SOME GENERALIZATIONS OF METRIC SPACES

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1. Introduction. This paper consists of a study of certain classes of topological spaces (called M_1-, M_2-, and M_3-spaces) which include metric spaces and CW-complexes and are included in the class of all paracompact and perfectly normal spaces. It is shown, for example, that like the case in metric spaces, a subset of an M_2- (or M_3-) space is an M_2- (or M_3-) space; a countable product of M_i-spaces ($i = 1, 2, 3$) is again an M_i-space; and separable is equivalent to Lindelöf in an M_i-space. Moreover, unlike the case in metric spaces, the quotient space obtained by identifying the points of a closed subset of an M_2- (or M_3-) space is again an M_2- (or M_3-) space (for metric spaces such a quotient space need not be first countable). Also, we have $M_1 \rightarrow M_2 \rightarrow M_3$, but whether $M_3 \rightarrow M_2$ or $M_2 \rightarrow M_1$ is unknown.[1]

These classes of spaces are derived from generalizations of the following well-known characterization of metrizability in terms of specific properties of the base:

THEOREM 1.1. (Smirnov [14] or Nagata [12]). *A regular space is metrizable if and only if it has a σ-locally finite base.*

Recall that a σ-locally finite family is a union of countably many locally finite families. It is easily checked that a locally finite family $\boldsymbol{U}$ of sets has the property, called *closure preserving*, that for any

$$\boldsymbol{V} \subset \boldsymbol{U}, \quad (\cup \{V \in \boldsymbol{V}\})^- = \cup \{V^- : V \in \boldsymbol{V}\} .$$

This, then, suggests we consider spaces having a *σ-closure preserving base* (that is, a base which is the union of countably many closure preserving families).

DEFINITION 1.1. An M_1-space is a regular space having a σ-closure preserving base.

Although conceptually simple, M_1-spaces prove unsatisfactory in some respects, so we weaken the condition of having a σ-closure preserving base. We begin by calling a collection $\boldsymbol{B}$ of (not necessarily open!) subsets of X a *quasi-base* if, whenever $x \in X$ and U is a neighborhood of

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[1] Nearly all topological terminology appearing in this paper is consistent with that used in Kelley [4]. Exceptions are that our regular, and normal spaces are assumed to be T_1-spaces.

x, then there exists a $B \in \boldsymbol{B}$ such that $x \in B^0 \subset B \subset U$ where B^0 denotes the interior of B).

DEFINITION 1.2. An M_2-space is a regular space with a σ-closure preserving quasi-base.

Now we proceed to weaken the condition of having a σ-closure preserving quasi-base. Let $\boldsymbol{P}$ be a collection of ordered pairs $P = (P_1, P_2)$ of subsets of X, with $P_1 \subset P_2$ for all $P \in \boldsymbol{P}$. Then $\boldsymbol{P}$ is called a *pair-base* for X if P_1 is open for all $P \in \boldsymbol{P}$ and if, for any $x \in X$ and neighborhood U of x, there exists a $P \in \boldsymbol{P}$ such that $x \in P_1 \subset P_2 \subset U$. Moreovor, $\boldsymbol{P}$ is called *cushioned* if for every $\boldsymbol{P}' \subset \boldsymbol{P}$,

$$(\bigcup\{P_1 : P \in \boldsymbol{P}'\})^- \subset \bigcup\{P_2 : P \in \boldsymbol{P}'\} .$$

$\boldsymbol{P}$ is called *σ-cushioned* if it is the union of countably many cushioned subcollections.

DEFINITION 1.3. An M_3-space is a T_1-space with a σ-cushioned pair-base.

2. Properties of M_i-spaces.

THEOREM 2.1. (Michael [6]). *A T_1-space is paracompact if and only if every open cover $\boldsymbol{U}$ has a σ-cushioned open refinement $\boldsymbol{V}$ (that is, $\boldsymbol{V} = \bigcup_{n=1}^{\infty} \boldsymbol{V}_n$, where for each n, and $V \in \boldsymbol{V}_n$ one can assign a $U_{V,n} \in \boldsymbol{U}$ such that $\{(V, U_{V,n}) : V \in \boldsymbol{V}_n\}$ is cushioned).*

THEOREM 2.2. *The following implications hold: Metrizable $\rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow$ paracompact and perfectly normal.*

Proof. Metrizable $\rightarrow M_1$ and $M_1 \rightarrow M_2$ are obvious.

To show $M_2 \rightarrow M_3$, let $\bigcup_{n=1}^{\infty} \boldsymbol{B}_n$ be a σ-closure preserving quasi-base. For each n, put $\boldsymbol{P}_n = \{(B^0, \bar{B}) : B \in \boldsymbol{B}_n\}$. Then clearly $\bigcup_{n=1}^{\infty} \boldsymbol{P}_n$ becomes a σ-cushioned pair-base.

To show $M_3 \rightarrow$ paracompactness, let $\bigcup_{n=1}^{\infty} \boldsymbol{P}_n$ be a σ-cushioned pair-base. Let $\boldsymbol{U}$ be an open cover and for each n, let $\boldsymbol{W} = \{P_1 \subset P_2 \subset U_{W,n}$ for some $U \in \boldsymbol{U}$, $U \in \boldsymbol{P}_n\}$. For $W \in \boldsymbol{W}_n$, pick $U_{W,n} \in \boldsymbol{U}$ such that for some $P \in \boldsymbol{P}_n$, $W = P_1 \subset P_2\, U_{W,n}$. Then $\boldsymbol{W} = \bigcup_{n=1}^{\infty} \boldsymbol{W}_n$ becomes a σ-cushioned open refinement of $\boldsymbol{U}$ and hence, by Theorem 2.1, X is paracompact.

To show $M_3 \rightarrow$ perfectly normal, let G be an open set in X. For each n, put $F_n = (\cup\{P_1 : P_2 \subset G, P \in \boldsymbol{P}_n\})^-$. Then $G = \bigcup_{n=1}^{\infty} F_n$, so every open set is an F_σ, whence X is perfectly normal since X is normal by paracompactness, thus completing the proof of Theorem 2.2.

Example 9.2 furnishes us with a separable and first countable M_1-space which is non-metrizable. The "half-open interval" space R (the real line R with base the family $\{[x, y) : x, y \in R\}$ is paracompact and perfectly normal and $R \times R$ is not paracompact (Sorgenfrey [16] or Kelley [4]). Hence, by Theorem 2.2, $R \times R$ is not M_3, and by Theorem 2.4 it follows that R is not M_3. The questions of whether $M_2 \to M_1$ or $M_3 \to M_2$ remain unsolved. However, see Proposition 7.7 for a partial result.

The following three theorems exhibit properties which metric spaces have in common with M_i-spaces.

THEOREM 2.3. *If A is a subset of an M_2- (or M_3-) space X, then A is M_2 (or M_3).*

Proof. We prove it only for the M_2-case. Let $\bigcup_{n=1}^{\infty}\boldsymbol{B}_n$ be a σ-closure preserving quasi-base for X. For each n, put $\boldsymbol{B}'_n = \{A \cap \bar{B} : B \in \boldsymbol{B}_n\}$. To show $\boldsymbol{B}'_n$ is closure preserving in A it suffices to show for $x \in A$ and $\boldsymbol{A} \subset \boldsymbol{B}_n$, that $x \notin \cup\{(A \cap \bar{B})^- : B \in \boldsymbol{A}\}$ implies $x \notin (\cup\{A \cap \bar{B} : B \in \boldsymbol{A}\})^-$. But for any $B \in \boldsymbol{A}, x \notin (A \cap \bar{B})^-$ implies $x \notin A \cap \bar{B}$ and $x \notin \bar{B}$. So $x \notin \cap\{\bar{B} : B \in \boldsymbol{A}\} = (\cup\{\bar{B} : B \in \boldsymbol{A})^-$ and hence, $x \notin (\cup\{A \cap \bar{B} : B \in \boldsymbol{A}\})^-$ and $\boldsymbol{B}'_n$ is closure preserving. Let U be open about x in A. Then for some U' open in X we have $U = U' \cap A$, so there exists B in some $\boldsymbol{B}_n$ so that $x \in B^0 \subset B \subset \bar{B} \subset U'$. Then with $A \cap \bar{B} \in \boldsymbol{B}'_n$, we have $x \in (\boldsymbol{B}^0 \cap A) \subset (A \cap \bar{B})^0 \subset (A \cap \bar{B}) \subset (U' \cap A) = U$. Hence A is M_2, which completes the proof.

The foregoing proof breaks down in the case of an M_1-space (since in general $(B^0 \cap A)^- \neq (A \cap \bar{B})$), and it is unsolved whether a subspace, or even a closed subspace, of an M_1-space is M_1.

THEOREM 2.4. *A countable product of M_i-spaces is M_i.*

Proof. We prove it only for the M_1 case; the other cases follow similarly. For each n, let X_n be an M_1-space with a σ-closure preserving base $\bigcup_{m=1}^{\infty}\boldsymbol{B}_n^m$. Without loss of generality we can assume that, for all m, n, $X_n \in \boldsymbol{B}_n^m$ and $\boldsymbol{B}_n^m \subset \boldsymbol{B}_n^{m+1}$. Now put $X = \prod_{n=1}^{\infty} X_n$ and, for each n, let

$$\boldsymbol{B}_n = \prod_{i=1}^{n}\left\{B_i : B_i \in \boldsymbol{B}_i^n\right\},$$

where

$$\prod_{i=1}^{n} B_i = \{x \in X : x_i \in B_i \text{ for } i \leqq n\}.$$

Then $\bigcup_{n=1}^{\infty}\boldsymbol{B}_n$ becomes a σ-closure preserving base for X, making X an M_1-space.

We can also prove the following result:

THEOREM 2.5. *Let X be an M_i-space. Then the following are equivalent*:

(1) *X is separable,*

(2) *X is Lindelöf,*

(3) *X is satisfies the countable chain condition* (that is, every disjoint family of open sets is countable).

A separable M_1-space need not have a countable base; for example, see Example 9.2.

Smirnov [15] has shown that any locally metrizable paracompact space is metrizable. And Nagata [13] has obtained the stronger result that a space which is the union of a locally finite family of closed metrizable subsets in metrizable. We can obtain analogous results as follows:

THEOREM 2.6. *If X is paracompact and locally M_i, then X is M_i.*

Proof. We prove it only for the M_1 case, and note that the others follow analogously. For each $x \in X$, there exists an open neighborhood $W(x)$ of x such that $W(x)$ is M_1. By paracompactness, let $\{U_\alpha : \alpha \in A\}$ be an open locally finite refinement of $\{W(x) : x \in X\}$. Then, since an open subset of an M_1-space is clearly M_1, each U_α is M_1. Let $\boldsymbol{B}^\alpha = \bigcup_{n=1}^\infty \boldsymbol{B}_n^\alpha$ be a σ-closure preserving base for U_α such that, for each $B \in \boldsymbol{B}^\alpha$, $\bar{B} \subset U_\alpha$. For each n, put $\boldsymbol{C}_n = \cup\{\boldsymbol{B}_n^\alpha : \alpha \in A\}$. Then it easily follows that each $\boldsymbol{C}_n$ is closure preserving and $\bigcup_{n=1}^\infty \boldsymbol{C}_n$ is a base for X.

LEMMA 2.7. *If $X = A_1 \cup A_2$, where A_1 and A_2 are closed M_2- (or M_3-) subspaces, then X is M_2 (or M_3).*

Proof. First we get X to be regular (Nagata [12]). For the M_2 case, let $\bigcup_{n=1}^\infty \boldsymbol{B}_n^1$ and $\bigcup_{n=1}^\infty \boldsymbol{B}_n^2$ be σ-closure preserving quasi-bases for A_1 and A_2 respectively, with $\phi \in \boldsymbol{B}_n^1 \cap \boldsymbol{B}_n^2$ for all n. Now for each n, m, we put $\boldsymbol{B}_{n,m} = \{B_1 \cup B_2 : B_1 \in \boldsymbol{B}_n^1, B_2 \in \boldsymbol{B}_m^2\}$. Then it is easily checked that $\bigcup_{n,m=1}^\infty \boldsymbol{B}_{n,m}$ is a σ-closure preserving quasi-base for X. Hence X is M_2. The M_3 case is similar.

THEOREM 2.8. *If X is a locally finite union of closed M_2- (or M_3-) spaces, then X is M_2 (or M_3).*

Proof. First we apply a theorem of Michael [7, pp. 379–380] and Morita [10] (see Theorem 8.1 of this paper) to get X paracompact. Let X be the union of a locally finite family $\boldsymbol{A}$ of closed M_2- (or M_3-) spaces. Then, for each $x \in X$, there exists an open U_x containing x which intersects only finitely many members of $\boldsymbol{A}$, say $F_1, \cdots, F_n$. Then $x \in U_x \subset \bigcup_{i=1}^n F_i$. But by Lemma 2.7 $\bigcup_{i=1}^n F_i$ is M_2 (or M_3), and then by Theorem 2.3 we see that U_x is M_2 (or M_3). Now, since X is paracompact and locally M_2 (or M_3), we get X to be M_2 (or M_3) by Theorem 2.6, which completes the proof.

Whether Theorem 2.9 is true for M_1-space is unknown.

3. Nagata spaces.

DEFINITION 3.1. A *Nagata* space X is a T_1-space such that for each $x \in X$ there exist sequences of neighborhoods of x, $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$, such that:

(1) for each $x \in X$, $\{U_n(x)\}_{n=1}^{\infty}$ is a local base of neighborhoods of x,

(2) for all $x, y \in X$, $S_n(x) \cap S_n(y) \neq \phi$ implies $x \in U_n(y)$.

The order pair $\langle\{U_n(x)\}_{n=1}^{\infty}, \{S_n(x)\}_{n=1}^{\infty}\rangle$ is said to be a *Nagata structure* for X if and only if, for each x, $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$ are sequences of neighborhoods of x satisfying the above two conditions.

Now having defined Nagata spaces, we get the following relation between a Nagata space and an M_3-space:

THEOREM 3.1. *A topological space is a Nagata space if and only if it is first countable and M_3.*

Proof. Let X be a Nagata space with a Nagata structure $\langle\{U_n(x)\}_{n=1}^{\infty}, \{S_n(x)\}_{n=1}^{\infty}\rangle$. Define $\boldsymbol{P}_n = \{(S_n(x)^0, U_n(x)) : x \in X\}$ for each n. Then obviously $\bigcup_{n=1}^{\infty}\boldsymbol{P}_n$ is a pair-base. To show that each $\boldsymbol{P}_n$ is cushioned, we must show, for any index set A, that $(\cup\{S_n(x_\alpha)^0 : \alpha \in A\})^- \subset \cup\{U_n(x_\alpha) : \alpha \in A\}$. Suppose $y \notin \cup\{U_n(x_\alpha) : \alpha \in A\}$. Then $S_n(y)^0 \cap S_n(x_\alpha)^0 = \phi$ for all α in A. Hence, $S_n(y)^0 \cap (\cup\{S_n(x_\alpha)^0 : \alpha \in A\}) = \phi$ and $y \notin (\cup\{S_n(x_\alpha)^0 : \alpha \in A\})^-$. Thus X is M_3 and first countable.

Now let X be M_3 and first countable. For each $x \in X$, let $\{W_n(x)\}_{n=1}^{\infty}$ be a local base at x. Suppose $\bigcup_{n=1}^{\infty}\boldsymbol{P}_n$ is a σ-cushioned pair-base for X. We can assume that for all n, $(X, X) \in \boldsymbol{P}_n$. For m, n and $x \in X$ define

$$U_{m,n}(x) = \cap\{\bar{P}_2 : W_m(x) \subset P_1,\ P \in \boldsymbol{P}_n\}$$

and

$$S_{m,n}(x) = \cap\{P_1 : W_m(x) \subset P_1,\ P \in \boldsymbol{P}_n\} - \cup\{\bar{P}_1 : x \notin P_2,\ P \in \boldsymbol{P}_n\}\ .$$

We wish to show that $\langle\{U_{m,n}(x)\}_{m,n=1}^{\infty}, \{S_{m,n}(x)\}_{m,n=1}^{\infty}\rangle$ is a Nagata structure for X. Obviously $\{U_{m,n}(x)\}_{m,n=1}^{\infty}$ and $\{S_{m,n}(x)\}_{m,n=1}^{\infty}$ are sequences of neighborhoods of x satisfying condition (1) in Definitition 3.1. To show (2), suppose $y \notin U_{m,n}(x)$. Then there exists a $P \in \boldsymbol{P}_n$ such that $W_m(x) \subset P_1$ and $y \notin \bar{P}_2$. Then, by definition of $S_{m,n}(x)$, we have $S_{m,n}(y) \cap \bar{P}_1 = \phi$. But $S_{m,n}(x) \subset P_1$, so $S_{m,n}(x) \cap S_{m,n}(y) = \phi$, which completes the proof.

Now by virture of Theorem 3.1 and the fact subsets and countable products of first countable spaces are first countable, we obtain the results that: any subspace of a Nagata space is a Nagata space; a countable product of Nagata spaces is Nagata; and in a Nagata space, separable $\longleftrightarrow$ Lindelöf $\longleftrightarrow$ the countable chain condition.

We can also get the following generalization (from X being metric to X being Nagata) of a well known extension theorem of Dugundji [3]:

THEOREM 3.2. *Let A be a closed subset of a Nagata space X and let f be a continuous map from A into a convex subset K of a locally convex topological linear space Y. Then f can be extended to a continuous g from X to K.*

Proof. Let $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)\}_{n=1}^\infty\rangle$ be a Nagata structure for X. Without loss of generality we can suppose that, for $n < m$ and $y \in X$, we have $S_m(y) \subset S_n(y)$, $U_m(y) \subset U_n(y)$, and $S_1(y) = U_1(y) = X$. Now for $x \in X - A$, put $n_x = \max\{n : \text{for some } y \in A, x \in S_n(y)\}$ and $m_x = \min\{n : U_n(x) \cap A = \phi\}$. By the paracompactness of $X - A$, let $\boldsymbol{V}$ be an open locally finite refinement of $\{S_{m_x}(x) : x \in X - A\}$. For each $V \in \boldsymbol{V}$ pick x_V such that $V \subset S_{m_{x_V}}(x_V)$, and pick a_V such that $x_V \in S_{n_{x_V}}(a_v)$. Now let $\{p_V : V \in \boldsymbol{V}\}$ be a partition of unity subordinate to $\boldsymbol{V}$, and define $g: X \to Y$ by

$$g(x) = f(x) \quad \text{for } x \in A$$

and

$$g(x) = \sum_{V \in \boldsymbol{V}} p_V(x) f(a_V) \quad \text{for } x \notin A .$$

Then it can be shown without difficulty that g is the desired extension of f.

4. Some metrization theorems. The following is a recent characterization of metrizability by Nagata [13], which has the dual virtue of being obviously satisfied by a metric space and of easily implying many other known metrization theorems. (The concept of a Nagata space was actually abstracted from this characterization.)

THEOREM 4.1. (Nagata [13]). *A T_1-space X is metrizable if and only if X is a Nagata space with a Nagata structure $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)\}_{n=1}^\infty\rangle$ with the property that $x \in S_n(y)$ implies $S_n(x) \subset U_n(y)$ for all $x, y \in X$.*

The following theorems are consequences of this result:

THEOREM 4.2. *A regular space X is metrizable if and only if X is an M_1-space with σ-closure preserving base $\boldsymbol{B} = \bigcup_{n=1}^\infty \boldsymbol{B}_n$ such that, for each $x \in X$ and each n, $\bigcap\{B : x \in \boldsymbol{B}_n\}$ is neighborhood of x.*

Proof. The sufficiency follows easily from Theorem 1.1. For the necessity, we put, for $x \in X$ and m,

$$U_m(x) = \bigcap\{\bar{B} : x \in B \in \boldsymbol{B}_m\} ,$$

and

$$S_m(x) = \bigcap\{B : x \in B \in \boldsymbol{B}_m\} - \bigcup\{\bar{B} : x \notin \bar{B} \text{ and } B \in \boldsymbol{B}_m\} .$$

Then it is easily checked that $\langle\{U_m(x)\}_{m=1}^{\infty}, \{S_m(x)\}_{m=1}^{\infty}\rangle$ is a Nagata structure for X with the property that $x \in S_n(y)$ implies $S_n(x) \subset U_n(y)$ for all $x, y \in X$. Hence, according to Theorem 4.1, X is metrizable.

COROLLARY 4.3. *A regular space X is metrizable if and only if X has a σ-closure preserving base $\boldsymbol{B} = \bigcup_{n=1}^{\infty}\boldsymbol{B}_n$ where each $\boldsymbol{B}_n$ is point finite.*

Proof. The sufficiency follows from Theorem 1.1 and the necessity from Theorem 4.2.

The above theorem and corollary have analogues for the case of M_2- and M_3-spaces.

An interesting but unsolved problem poses itself here, namely: is an M_1-space with a σ-closure preserving base $\boldsymbol{B} = \bigcup_{n=1}^{\infty}\boldsymbol{B}_n$, where each $\boldsymbol{B}_n$ is point countable, necessarily metrizable?

We also have the following metrization theorem on M_1-spaces:

THEOREM 4.4. (Bing [1]). *A T_1-space X is metrizable if and only if X is an M_1-space with a σ-closure preserving base $\bigcup_{n=1}^{\infty}\boldsymbol{B}_n$ such that, for any $x \in X$ and open set U containing x, there exists an n such that $\phi \neq \bigcup\{B : x \in B \in \boldsymbol{B}_n\} \subset U$.*

We can easily generalize this result to the following:

THEOREM 4.5. *A T_1-space X is metrizable if and only if X is an M_3-space with a σ-cushioned pair-base $\bigcup_{n=1}^{\infty}\boldsymbol{P}_n$ with the property that for each $x \in X$ and open set U containing x, there exists an n such that $\phi \neq \bigcup\{P_1 . x \in P_1, P \in \boldsymbol{P}_n\} \subset U$.*

5. Completeness. According to Čech [2], a Hausdorff space is *topologically complete* if it is a G_δ in some compact Hausdorff space, and a Hausdorff space is *completely metrizable* if it has a compatible complete metric. Čech then proves that a metrizable space is completely metrizable if and only if it is topologically complete. In this section we investigate topologically complete M_i-spaces.

THEOREM 5.1. (Nagata [13]). *A topologically complete Nagata space is completely metrizable.*

Actually Nagata's proof of Theorem 5.1 establishes the following result.

THEOREM 5.2. *Let X be a paracompact topologically complete space, and suppose there exists a sequence of open converings $\{S_n\}_{n=1}^{\infty}$ such that, for every $x, y \in X$, $x \neq y$ implies there exists an m such that $y \notin (\bigcup\{S : x \in S \in \mathbf{S}_m\})^-$. Then X is completely metrizable.*

We can generalize this result by virture of the following lemmas:

LEMMA 5.3. *Let X be a paracompact space. Then, if there exists a sequence of open coverings $\{\mathbf{V}_n\}_{n=1}^{\infty}$ such that $x \neq y$ implies there exists an m such that $y \notin \bigcup\{V : x \in V \in \mathbf{V}_m\}$, then there exists a sequence of open coverings $\{\mathbf{S}_n\}_{n=1}^{\infty}$ such that $x \neq y$ implies there exists an m such that $y \notin (\bigcup\{S : x \in S \in \mathbf{S}_m\})^-$.*

Proof. Let $\mathbf{W}_m$ be an open locally finite refinement of $\mathbf{V}_m$ such that, if $W \in \mathbf{W}_m$, then $\bar{W} \subset$ some $V \in \mathbf{V}_m$. For $V \in \mathbf{V}_m$, define $S_V = \bigcup\{W \in \mathbf{W}_m : \bar{W} \subset V\}$. Let $\mathbf{S}_m = \{S_V : V \in \mathbf{V}_m\}$. Then $\mathbf{S}_m$ is cushioned in $\mathbf{V}_m$ and in particular, if $x \notin \bigcup\{V \in \mathbf{V}_m : y \in V\}$, then $x \notin (\bigcup\{S_V \in \mathbf{S}_m : y \in V\})^-$, and the conclusion of the lemma follows.

LEMMA 5.4. *The diagonal is a G_δ in $X \times X$ if and only if there exists a sequence of open coverings $\{\mathbf{S}_n\}_{n=1}^{\infty}$ of X such that for each $x, y \in X$ $x \neq y$ implies there exists an m such that $y \notin \bigcup\{S : x \in S \in \mathbf{S}_m\}$.*

Proof. Let Δ be the diagonal in $X \times X$. Suppose $\Delta = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open in $X \times X$. For each n, put $\mathbf{S}_n = \{S : S$ open in $X, S \times S \subset G_n\}$. Then if $x \neq y$, there exists an m such that $(x, y) \notin G_m$ and hence $y \notin \bigcup\{S : x \in S \in \mathbf{S}_m\}$.

Now assume we have such a sequence of open coverings $\{\mathbf{S}_n\}_{n=1}^{\infty}$. For each n, put $G_n = \bigcup\{S \times S : S \in \mathbf{S}_n\}$. Then clearly $\Delta = \bigcap_{n=1}^{\infty} G_n$, which completes the proof.

Then obviously we can strengthen Theorem 5.2 to:

THEOREM 5.5. *A paracompact topologically complete space whose diagonal is a G_δ in $X \times X$ is completely metrizable.*

Now we generalize Theorem 5.1. to:

THEOREM 5.6. *A topologically complete M_i-space is completely metrizable.*

Proof. Let X be an M_i-space. Then $X \times X$ is an M_i-space and thus perfectly normal; so the diagonal is a G_δ. Now applying the previous theorem we complete the proof.

COROLLARY 5.7. *A locally compact M_i-space is completely metrizable.*

Proof. It is well known that a locally compact space is open in any Hausdorff space in which it is densely embedded (Kelly [4], p. 163). Hence X is open in $\beta(X)$, the Stone-Čech compactification of X, and, by Theorem 5.6, X is completely metrizable.

Now we proceed to establish a "completeness-like" condition that will make a Nagata space topologically complete.

DEFINITION 5.1. Let X be a Nagata space. Then the Nagata structure $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)\}_{n=1}^\infty\rangle$ is *complete* if, whenever $\{A_n\}_{n=1}^\infty$ is a decreasing sequence of nonempty closed sets such that for every n there exists x_n and k_n such that $A_{k_n} \subset S_n(x_n)$, we have $\bigcap_{n=1}^\infty A_n \neq \phi$.

First we note without proof that:

THEOREM 5.8. *Let X be a Nagata space with Nagata structure $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)\}_{n=1}^\infty\rangle$. Then the following are equivalent:*

(1) $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)]_{n=1}^\infty\rangle$ *is complete.*

(2) *Whenever $\boldsymbol{A}$ is a family of closed sets having the finite intersection property such that for every n, there exists $A_n \in \boldsymbol{A}$ and $x_n \in X$ so that $A_n \subset S_n(x_n)$, then $\bigcap \boldsymbol{A} \neq \phi$.*

(3) *If $\{x_m\}_{m=1}^\infty$ is a sequence such that for any n there exists k_n, y_n such that $k_n \leqq m$ implies $x_m \subset S_n(y_n)$, then $\{x_m\}_{m=1}^\infty$ has a cluster point.*

THEOREM 5.9. *A Nagata space with a complete Nagata structure is completely metrizable.*

Proof. For the proof, we need the concept of the Wallman compactification of a normal space (Wallman [18], Kelly [4, pp. 167–168]). Let X be normal and let $\boldsymbol{F}$ be the family of all closed subsets of X. Define $w(X)$ to be the collection of all subfamilies of $\boldsymbol{F}$ which have the finite intersection property and are maximal with respect to this property. For U open in X, we put $U^+ = \{\boldsymbol{A} \in w(X)$: for some $A \in \boldsymbol{A}, A \subset U\}$. Then $\{U^+ : U \text{ open in } X\}$ is a base for some topology τ. Then $\langle w(X), \tau\rangle$ is called the *Wallman compactification* of X. Then $w(X)$ is compact Hausdorff and X is densely embedded in $w(X)$ by the homeomorphism $\phi(x) = \{A \in \boldsymbol{F} : x \in A\}$.

To show that X is completely metrizable we need only show that X is a G_δ in $w(X)$. Let $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)\}_{n=1}^\infty\rangle$ be the complete Nagata structure for X. For each n, put $G_n = \cup\{S_n(x)^+ : x \in X\}$. Then G_n is open and obviously $\phi(X) \subset \bigcap_{n=1}^\infty G_n$. Now suppose $\boldsymbol{A} \in \bigcap_{n=1}^\infty G_n$. Then for each n there exists an $x_n \in X$ such that $\boldsymbol{A} \in S_n(x_n)^+$, which means that for each n there exists $x_n \in X$ and $A_n \in \boldsymbol{A}$ so that $A_n \subset S_n(x_n)$. Hence by completeness $\bigcap \boldsymbol{A} \neq \phi$. So let $x \in \bigcap \boldsymbol{A}$, then since $\boldsymbol{A}$ is maximal with respect to the finite intersection property we must have $\boldsymbol{A} = \phi(x) \in \phi(X)$. Hence, $\phi(X) = \bigcap_{n=1}^\infty G_n$, showing that X is a G_δ in $w(X)$.

6. Semi-metric spaces.

DEFINITION 6.1. Let d be a real-valued nonnegative function defined on $X \times X$. Then d is a *semi-metric* for X provided:

(1) $$d(x, y) = 0 \text{ if and only if } x = y ,$$

(2) $$d(x, y) = d(y, x) \text{ for all } x, y \in X .$$

If d is a semi-metric for X, the semi-metric topology is that determined by: p is a limit point of $A \subset X$ if and only if $\inf \{d(p, x) : x \in A\} = 0$. A topological space $\langle X, \tau \rangle$ is *semi-metrizable* if and only if there is a semi-metric d such that the semi-metric topology agrees with τ.

We can characterize semi-metric spaces as follows:

THEOREM 6.1. *A Hausdorff space X is semi-metrizable if and only if for all $x \in X$, there exists sequences of neighborhoods $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$ such that $\{U_n(x)\}_{n=1}^{\infty}$ is a nested local base of neighborhoods of x, and for each n and $x, y \in X$, $S_n(x) \subset U_n(x)$ and $y \in S_n(x)$ implies $x \in U_n(y)$.*

Proof. For the sufficiency, put $S_n(x) = U_n(x) = \{y : d(x, y) \leqq 1/n\}$. For the necessity, define $d(x, y) = \inf \{1/n : x \in U_n(y) \text{ and } y \in U_n(x)\}$ where we assume $U_1(x) = X$ for all $x \in X$.

Now by virtue of the preceding characterization of semi-metrizability, we obviously have:

THEOREM 6.2. *A Nagata space is semi-metrizable.*

McAuley [5] has given an example of a regular separable semi-metric space X which is not hereditarily sparable (that is, subsets are not necessarily separable). It follows by Theorems 2.3 and 2.5 that X is not a Nagata space. In fact, it can be shown that X is not even paracompact. An interesting unsolved problem is whether a paracompact (or even a regular Lindelöf) semi-metric space must be a Nagata space.

McAuley [5] has defined a semi-metric space to be *strongly-complete* if, whenever $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty closed sets such that for every n there exists k_n and x_n such that $A_{k_n} \subset \{y : d(x_n, y) \leqq 1/n\}$, then we have $\bigcap_{n=1}^{\infty} A_{k_n} \neq \phi$. (Theorem 5.8 has an analogue for semi-metric spaces). McAuley has proved the following result concerning strongly complete semi-metric spaces:

THEOREM 6.3. (McAuley [5]). *A regular, hereditarily separable, strongly complete semi-metric space is metrizable.*

The following two theorems, taken together, clarify and improve the above theorem of McAuley.

THEOREM 6.4. *A regular, hereditarily separable, semi-metric space is hereditarily Lindelöf (hence paracompact).*

Proof. Let U be an open cover of X. For each $x \in X$, there exists n_x and $U_x \in U$ such that $S_{n_x}(x) = \{y : d(x, y) < 1/n_x\} \subset U_x$. Put $A_n = \{x \in X : n_x = n\}$. Then A_n has a separable subset $\{d_n^m\}_{m=1}^{\infty}$ and it follows that $A_n \subset \bigcup_{m=1}^{\infty} S_n(d_n^m)$. Now choose $U_n^m \in U$ such that $S_n(d_n^m) \subset U_n^m$. Then

$$X = \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n,m=1}^{\infty} S_n(d_n^m) \subset \bigcup_{n,m=1}^{\infty} U_n^m .$$

So $\{U_n^m\}_{n,m=1}^{\infty}$ is a countable subcover of U. So X is Lindelöf and hence normal, but a normal semi-metric space is easily seen to be perfectly normal, and a perfectly normal Lindelöf space is easily seen to be hereditarily Lindelöf. So we conclude that X is hereditarily Lindelöf and hence paracompact, which completes the proof.

THEOREM 6.5. *A paracompact, strongly complete semi-metric space is completely metrizable.*

Proof. Exactly analogously to the proof of Theorem 5.9 we show that X is a G_δ in $w(X)$. Then we apply Lemma 5.4 and Theorem 5.5, where we take $S_m = \{(S_m(x))^0 : x \in X\}$ and $S_m(x) = \{y : d(x, y) < 1/m\}$, which completes the proof.

7. Closed continuous images.

We have the following theorem about closed continuous images of metric spaces:

THEOREM 7.1. (Stone [17], Morita and Hanai [11]). *Let f be a closed continuous map of a metric space X onto a topological space Y. Then the following are equivalent:*

(1) *Y is first countable,*
(2) *for each $y \in Y$, the boundary of $f^{-1}(y)$, $\partial f^{-1}(y)$, is compact,*
(3) *Y is metrizable.*

A special case of a closed continuous image of a space X is X/A, the quotient space of X formed by identifying the points of a closed subset A. Here, the natural map is clearly closed and continuous. Then, according to Theorem 7.1, if X is a metric space and A is a closed subset of X with a non-compact boundary, then X/A is not metrizable.

We have the following partial analogue to Theorem 7.1:

THEOREM 7.2. *Let X be an M_2- (or M_3-) space and f a closed continuous function from X onto any space Y. Then*

(1) *if Y is first countable, then for all $y \in Y, \partial f^{-1}(y)$ is compact,*
(2) *if for all $y \in Y, \partial f^{-1}(y)$ is compact, then Y is M_2 (or M_3).*

Proof. The proof of (1) is somewhat similar to Stone's proof of (1) $\rightarrow$ (2) in Theorem 7.1. To prove (2) for the M_2-case let $\bigcup_{n=1}^{\infty}\boldsymbol{B}_n$ be a σ-closure preserving quasi-base for X. Then $\bigcup_{n=1}^{\infty}A_n$ becomes a σ-closure preserving quasi-base for Y, where $\boldsymbol{A}_n = \{f(\bigcup_{i=1}^{k}A_i) : A_1, \cdots, A_k \in \boldsymbol{B}_n\}$. The M_3-case is similar.

The converse of (1) is easily seen to be false by taking the identity map from a non-first countable M_2- (or M_3-) space onto itself. Also, Example 9.2 shows that the converse of (2) is false. It is unknown whether Theorem 7.2 is true for M_1-spaces.

It is also unsolved whether an arbitrary closed continuous image of an M_i-space is again M_i. However we can obtain the partial result that the quotient space of an M_2- (or M_3-) space with respect to a closed subset is again M_2 (or M_3).

For the M_2 case this result would follow if every closed subset A of X had a "local σ-closure preserving quasi-base" in the sense that there exists a σ-closure preserving family $\boldsymbol{V}$ such that for every open U containing A, $A \subset V^0 \subset V \subset U$ for some $V \in \boldsymbol{V}$. For then, if $\boldsymbol{B}$ were a σ-closure preserving quasi-base for X, the image under the natural map of the family $\boldsymbol{V} \cup \{B \in \boldsymbol{B} : \bar{B} \cap A = \phi\}$ would be a σ-closure preserving quasi-base for X/A. As it turns out, we have the stronger result that every closed subset has a "local closure preserving quasi-base" as follows:

LEMMA 7.3. *Let A be a closed subset of an M_2-space X. Then there exists a closure preserving family $\boldsymbol{V}$ of neighborhoods of A such that for every open U containing A, $A \subset V^0 \subset V \subset U$ for some $V \in \boldsymbol{V}$.*

Proof. Let $\boldsymbol{B} = \bigcup_{n=1}^{\infty}\boldsymbol{B}_n$ be a σ-closure preserving quasi-base for X. Without loss of generality we can assume that the members of B are closed and $\boldsymbol{B}_n \subset \boldsymbol{B}_m$ for $n < m$. For each $B \in \boldsymbol{B}_n$ we put

$$R(B, n) = B - \bigcup\{W^0 : A \cap W = \phi,\ W \in \boldsymbol{B}_n\}\ .$$

Now let $\{S_\alpha : \alpha \in E\}$ be the family of all subcollections of $\boldsymbol{B}$. For each $\alpha \in E$ and n, we put

$$V_{\alpha,n} = \bigcup\{R(B, n) : B \in (\boldsymbol{S}_\alpha \cap \boldsymbol{B}_n)\}$$
$$V_\alpha = \bigcup_{n=1}^{\infty} V_{\alpha,n} \text{ and } D = \{\alpha \in E : A \subset V_\alpha\}\ .$$

To show $\boldsymbol{V} = \{V_\alpha : \alpha \in D\}$ is closure preserving, let $C \subset D$ and suppose $x \notin \bigcup\{\bar{V}_\alpha : \alpha \in C\}$. Then clearly $x \notin A$; so let k be the least integer for which there exists a $B \in \boldsymbol{B}_{k+1}$ such that $x \in B^0$ and $B \cap A \neq \phi$. Then we have $V_{\alpha n} \cap B^0 = \phi$ for every $n > k$ and $\alpha \in C$. Hence

$x \notin (\bigcup\{V_{\alpha,n} : n > k, \alpha \in C\})^-$. If $k \geqq 1$ (otherwise we are finished), then we also have $x \notin \cup\{W^0 : A \cap W = \phi, W \in \boldsymbol{B}_k\}$. From the facts that $x \notin \bigcup\{W^0 : A \cap W = \phi, W \in \boldsymbol{B}_k\}$ and $x \notin \bigcup\{R(B, k) : B \in (\boldsymbol{S}_\alpha \cap \boldsymbol{B}_k)\}$ it follows that $x \notin \bigcup(\boldsymbol{S}_\alpha \cap \boldsymbol{B}_k)$. Since

$$(\bigcup\{V_{\alpha m} : m \leqq k, \alpha \in C\})^- \subset (\bigcup(\boldsymbol{S}_\alpha \cap \boldsymbol{B}_k))^- = \bigcup(\boldsymbol{S}_\alpha \cap \boldsymbol{B}_k)$$

(because $\boldsymbol{B}_k$ is closure preserving), we have that $x \notin (\bigcup\{V_{\alpha,n} : n \leqq k, \alpha \in C\})^-$. Hence $x \notin (\bigcup\{V_\alpha : \alpha \in C\})^-$.

Finally, suppose U is an open set containing A. For each $x \in A$ there exists n_x and $B_x \in \boldsymbol{B}_{n_x}$ such that $x \in B_x^0 \subset B_x \subset U$. Then x is in the open set $B_x^0 - \bigcup\{W : x \notin W, W \in \boldsymbol{B}_{n_x}\}$ which is included in $R(B_x, n_x)^0$. Hence $x \in R(B_x, n_x)^0 \subset R(B_x, n_x) \subset U$. So putting $\boldsymbol{S}_\alpha = \{B_x : x \in A\}$ we clearly get $A \subset V_\alpha^0 \subset V_\alpha \subset U$ with $\alpha \in D$, which completes the proof.

Lemma 7.4 has an analogue for M_3-spaces. Now by virtue of the remarks preceding Lemma 7.3 we clearly obtain:

THEOREM 7.4. *Let X be an M_2- (or M_3-) space and A a closed subset of X. Then X/A is M_2 (or M_3).*

It is unknown whether the above theorem is true for M_1-spaces. However, we can get X/A to be M_1 if X is metrizable, as follows:

LEMMA 7.5. *Let A be a closed subset of the metric space X. Then there exists a closure preserving family $\boldsymbol{V}$ of open sets such that for every open U containing A, $A \subset V \subset U$ for some $V \in \boldsymbol{V}$.*

Proof. Let $\boldsymbol{B} = \bigcup_{n=1}^\infty \boldsymbol{B}_n$ be a σ-locally finite base for X such that $\boldsymbol{B}_n \subset \boldsymbol{B}_m$ for $n < m$. For each n put

$$A_n = \{y \in X : \text{dist}\,(y,A) < 1/n\} \text{ and } \boldsymbol{A}_n = \{B \cap A_n : B \in \boldsymbol{B}_n\}\ .$$

Then each $\boldsymbol{A}_n$ is locally finite. Let $\{\boldsymbol{W}_\alpha : \alpha \in D\}$ be the family of all subcollections of $\bigcup_{n=1}^\infty \boldsymbol{A}_n$ which cover A, and put $\boldsymbol{V} = \{V_\alpha : V_\alpha = \bigcup \boldsymbol{W}_\alpha, \alpha \in D\}$. Then obviously for every open U containing A there exists $\alpha \in D$ such that $A \subset V_\alpha \subset U$. Now consider any $C \subset D$ and suppose $x \notin \bigcup\{\bar{V}_\alpha : \alpha \in C\}$. Then $x \notin A$ and there exists a k such that $x \notin \bar{A}_k$; hence $(X - \bar{A}_k) \cap W = \phi$ for $W \in \boldsymbol{A}_m \cap \boldsymbol{W}_\alpha$ with $k \leqq m$ and $\alpha \in C$. Since $\bigcup_{i=1}^{k-1} \boldsymbol{A}_i$ is closure preserving, it follows that $x \in (\bigcup\{W \in \boldsymbol{A}_m \cap \boldsymbol{W}_\alpha : m < k, \alpha \in C\})^-$. Then we get $x \notin (\bigcup\{V_\alpha : \alpha \in C\})^-$, which completes the proof.

Now we obviously obtain the following:

THEOREM 7.6. *Let X be a metric space and A a closed subset of X. Then X/A is M_1.*

According to Lemma 7.3 every point in an M_2-space has a "local closure preserving quasi-base.'' It is unsolved, however, if every point in an M_1-space has a ''local closure preserving base'' (that is, an open local base which is closure preserving). Nevertheless, we can establish the following negative result:

PROPOSITION 7.7. Suppose there exists an M_1-space X with some point p at which there does not exist a closure preserving open local base. Then

(1) there exists an M_2-space which is not M_1,

(2) there exists an M_1-space Y with a closed subset A such that Y/A is not M_1.

Proof. Let $Y = \bigcup_{n=1}^{\infty} X_n$ where $n \neq m$ implies $X_n \cap X_m = \phi$ and each X_n is homeomorphic to X by a map i_n. Topologize Y by: O is open $\longleftrightarrow O \cap X_n$ is open in X_n for all n. Let $p_n = i_n(p)$ and $A = \{y \in Y : y = p_n$ for some $n\}$. Let i be the natural map from Y onto Y/A. Then clearly A is closed and Y is M_2; hence Y/A is M_2. Now suppose Y/A has a σ-closure preserving base $\boldsymbol{B} = \bigcup_{n=1}^{\infty} \boldsymbol{B}_n$. Then for each n, $\{i^{-1}(B) \cap X_n : A \in B \in \boldsymbol{B}_n\}$ is closure preserving in X_n. Hence, there exists an open V_n in X_n so that $p_n \in V_n$ and $A \in B \in \boldsymbol{B}_n$ implies $(i^{-1}(B) \cap X_n) \not\subset V_n$. Now put $V = \bigcup_{n=1}^{\infty} V_n$. Since $\boldsymbol{B}$ is a base there exists some B in some $\boldsymbol{B}_k$ such that $A \in B \subset i(V)$, whence $(i^{-1}(B) \cap X_k) \subset V_k$, which is a contradiction. Hence, Y/A is M_2 but not M_1.

8. The Topology of chunk-complexes. A *chunk-complex* is a topological space $\langle K, \tau \rangle$ having a family $\boldsymbol{K}$ of closed subsets, called *chunks*, such that

(1) $\bigcup \boldsymbol{K} = K$,

(2) for $S, T \in \boldsymbol{K}$, either $S \cap T = \phi$ or $S \cap T \in \boldsymbol{K}$,

(3) for $S \in \boldsymbol{K}$, $\{T \in \boldsymbol{K} : T \subset S\}$ is finite,

(4) each $S \in \boldsymbol{K}$ is a compact metric space $\langle S, \rho_S \rangle$,

(5) $U \in \tau$ if and only if for every $S \in \boldsymbol{K}$, $S \cap U$ is open in $\langle S, \rho_S \rangle$.

If $\boldsymbol{B}$ is a collection of closed subsets of a space X, then $\boldsymbol{B}$ *dominates* X provided that, for every subfamily $\boldsymbol{A}$ of $\boldsymbol{B}$, if $C \subset \bigcup \boldsymbol{A}$ and $A \cap C$ is closed in A for all $A \in \boldsymbol{A}$, then C is closed in X.

THEOREM 8.1. (Michael [7, pp. 379–380], Morita [10]). *If X is dominated by a collection of paracompact (or perfectly normal) subsets, then X is paracompact (or perfectly normal).*

Using Theorem 8.1, it is easy to show that.

LEMMA 8.2. *A chunk-complex is dominated by the set of its chunks, and hence is paracompact and perfectly normal.*

In this section we establish the stronger result that each chunk-complex is an M_1-space.

For the proof we establish the following notation: For $S \in \boldsymbol{K}$ define $\Delta(S) = \{T \in \boldsymbol{K} : T \subset S,\ T \neq S\}$. Define $\boldsymbol{K}_0 = \{S \in \boldsymbol{K} : \Delta(S) = \phi\}$ and, assuming $\boldsymbol{K}_m$ has been defined for $0 \leqq m < n$, we define

$$\boldsymbol{K}_n = \left\{S \in \boldsymbol{K} : \Delta(S) \subset \bigcup_{i=0}^{n-1} \boldsymbol{K}_i\right\} - \bigcup_{i=0}^{n-1} \boldsymbol{K}_i \ .$$

Then $\bigcup_{n=1}^{\infty} \boldsymbol{K}_n = \boldsymbol{K}$, by induction on the number of subchunks. For $S \in \boldsymbol{K}$ put $\partial S = \bigcup(\Delta(S))$, $S^0 = S - \partial S$, and $\boldsymbol{A}_S = \{T \in \boldsymbol{K} : S \subset T\}$. Then obviously $\bigcup\{S^0 : S \in \boldsymbol{K}\} = K$. Let N be the set of nonnegative integers and $M = \{1/n : n \in N - \{0\}\}$.

THEOREM 8.3. *A chunk-complex is an M_1-space.*

Proof. Let $\langle K, \tau \rangle$ be a chunk-complex with a set of chunks $\boldsymbol{K}$. First we observe that for each $n \in N$, $P \in \boldsymbol{K}_n$, there exists a countable family $\boldsymbol{B}(P) = \{P_m : m \in N\}$ of open sets in P^0 forming a base for points in P^0 so that $\bar{P}_m \in P^0$ for all $m \in N$. Fix $n \in N$, $P \in \boldsymbol{K}_n$ and $B \in \boldsymbol{B}(P)$. Let $g\colon \boldsymbol{A}_P \to M$. Then we define a candidate B_g for our base as follows: By normality, let W be an open set containing $\bar{B}$ and such that $\bar{W} \cap (\bigcup\{T \in \boldsymbol{K} : T \cap P^0 = \phi\}) = \phi$. Now, by induction, for any $T \in \boldsymbol{K}_n \cap \boldsymbol{A}_P$ we necessarily have $T = P$ and we define $B_g^P = B$ and $\dot{B}_g^P = \phi$. Now assume we have defined B_g^S for all $S \in \boldsymbol{K}_{n+k} \cap \boldsymbol{A}_P$ with $k < m$. Then for any $T \in \boldsymbol{K}_{n+m} \cap \boldsymbol{A}_P$ we put

$$\dot{B}_g^T = \bigcup\{B_g^S : S \in \Delta(T) \cap \boldsymbol{A}_P\}$$

and

$$B_g^T = W \cap \{y \in T : \rho_T(\dot{B}_g^T, y) < \min\,[g(T), \rho_T(y, \partial T \dot{-} \dot{B}_g^T)]\} \ .$$

Finally we put

$$B_g = \bigcup\{B_g^T : T \in \boldsymbol{A}_P\} \ .$$

We note that for all $T \in \boldsymbol{A}_P$ we have $(B_g^T \cap \partial T) \subset \dot{B}_g^T$, $((B_g^T)^- \cap \partial T) \subset (\dot{B}_g^T)^-$, and if $S \notin \boldsymbol{A}_P$, $(B_g^T)^- \cap S = \phi$.

Now we need to establish the following lemma:

LEMMA 8.4. *For all $P \in \boldsymbol{K}_n$ and $S, T \in \bigcup_{k=0}^{m} \boldsymbol{K}_{n+k} \cap \boldsymbol{A}_P$,*

(a) *$\dot{B}_g^S$ is open in ∂S,*

(b) *$\dot{B}_g^S \subset B_g^S$,*

(c) *$(B_g^S \cap T) \subset B_g^T$,*

(d) *$((B_g^S)^- \cap T) \subset (B_g^T)^-$.*

Proof. By induction on m: if $m = 0$, then $S = T = P$ and all conditions are obviously satisfied. Now assume that (a), (b), (c) and (d) hold for all $k < m$, and let us prove this for m.

(a) Applying the induction hypothesis on (c) we get for all $R, Q \in \Delta(S) \cap \boldsymbol{A}_P$ that $(B_g^R \cap Q) \subset B_g^Q$. Hence

$$\partial S - \dot{B}_g^S = \partial S - \bigcup\{B_g^T \in \Delta(S)\} = \bigcup\{R - B_g^R : R \in \Delta(S)\} .$$

But each $R - B_g^R$ is closed in R which is in turn closed in ∂S. Hence $\partial S - \dot{B}_g^S$ is closed in ∂S for $S \in \boldsymbol{K}_{n+m}$.

(b) Then if $y \in \dot{B}_g^S$, $\rho_S(y, \dot{B}_g^S) = 0$ and $\rho_S(y, S - \dot{B}_g^S) > 0$, so $y \in B_g^S$. Hence we have $\dot{B}_g^S \subset B_g^S$ for all $S \in \boldsymbol{K}_{n+m}$.

(c) If $S \not\subset T$, then $(B_g^S \cap T) \subset (B_g^S \cap (T \cap S)) \subset (B_g^S \cap \partial S) \subset \dot{B}_g^S$. So if $x \in B_g^S \cap T$, then $x \in$ some B_g^R with $R \in \Delta(S)$, and then $x \in (B_g^R \cap (T \cap S)) \subset B^{T \cap S}$ by the induction hypothesis on (c). If $S \cap T = T$, then $B_g^{S \cap T} = B_g^T$. If $S \cap T \neq T$, then $S \cap T \in \Delta(T)$, and by (b) we have $B_g^{S \cap T} \subset B_g^T$. Hence if $S \not\subset T$, $(B_g^S \cap T) \subset B_g^T$. If $S \subset T$, then $B_g^S \subset B_g^T$ by (b). Hence $(B_g^S \cap T) \subset B_g^T$ for all $S, T \in \boldsymbol{K}_{n+m} \cap \boldsymbol{A}_P$.

(d) The proof of (d) is exactly similar to (c) above; but here we use the fact that $((B_g^S)^- \cap S) \subset (\dot{B}_g^S)^-$.

This completes the proof of Lemma 8.4.

For $m, n \in N$, $P \in \boldsymbol{K}_n$, define $\boldsymbol{V}_P^m = \{(P_m)_g : g : \boldsymbol{A}_P \to M\}$ and $\boldsymbol{U}_n^m = \bigcup\{\boldsymbol{V}_P^m : P \in \boldsymbol{K}_n\}$. Now we will show that

(a) each $(P_m)_g$ is open,

(b) $\bigcup\{\boldsymbol{V}_P^m : m \in N\}$ is a base for points in P^0,

(c) each $\boldsymbol{V}_P^m$ is closure preserving,

(d) each $\boldsymbol{U}_n^m$ is closure preserving.

Then, since $\bigcup\{P^0 : P \in \bigcup_{n=1}^\infty \boldsymbol{K}_n\} = K$, $\boldsymbol{B} = \bigcup\{\boldsymbol{U}_n^m : n, m \in N\}$ will be the desired σ-closure preserving base for K.

(a) each $(P_m)_g$ is open. Let $P_m = B$. It then suffices to show that for every $S \in \boldsymbol{A}_p$, $S \cap B_g$ is open in S. But by Lemma 8.4, $S \cap B_g = \bigcup\{S \cap B_g^T : T \in \boldsymbol{A}_P\} = S \cap B_g^S$, which is open in S by construction.

(b) $\bigcup\{\boldsymbol{V}_P^m : m \in N\}$ is a base for points in P^0. Let $P \in \boldsymbol{K}_n$, $x \in P^0$, and U be on open set containing x. Choose $B \in \boldsymbol{B}(P)$ such that $x \in B \subset \bar{B} \subset (U \cap P^0)$. We want to find $g : \boldsymbol{A}_P \to M$ so that $x \in B_g \subset U$. By induction on m, we define $g(T)$ for $T \in \boldsymbol{K}_{n+m} \cap \boldsymbol{A}_P$ so that $(B_g^T)^- \subset U$. For $m = 0$ we have $T = P$ and $(B_g^T)^- = \bar{B} \subset (P \cap U)$ for any $g : \boldsymbol{A}_P \to M$, so put $g(P) = 1$. Now assume we have defined $g(S)$ for every $S \in \boldsymbol{K}_{n+k} \cap \boldsymbol{A}_P$ with $k < m$ so that $(B_g^S)^- \subset U$. Let $T \in \boldsymbol{K}_{n+m} \cap \boldsymbol{A}_P$. Then, by the induction hypothesis, $(\dot{B}_g^T)^- = \bigcup\{(B_g^S)^- : S \in \Delta(T)\} \subset (U \cap T)$. So by the compactness of T there exists $\beta \in M$ so that $\{y \in T : \rho_T(y, \dot{B}_g^T) \leqq \beta\} \subset (T \cap U)$. Then put $g(T) = \beta$. Then we have

$$\begin{aligned}(B_g^T)^- &= (W \cap \{y \in T : \rho_T(y, \dot{B}_g^T) < \min\,[g(T), \rho_T(y, \partial T - \dot{B}_g^T)]\})^- \\ &\subset \{y \in T : \rho_T(y, \dot{B}_g^T) \leqq g(T)\} \subset (T \cap U) .\end{aligned}$$

Hence $x \in B_g = \bigcup\{B_g^T : T \in \boldsymbol{A}_P\} \subset U$, with $B_g \in \boldsymbol{V}_P^m$ and $B = P_m$.

(c) each V_P^m is closure-preserving. First we need the following result:

LEMMA 8.5. (Michael [8]). *Let* $D = \prod_{i=1}^j M_i$, *where* $M_i = M$ *for all* i. *For all* $x, y \in D$, *define* $x \leqq y$ *if and only if* $x_i \leqq y_i$ *for all* i. *Then* $\langle D, \leqq \rangle$ *is a partially ordered set with the property that, for each* $C \subset D$, *there exist* $c_1, \cdots c_m \in C$ *so that, for all* $c \in C$, *there exists* c_k $(1 \leqq k \leqq m)$ *such that* $c \leqq c_k$.

Now let $\{B_g : g \in G\}$ be a subfamily of $\boldsymbol{V}_P^m$ with $P_m = B$. For every $T \in \boldsymbol{A}_P$ we must show $T \cap (\bigcup\{\bar{B}_g : g \in G\})$ is closed. First we show that $\bar{B}_g = \bigcup\{(B_g^S)^- : S \in \boldsymbol{A}_P\}$. For this it suffices to show, for every $T \in \boldsymbol{A}_P$, that $T \cap (\bigcup\{B_g^S)^- : S \in \boldsymbol{A}_P\})$ is closed. But by part (d) of Lemma 8.4, $T \cap (\bigcup\{(B_g^S)^- : S \in \boldsymbol{A}_P\}) = (B_g^T)^-$. Then

$$T \cap (\bigcup\{\bar{B}_g : g \in G\}) = T \cap (\bigcup\{(B_g^S)^- : g \in G, S \in \boldsymbol{A}_P\}) = \bigcup\{(B_g^T)^- : g \in G\} .$$

Now we apply Lemma 8.5 above to the subset $A = \{(g(S_1), \cdots, g(S_k)) : g \in G\}$ of the partially ordered set $\prod_{i=1}^k M_i$, where $\{S_1, \cdots, S_k\} = \Delta(T) \cap \boldsymbol{A}_P$. Notice that, if $g(S_i) \leqq h(S_i)$ for all i with $g, h \in G$, then we have $(B_g^T)^- \subset (B_h^T)^-$. Hence by Lemma 8.5 we get $g_1, \cdots, g_n \in G$ such that

$$T \cap (\bigcup\{\bar{B}_g : g \in G\}) = \bigcup\{(B_g^T)^- : g \in G\} = \bigcup_{i=1}^n\{(B_{g_i}^T)^-\} ,$$

which is closed.

(d) each $\boldsymbol{U}_n^m$ is closure preserving. Let $\boldsymbol{U}$ be a subfamily of $\boldsymbol{U}_n^m$. Then we can express $\boldsymbol{U}$ as $\{(P_m)_g : g \in G_P, P \in \boldsymbol{P}\}$ for some $\boldsymbol{P} \subset \boldsymbol{K}_n$ and $G_P \subset \{g : g : \boldsymbol{A}_P \rightarrow M\}$. Let $T \in \boldsymbol{K}$. If $P \not\subset T$, then $T \notin \boldsymbol{A}_P$ and $((P_m)_g)^- \cap T = \phi$. But there are only finitely many $P \in \boldsymbol{P}$ contained in T. Hence there exist $P^1, \cdots, P^k \in \boldsymbol{P}$ so that

$$T \cap (\bigcup\{\bar{B}_g : B_g \in \boldsymbol{U}\}) = T \cap (\bigcup\{((P_m^i)g)^- : 1 \leqq i \leqq k, g \in G_{Pi}\})$$

which is closed by part (c) above.

This completes the proof of the theorem.

COROLLARY 8.6. *A CW-complex* (Whitehead [19]) *is an* M_1*-space.*

Proof. Let $\langle K, \tau \rangle$ be a CW-complex. Then the family of finite subcomplexes is a family of chunks, whence the CW-complex $\langle K, \tau \rangle$ is M_1. (See Whitehead [19] for terminology).

COROLLARY 8.7. *A countable product of CW-complexes is an* M_1*-space; hence; both paracompact and perfectly normal.*

Proof. Apply Theorems 2.2 and 2.4 and Corollary 8.6.

9. Some examples. In the sequel, R will denote the real numbers and N the natural numbers. We will also use the notation $\langle x, y\rangle$ for the point $(x, y) \in R \times R$ to distinguish it from (s, t) which will mean the open interval $\{x \in R : s < x < t\}$ and $[s, t]$ which will be the closed interval $\{x \in R : s \leqq x \leqq t\}$.

EXAMPLE 9.1. A non-metrizable first countable M_1-space.

Let R' be the rational numbers. For $x \in R$, put $L_x = \{\langle x, y\rangle : \langle x, y\rangle \in R \times R, 0 < y\}$ and $X = R \cup (\bigcup\{L_x : x \in R\})$. Then we will define a base for X as follows: For $s, t \in R'$ and $z = \langle x, w\rangle \in L_x$ such that $0 < s < w < t$ we put $\bigcup^x_{s,t}(z) = \{\langle x, y\rangle : S < y < t\}$ and let $\boldsymbol{U}$ be the set of all such $U^x_{s,t}(z)$. For $r, s, t \in R'$ and $z \in R$ such that $s < z < t$ and $r > 0$, we put

$$V_{r,s,t}(z) = (s, t) \cup (\bigcup\{\langle w, y\rangle : 0 < y < r, w \in (s, t) - \{z\}\}) ,$$

and let $\boldsymbol{V}$ be the set of all such $V_{r,s,t}(z)$. Now put $\boldsymbol{B} = \boldsymbol{U} \cup \boldsymbol{V}$. Then it can be easily shown that $\boldsymbol{B}$ is a σ-closure preserving base making X into a non-metrizable first countable M_1-space.

The following example is more powerful than Example 9.1. But here the proof of M_1-ness, which is due to Jun-iti Nagata, is far from being straightforward. (The space of the example seems to have first appeared in McAuley [5]; Nagata [13] gives it without proof of M_1-ness as an example of a non-metrizable, separable Nagata space.)

EXAMPLE 9.2. [Nagata]. A non-metrizable, separable, first countable M_1-space.

Let $X = \{\langle x, y\rangle : \langle x, y\rangle \in R \times R, 0 < x < 1, 0 \leqq y\}$. Clearly $X - (0, 1)$, as a subset of $R \times R$, has a σ-closure preserving base $\boldsymbol{B}$. For $n \in N$ and $\langle p, 0\rangle \in X$, we define

$$U_n(p) = \{p\} \cup \{\langle x, y\rangle \in X : y < n - (n^2 - (x - p)^2)^{1/2}, |x - p| < 1/n\} .$$

Then $\boldsymbol{B} \cup \{U_n(p) : n \in N, \langle p, 0\rangle \in X\}$ is a base which clearly generates a regular topology. Obviously X is separable, first countable, and not second countable; hence X is not metrizable.

To show the existence of a σ-closure preserving base for X, it suffices to show one for points in $(0, 1)$. For $m, q \in N, m < q$, and $0 \leqq k \leqq 2^{m+1} - 2$, we define

$$W_{q,m,k} = \{\langle x, y\rangle : (k)2^{-m-1} < x < (k + 2)2^{-m-1}, 0 < y \leqq 2^{-q}\} .$$

Now consider any $U_n(p)$. Then we can choose $m, k \in N$ so that

$$(k)2^{-m-1} < n^{-1} + p \quad\text{and}\quad (k - 4)2^{-m-1} \leqq p < (k - 3)2^{-m-1} .$$

For this m, k, we put

$$q = \min\{j : W_{j,m,k-2} \subset U_n(p)\} ,$$

$$I_1 = W_{q,m,k-2} ,$$
$$a_1 = (k)2^{-m-1} ,$$
$$a_2 = (k-2)2^{-m-1} ,$$
$$b_1 = 2^{-q} .$$

Now for each $i \in N$, we choose k_i so that

$$(k_i - 4)2^{-m-i-1} \leqq p < (k_i - 3)2^{-m-i-1} .$$

Then we put

$$q_i = \min \{j : W_{i,m+i,k_i-2} \subset U_n(p)\} ,$$
$$I_{i+1} = W_{q_i,m+i,k_i-2} ,$$
$$a_{i+2} = (k_i - 2)2^{-m-i-1} ,$$
$$b_{i+1} = 2^{-q_i} .$$

Now it follows that for each $i, j \in N$, $i < j$ implies $a_j < a_i$ and $b_j < b_i$, and obviously $b_i \to 0$ and $a_i \to p$.

We also choose $m', k' \in N$ such that

$$p - n^{-1} < (k')2^{-m'-1} \quad \text{and} \quad (k'+3)2^{-m'-1} < p \leqq (k'+4)2^{-m'-1} .$$

Then we put

$$q' = \min \{j : W_{j,m',k'} \subset U_n(p)\} ,$$
$$I_1' = W_{q',m',k'} ,$$
$$a_1' = (k')2^{-m'-1} ,$$
$$a_2' = (k'+2)2^{-m'-1} ,$$
$$b_1' = 2^{-q'} .$$

Now for $i \in N$, we choose k_i' so that

$$(k_i' + 3)2^{-m'-i-1} < p \leqq (k_i' + 4)2^{-m'-i-1} .$$

Then put

$$q_i' = \min \{j : W_{j,m'+i,k_i'} \subset U_n(p)\} ,$$
$$I_{i+1}' = W_{q_i',m+i,k_i'} ,$$
$$a_{i+2}' = (k_i' + 2)2^{-m'-i-1} ,$$
$$b_{i+1}' = 2^{-q_i'} .$$

Then for each $i, j \in N$, $i < j$ implies $a_i' < a_j'$ and $b_i' < b_j'$, and obviously $b_i' \to 0$ and $a_i' \to p$.

Now putting

$$N_n(p) = \left(\left(\left(\bigcup_{j=1}^{\infty} I_j\right) \cup \left(\bigcup_{j=1}^{\infty} I_j'\right)\right)^-\right)^0 ,$$

it can be shown that $p \in N_n(p) \subset U_n(p)$.

Now consider the countable set

$$T = \{\langle (k')2^{-m'}, (k)2^{-m}\rangle : k, k', m, m' \in N, (k')2^{-m'} < (k)2^{-m}\} .$$

For $t = \langle (k')2^{-m'}, (k)2^{-m}\rangle \in T$, put

$$\boldsymbol{B}_t = \{N_n(p) : a_1' = (k')2^{-m'}, a_1 = (k)2^{-m}\} .$$

Then obviously $\bigcup\{\boldsymbol{B}_t : t \in T) = \{N_n(p) : n \in N, p \in (0, 1)\}$, which is a base for points in $(0, 1)$. Finally, it can be shown that each $\boldsymbol{B}_t$ is closure preserving. Hence $\bigcup\{\boldsymbol{B}_t : t \in T\}$ is a σ-closure preserving base and X is an M_1-space.

If X is the space in Example 9.2, then it can be shown without difficulty that $X/(0, 1)$ is an M_1-space with $(0, 1)$ having a closure preserving local base.

EXAMPLE 9.3. There exists a non-metrizable M_1-space X with $p \in X$ such that p has an uncountable closure preserving local base and $X - \{p\}$ is homeomorphic to R.

Let $p \notin R$ and put $X = R \cup \{p\}$. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the integers and put $B = \{1/n : n \in N - \{1\}\} \cup \{0\}$. Let F be the set of all functions from the integers I to B such that either there exists $r \in I$ such that if $s < r$, then $f(s) = 0$ and if $r \leqq s$, then $f(s) \neq 0$; or for all $r \in I$, $f(r) \neq 0$. For $f \in F$, put $U_f = \bigcup_{n=1}^{\infty}(r_n - f(r_n), r_n + f(r_n))$ where if $f(r_n) = 0$, $(r_n, r_n) = \phi$. Let $\boldsymbol{U} = \{\{p\} \cup U_f : f \in F\}$ and $\boldsymbol{B}$ be a countable base for R. Then it is obvious that $\boldsymbol{U} \cup \boldsymbol{B}$ is a σ-closure preserving base for X. Moreover, it is easy to see that X is not first countable at p and R is homeomorphic to $X - \{p\}$.

It is clear that this construction can be carried out for any non-compact metric space without isolated points. In particular, carrying it out for the rational numbers we get a countable non-metrizable M_1-space.

EXAMPLE 9.4. (Michael [9]). We can get another countable non-metrizable M_1-space by taking the subspace $I \cup \{p\}$ of $\beta(I)$, where I is the integers and $\beta(I)$ is the Stone-Čech compactification of I and $p \in \beta(I) - I$. Here the family of all open sets containing p is closure preserving.

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