the only case in which this construction will be used. All the dimensions of 1(c) may be determined by elementary methods. The first condition on λ is

$$\lambda \geqq 3\sqrt{\theta(\pi - \theta)}\,. \tag{6}$$

For $0 < \theta \leqq \pi/2$, the model described may be used when λ satisfies this inequality.

Another model may be constructed as indicated in figure 2. The configuration $BFBCGC$ in 2(a) is determined by the angle

$$\alpha = ctn^{-1}[1 - \sqrt{2(1 - \theta/\pi)}]$$

which is marked in six locations. The condition $AB = CD$ determines the rest of the construction of 2(a). Figure 2(b) is analogous to 1(c). The central quadrilateral is now a trapezoid instead of a rectangle, and the lines AB, CD in 2(b) are perpendicular. The condition that AE and DH do not enter $FGBC$ in this diagram allows the construction of the singular surface to be completed as before. This gives the following condition on λ:

$$\lambda \geqq 2\pi - \theta \tag{7}$$

This model will be used for $\theta > \pi/2$. Note that the two models are the same for $\theta = \pi/2$.

By the two constructions just described, we have a family of singular surfaces associated one-to-one with the values of ω in the region

$$\mathrm{R}: \begin{cases} 0 < \theta < \pi, \\ \lambda \geqq \begin{cases} 3\sqrt{\theta(\pi - \theta)}\,, \theta \leqq \pi/2 \\ 2\pi - \theta,\ \theta \geqq \pi/2 \end{cases} \end{cases} \tag{8}$$

The surface $S_0(\omega)$ corresponding to ω has modulus ω. The curve which forms the lower boundary of this region is below the circular arc $\lambda = \sqrt{4\pi^2 - \theta^2}$ which forms the lower boundary of (1′). Thus all the points of (1′) for which $\theta \neq \pi$ are in R. Another construction to be used later for the case $\theta = \pi$ depends on the fact that R is not bounded away from 0, and the lower boundaries of (1′) and R do not meet even at $\theta = \pi$. Note that $S_0(\omega)$ depends continuously on ω.

3. **The C^∞ surfaces.** The singular surface $S_0(\omega)$ will be transformed into a family of C^∞ surfaces $S(\omega, \vartheta)$, $0 < \vartheta < \vartheta_1(\omega)$ with modulus $\Omega(\omega, \vartheta)$ which approaches ω as $\vartheta \to 0$. These surfaces will also depend on a large positive constant K.

For convenience, suppose that $S_0(\omega)$ is situated in a horizontal plane. In this plane, around each vertex V_k of $S_0(\omega)$ construct a circular disc γ_k^0 of radius $K\vartheta$. For each edge of $S_0(\omega)$, consider the strip of the plane extending the distance ϑ to each side of the edge. Construct the regions ζ_e^0 interior to these strips and exterior to the circles γ_k^0. These are well defined if $K > 1$. For fixed K, the γ_k^0's and ζ_e^0's will be disjoint for sufficiently small $\vartheta : \vartheta < \vartheta_1(\omega)$, if none of the angles in the figure are too small. This will be true if θ is not too close to 0 or $\pi : \theta_1(K) \leqq \theta \leqq \theta_2(K)$, where $\theta_1(K) \to 0$ and $\theta_2(K) \to \pi$ as $K \to \infty$.

Form ζ_e from ζ_e^0 by replacing the bounding circular arcs by chords. Form γ_k from γ_k^0 by removing the segments which have been added to the ζ_e's.

Let C_k be the cylindrical region of space with base γ_k, and let R_e be the cylindrical region with base ζ_e. The C^∞ surface $S(\omega, \vartheta)$ will be constructed above the plane so that in the mapping from $S_0(\omega)$ to $S(\omega, \vartheta)$, (1) the parts of the faces of $S_0(\omega)$ which lie outside the γ_k's and the ζ_e's are each translated upwards by a certain amount, (2) the pieces of $S_0(\omega)$ in ζ_e are mapped isometrically into R_e, and (3) the pieces in γ_k are mapped into C_k.

Let the twelve faces of $S_0(\omega)$ be numbered in order from bottom to top. The jth face is cut into a number of sections by the boundaries of the γ_k's and ζ_e's. Raise each section bounded entirely by these boundaries to the height $j/12$ above the plane of $S_0(\omega)$. This includes the vertical translation referred to above. The remainder of $S_0(\omega)$ consists of strips of surface containing edges, and a finite number of regions in each γ_k.

Each edge strip S_m has width 2ϑ. In extending the mapping to S_m, the sides of the strip must go into parallel horizontal lines at a distance less than ϑ, one above the other. Opposing points on the two sides are to go into points of the parallel lines which lie on the same vertical line, and the mapping is to be isometric along each side.

Such a mapping can be constructed by mapping S_m isometrically onto a cylindrical surface of width 2ϑ bounded by the two parallel lines as generators. For $S_m \subset \zeta_e$, its image shall lie in R_e. This surface can be so chosen that the image of S_m and the adjacent sections of faces of $S_0(\omega)$ lies on a C^∞ cylindrical surface. If several of the S_m''s lie along the same edge of $S_0(\omega)$, use the same cylindrical surface for each of their images. Then the cylindrical mapping may be extended through the intervening regions which lie in the γ_k's.

The image of the region around each edge of $S_0(\omega)$ may be constructed in turn, so as not to meet any of the parts of $S(\omega, \vartheta)$ previously constructed, if we take the edges in the right order, for example first all the edges of the top face, than all the unconnected edges of the next face, and so on. It remains to construct the images of the regions about vertices of $S_0(\omega)$.

Let v_j be the region about a vertex of $S_0(\omega)$, bounded by the boundary of γ_k. It is to be mapped into a piece of C^∞ surface in C_k which joins to previously constructed parts of $S(\omega, \vartheta)$ at the boundary of C_k, so as to make a C^∞ surface. Let V_j be any piece of surface with these properties. Then a mapping from v_j to V_j can be made which agrees at the boundary with the mapping already constructed, and has a bounded dilation quotient. $S(\omega, \vartheta)$ and the mapping from $S_0(\omega)$ will be completed when this has been done for each v_k.

The resulting mapping is isometric, hence conformal, except in the regions of v_k. The dilation quotient has an upper bound $D_0(\omega, \vartheta)$. If $\Omega = \Omega(\omega, \vartheta)$ is the modulus of $S(\omega, \vartheta)$, then by the lemma

$$|\Omega - \omega| < \sqrt{\eta(\eta + 2\lambda)}\,, \tag{9}$$

where

$$\eta = \frac{(D_0 - 1)^2}{2\pi D_0} \sum_k \text{Area}\,(v_k) < \frac{(D_0 - 1)^2}{2\pi D_0} \cdot 12\pi K^2\vartheta^2\,,$$

$$< 6K^2\vartheta^2 \frac{(D_0 - 1)^2}{D_0}\,, \tag{10}$$

since there are twelve vertices, and each v_k is a subregion of a disc of radius $K\vartheta$.

Suppose that this construction is made only for particular values of ω and ϑ, e.g. $\omega = \omega_1 = \theta_1 + i\lambda_1$, $\vartheta = \vartheta_1$. Then from this we can derive $S(\omega, \vartheta)$ for other values of ω and ϑ in a useful way.

First, let ϑ vary. For all sufficiently small ϑ, we may take the vertex pieces V_k to be similar to those for ϑ_1, and take the cylindrical strips which contain the images of the edges to have a cross-section which is similar to the cross-section for $\vartheta = \vartheta_1$. If we take the mapping from v_k to V_k to be that which is induced by this similarity, then D_0 in (10) is independent of ϑ.

Next let λ vary. In $S_0(\omega)$, certain sides change in length, but all angles at the vertices are unchanged. Looking at the skeleton of $S(\omega, \vartheta)$, consisting of the V_k's and the cylindrical strips which contain the images of edges, to get a skeleton for $S(\theta_1 + i\lambda, \vartheta)$ it is sufficient to change appropriately the lengths of some of the strips. If $S(\theta_1 + i\lambda, \vartheta)$ is formed in this way, D_0 can be made independent λ.

To go to a value of ω with $\theta \neq \theta_1$, the angles at the vertices must change as well as lengths. Looking again at the skeleton of $S(\omega_1, \vartheta)$, $V_j \subset C_k$ must be transformed so that the adjacent cylindrical strips are rotated into different positions about the axis of C_k. We may impose on V_j the condition that it does not intersect the axis of C_k. Then the desired result may be obtained by performing a transformation of C_k which rotates points about the axis through a variable angle, is isometric in the sectors adjacent to the cylindrical strips, and is C^∞ with non-vanishing Jacobian at all points off the axis.

Let t_θ be the transformation which takes the vertex sections $V_k(\theta_1)$ into $V_k(\theta)$. Since all the angles involved are continuous functions of θ, we may choose t_θ so that $t_{\theta'}\ t_\theta^{-1}$ is a mapping whose dilation quotient has a bound $D(\theta, \theta')$ such that

$$\lim_{\theta' \to \theta} D(\theta, \theta') = 1$$

and $t_{\theta'}\ t_\theta^{-1}$ approaches the identity as $\theta' \to \theta$.

This transformation of the V_k's may be extended to a transformation of $S(\omega_1, \vartheta)$ into $S(\omega, \vartheta)$, by transforming the cylindrical strips of $S(\omega_1, \vartheta)$ linearly into the strips of $S(\omega, \vartheta)$, and then extending the mapping over the plane faces. Since the lengths of the edges of $S_0(\omega)$ are continuous functions of θ, if the extension to the faces is done properly the induced mapping $T_{\omega,\omega'}$ of $S(\omega, \vartheta)$ into $S(\omega', \vartheta)$ will have a dilation with a bound $D(\omega, \omega', \vartheta)$ such that

$$\lim_{\omega' \to \omega} D(\omega, \omega', \vartheta) = 1 \tag{11}$$

The mapping from $S_0(\omega)$ to $S(\omega, \vartheta)$ may be chosen so that $D_0(\omega, \vartheta)$ is a continuous function of ω. For this, it is necessary to map v_j on V_j in the right way.

A mapping of $S_0(\omega_1)$ onto $S_0(\omega)$ is associated with the mappings already described:

$$S_0(\omega_1) \longrightarrow S(\omega_1, \vartheta) \xrightarrow{T_{\omega_1\omega}} S(\omega, \vartheta) \longrightarrow S_0(\omega) .$$

Thus the map of $v_j(\omega)$ on $V_j(\omega)$ will be determined by that for $\omega = \omega_1$ and a mapping of $v_j(\omega_1)$ on $v_j(\omega)$. Since $T_{\omega,\omega'}$ has the property (11), $D_0(\omega, \vartheta)$ will be a continuous function of ω if the transformation between the v_k's also has this property.

To transform a v_j it is convenient to look at this region unfolded. It is a disc with several segments cut off, and is to be transformed into another such disc. The mapping is given on the boundary, and is to be extended to the interior so that the bound $D'(\omega, \omega')$ for the dilation quotient of the mapping from $v_j(\omega)$ to $v_j(\omega')$ approaches 1 as $\omega' \to \omega$. One way to get such an extension is to take the mapping given in rectangular coordinates by harmonic functions with the proper boundary values.

4. The existence theorem for $0 < \theta < \pi$. Any modulus $\omega_0 = \theta_0 + i\lambda_0$ in (1′) with $0 < \theta < \pi$ lies in the interior of R. Pick K and Λ so large that $\theta_1(K) < \theta_0 < \theta_2(K)$ and $\Lambda > \lambda_0$. Then ω_0 lies in the interior of the closed subregion R_1 of R for which

$$\theta_1(K) \leqq \theta \leqq \theta_2(K) ,$$
$$\lambda \leqq \Lambda$$

Let the distance from ω_0 to the boundary of R_1 be greater than ε ($\varepsilon > 0$)

In §3, we have constructed surfaces $S(\omega, \vartheta)$ for each $\omega \in R_1$ and $\vartheta < \bar{\vartheta} = \min \vartheta_1(\omega)$, $\omega \in R_1$. By (11) and the lemma, $\Omega(\omega, \vartheta)$ is a continuous function of ω, so $|\Omega(\omega, \vartheta) - \omega|$ is bounded on R_1, for each ϑ. An explicit bound is given by (9):

$$|\Omega(\omega, \vartheta) - \omega| < C\vartheta ,$$

where, setting $D_1 = \max D_0(\omega, \vartheta)$, $\omega \in R_1$,

$$C = \sqrt{6K^2 \frac{(D_1 - 1)^2}{D_1} [6K^2\bar{\vartheta}^2 \frac{(D^1 - 1)^2}{D_1} + \Lambda]} \,.$$

Take $\vartheta < \varepsilon/C$. Then in the mapping of R_1 by $\omega \to \Omega(\omega, \vartheta)$ each point is moved by less than ε. Hence the image of the boundary is a curve which winds around ω_0 once. It follows that $\Omega(\omega, \vartheta) = \omega_0$ for some $\omega \in R_1$.

5. The existence theorem for $\theta = \pi$. If a family of singular surfaces is available which varies continuously over a region of moduli containing the right-hand boundary of (1′) in its interior, then the procedure of §§ 3 and 4 may be applied to prove the existence theorem for $\theta = \pi$.

As observed in § 2, the lower boundary of R is at a positive distance from the lower boundary of (1′). Let this distance be d. Also, R contains

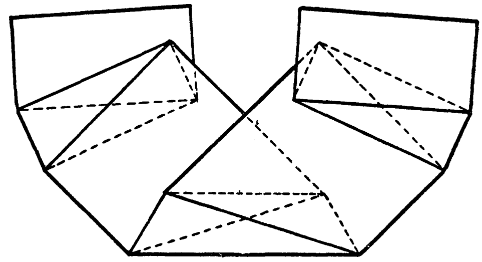

Figure 3

values of ω for which $|\omega| < d/2$. Let $\omega_2 = \theta_2 + i\lambda_2$ be such a value. Now take the singular surface of § 2 for modulus ω, constructed up to the point of figure 1(c) or 2(b). Join to each of its open ends the corresponding figure for $\omega = \omega_2$, as illustrated in figure 3. This gives a folded cylinder of length $\lambda + 2\lambda_2$. When it is folded over the vertical center line of figure 3 and has its ends joined properly, it is a singular surface with modulus $\omega + 2\omega_2$. If this is done for each $\omega \in R$, we get for a region of moduli the region R shifted up and to the right by a positive distance less than d, with singular surfaces varying continuously over this region. The line $\theta = \pi$, $\lambda \geqq \sqrt{3}\,\pi$ lies in the interior, as required.

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