ON SOME CLASSES OF SCALAR-PRODUCT ALGEBRAS

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1. Introduction. In the second author's previous paper [10] a *two-sided H^*-algebra* was defined as a complex Banach algebra which is a Hilbert space, and which possesses two conjugate-linear bounded mappings $x \to x^l$ and $x \to x^r$ with the property that for any x, y, and z in the algebra, $(xy, z) = (y, x^l z) = (x, zy^r)$. This concept generalized the original definition of an H^*-algebra given by Ambrose [1]. It may readily be seen that in a two-sided H^*-algebra the orthogonal complement of a right (left) ideal is again an ideal of the same kind. It is shown in [10], moreover, that this "right (left) complementation" property is sufficient to characterize a two-sided H^*-algebra A without the assumption of the mappings $x \to x^l$ and $x \to x^r$, provided that A is an annihilator algebra in the sense of Bonsall and Goldie [5], that is, provided that every proper right (left) ideal of A has a nonzero left (right) annihilator.

The present paper will carry out a study that bears somewhat the same relationship to the Hilbert algebras of Nakano [7] as does the above-mentioned investigation in [10] to Ambrose's H^*-algebras. The results here, however, will be more restricted, since Hilbert algebras (and the systems similar to them: see the papers of Ambrose [2], Segal [12], Godement [6], and Pallu de la Barrière [8]) are much more general and less manageable than H^*-algebras. In particular, we shall have neither joint continuity of multiplication in the algebra nor completeness of the metric space formed by its elements under the scalar-product norm. These strong properties are lacking for Hilbert algebras in general; in addition, however, we shall replace the standard assumption of the existence of a conjugate-linear isometry and the adjoint character of this mapping by the requirement that in our algebras the orthogonal complement of a right ideal shall be a right ideal. To compensate somewhat for this loss, our considerations will be restricted to a class of algebras that may be described as symmetric, maximal, and topologically semi-simple. We shall define these terms in the following section, in which we discuss some matters corresponding for our case to the theory of regular ideals fundamental in the study of Banach algebras.

2. Preliminary theory. We shall deal with algebras possessing some of the properties of Hilbert algebras, apart from the $*$-mapping.

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DEFINITION 2.1. Let A be a complex associative algebra that is a pre-Hilbert space under a given scalar-product. Denote by H the Hilbert space completion of A. A will be called a *scalar-product algebra* (*SP-algebra*) if the following postulates hold:

(1) The operators $L'_a: b \rightarrow ab$ ($R'_a: b \rightarrow ba$) are bounded for all a and b in A. We denote their extensions to H by L_a (R_a).

(2) Each operator $L'_x: b \rightarrow R_b x$ ($R'_x: b \rightarrow L_b x$), where $b \in A$, $x \in H$, has a closed linear extension L_x (R_x) which is the closure of the graph of L'_x (R'_x).

(3) A is symmetric: for each x in H, L_x and R_x are both bounded, or both unbounded.

(4) A is maximal: if L_x (or R_x) is bounded, then $x \in A$.

(5) If x in H is such that $L_x a = 0$ or $R_x a = 0$ for all a in A, then $x = 0$.

A Hilbert algebra has all these properties, or may readily be taken to have them. Property 2 follows from the nature of the *-operation, along with a standard theorem [9, p. 305] which states that a linear transformation T with domain dense in a Hilbert space H has a closed linear extension if and only if the domain of its adjoint T^* is dense in H. The maximality property is not automatically verified in a Hilbert algebra, but a given Hilbert algebra may be extended to a maximal one, as is shown by Takenouchi [14, Theorems 1 and 2] and by Segal [12, Theorem 16]. The remaining properties are easily seen to hold in Hilbert algebras.

The appropriate definition of an ideal in the present context must be more general than the ordinary algebraic concept, because of the interplay of the algebraic properties of A and the topological properties of its completion H.

DEFINITION 2.2. A *right ideal* R of A is a subspace of H such that $R_a(R) \subset R$ for every a in A. A similar definition holds for left and two-sided ideals. An ideal I is *proper* if $I \neq (0)$ and $I \neq H$.

It should be noted that we do not require an ideal of A to be *in* A, nor is it asserted at the outset that an ideal of A need even intersect A. Moreover, an ideal is not in general required to be closed. (In our discussion, *closure* will always mean *closure in* H.)

The following concepts are generalizations of standard ones.

DEFINITION 2.3. Let I be an ideal of A. The *left annihilator of*

I is the set $l(I) = \{x \in H \mid L_x y = 0$ for all y in $I\}$. The *right annihilator* $r(I)$ is similarly defined.

DEFINITION 2.4. A right ideal R is *regular* if there exists an element u in H such that $L_u a - a \in R$ for every a in A. In this case u is said to be a *relative identity for* R.

DEFINITION 2.5. An element x in H is *algebraically right quasi-regular* if there exists an element a in A such that $x + a - R_a x = 0$; x in H is *topologically right quasi-regular* if there exists a sequence $\{a_n\}$, where $a_n \in A$, such that $x + a_n - R_{a_n} x \to 0$. (In a Banach algebra, algebraic and topological quasi-regularity coincide.)

DEFINITION 2.6. A is *topologically semi-simple* if (0) is the only right ideal of A that consists entirely of topologically right quasi-regular elements.

The following example will illustrate the notion of topological semi-simplicity, which we shall hereafter assume for the SP-algebras with which we deal.

EXAMPLE 2.1. Let A be the complex matrix algebra consisting of all finite linear combinations of unit matrices (e_{ij}), where i and j belong to an arbitrary index set J. Let a scalar-product be defined as $(X, Y) = \operatorname{tr} XTY^* = \sum_{i,j} t_{jj} x_{ij} \bar{y}_{ij}$, where $T = (t_{ij})$, a positive definite diagonal matrix. Take R to be a nonzero right ideal, and X a matrix in R with the component $x_{ij} \neq 0$. Then right multiplication of X by $1/x_{ij}(e_{ji})$ yields a matrix $Y = (y_{ij})$ in R with a single non-zero column, the ith. Moreover, $y_{ii} = 1$. It is easy to see that Y is not algebraically right quasi-regular. Furthermore, since all matrices of the form $YA - A$ have a zero ith row, it is clear that Y cannot be a limit of such matrices, for denoting the (i, j) component of $YA - A - Y$ by u_{ij}, we have $\| YA - A - Y \|^2 = \sum_{i,j} t_{jj} \mid u_{ij} \mid^2 \geqq t_{ii} > 0$. Thus Y is not topologically right quasi-regular.

A trivial H^*-algebra of W. Ambrose [1] can be considered as an example of an SP-algebra which is not topologically semi-simple.

3. Left projections in right complemented SP-algebras. The remainder of this paper will be concerned with algebras of the following type.

DEFINITION 3.1. A is *right complemented* if the orthogonal complement R^p of every right ideal R is a right ideal.

It should be noted that if P is a projection operator whose range is a right ideal of A, then by the right complementation property the range subspaces of P and $I - P$ reduce R_a for every a in A; or equivalently, $PR_a = R_aP$. We may thus arrive at the following result.

LEMMA 3.1. *If P is a projection operator whose range is a right ideal, and $a \in A$, then $Pa \in A$.*

Proof. The lemma is an application of a more general result of Segal [12, Corollary 16.3] and Godement [6, Lemma 4], which tells us that L_{Pa} is bounded. By the maximality of A, $Pa \in A$.

LEMMA 3.2. *Let R be a closed right ideal. Then $A \cap R$ is dense in R.*

Proof. If x is arbitrary in R and $\{a_n\}$ is a sequence in A that converges to x, then letting P be the projection operator with range R, we have $Pa_n \to x$, $Pa_n \in R$. By the preceding lemma, $Pa_n \in A$.

DEFINITION 3.2. An element x in H is *left self-adjoint* if for any a, b in A we have $(L_xa, b) = (a, L_xb)$. If e is a nonzero left self-adjoint idempotent in A, e will be called a *left projection.*

THEOREM 3.1. *Every topologically semi-simple right complemented SP-algebra A contains a left projection. In fact, if u is any element of H that is not topologically right quasi-regular, then a left projection is obtained by projecting u upon R^p, where $R = \{L_ua - a \mid a \in A\}$.*

Proof. Details not given here may be found in Lemma 2 of [10]. Taking R as in the statement of the theorem, we see that $\bar{R}$ is a closed regular right ideal of A, with relative identity u. Moreover, $u \notin \bar{R}$. Now let $u = v + e$, where $v \in \bar{R}$, $e \in R^p$, $e \neq 0$. We shall show that the operator L'_e with domain A is bounded; it will then follow from the symmetry and maximality of an SP-algebra that $e \in A$. Since $L_ea - a \in \bar{R}$ for all a in A, we see that $L_e(\bar{R} \cap A) = 0$ and $L_eb = b$ for b in $R^p \cap A$, using the fact that R^p is a right ideal. From this it follows, if we write $a = a_1 + a_2$, where $a_1 \in R^p \cap A$, $a_2 \in \bar{R} \cap A$, that $L'_ea = a_1$, so that $\| L'_ea \| \leqq \| a \|$, $e \in A$, and $e^2 = e$. Finally, for arbitrary c, d in A, we have $(ec, d) = (c_1, d_1) = (c, ed)$, where c_1 and d_1 are the orthogonal projections of c and d on R^p.

Our next theorem will show that it is even possible to assert that certain left ideals of A contain left projections.

THEOREM 3.2. *If L is a nonzero left ideal such that $L \subset A$, then*

L contains a left projection.

Proof. We first note that for any a, b in A, ab is topologically (algebraically) right quasi-regular if and only if ba has the same property, since if $ab + u_n - abu_n \to 0$, where $u_n \in A$, then $ba + v_n - bav_n \to 0$, where $v_n = -ba + bu_na \in A$. Hence there exists in L a nonzero element a that is not topologically right quasi-regular; otherwise, for any nonzero b in L, the right ideal bA would consist only of topologically right quasi-regular elements, contradicting the topological semi-simplicity of A, since $bA \neq (0)$ by Property 5 of SP-algebras. According to the preceding theorem we obtain a left projection e by letting $a = u + e$, where $u \in \bar{R}$, $e \in R^p$, $R = \{ab - b \mid b \in A\}$. Since $eu = 0$, $e = ea \in L$.

COROLLARY. *If L is a left ideal such that $L \cap A \neq (0)$, then L contains a left projection.*

With the existence of left projections assured, we may proceed to introduce a relation of partial order among them.

DEFINITION 3.3. Let e and f be left projections. Then $e \leqq f$ if $L_e \leqq L_f$ in the standard ordering of projection operators. If for every left projection f, $f \leqq e$ only when $f = e$, then e will be called a *minimal left projection.*

It is clear that if $e \leqq f$, then $L_{ef} = L_eL_f = L_e = L_fL_e = L_{fe}$, so that $ef = fe = e$, and conversely. This follows from Property 5 of SP-algebras.

LEMMA 3.3. *If e is a minimal left projection, then Ae and eA are minimal left and right ideals, respectively.*

Proof. Suppose that $L \subset Ae$, where L is a left ideal. By Theorem 3.2 L contains a left projection f, and $fe = f$, so that $f \leqq e$. Since e is minimal, $f = e$ and $Ae \subset L$. To show that eA is minimal, we note that if R is a nonzero right ideal such that $R \subset eA$, then by the topological semi-simplicity of A there exists an element $u = eu$ in R that is not topologically right quasi-regular. Then $ue \in R$ and ue is not topologically right quasi-regular. Letting $Q = \{uea - a \mid a \in A\}$, we write $ue = v + f$, where $v \in \bar{Q}$, $f \in Q^p$. Then f is a left projection, and since $uee = ue = ve + fe$, $ve \in \bar{Q}$, $fe \in Q^p$, we have $fe = f$ so that as before, $f = e$. Finally, $ue = eue = ev + e = e$ (since $L_e\bar{Q} = 0$); thus $e \in R$ and $eA \subset R$.

Our final development of this section will show that a minimal left projection e has the property that eAe is isomorphic to the complex number field, as in the case of Hilbert algebras. We may first prove as in Theorem 4.3 of [1] that eAe is a division algebra with identity e;

this follows from the fact that for $0 \neq a = eae \in eAe$, $aA = eA$ and $Aa = Ae$. We then establish the following lemma.

LEMMA 3.4. *If e is a minimal left projection, then eAe is a complete metric space.*

Proof. Since H is complete, $\overline{eAe}$ is also complete. Moreover, $eAe \subset L_eR_eH = R_eL_eH$. If the sequence $\{c_n\}$ in eAe has limit x in H, then $c_n = L_eR_ec_n \to L_eR_ex$, so that $x = L_eR_ex \in L_eR_eH$. Hence $\overline{eAe} \subset L_eR_eH$. To complete the proof we shall show that $L_eR_eH \subset eAe$.

Suppose that $x \in L_eR_eH$. Then L_xA is a right ideal of A containing an element L_xa that is not topologically right quasi-regular. Using Theorem 3.1 we write $L_xa = v + f$, where $v \in \bar{R}$, $f \in R^p$, $R = \{L_{L_xa}b - b \mid b \in A\}$, and f is a left projection. Since $L_f\bar{R} = 0$, $L_fL_xa = L_fv + f = f \neq 0$, so that

$$\begin{aligned} R_{eae}L_fx &= \lim R_{eae}L_fc_n = \lim (fc_n)(eae) = \lim f((c_ne)a)e \\ &= \lim R_eL_fR_ac_ne = R_eL_fR_ax = R_eL_fL_xa = fe \;. \end{aligned}$$

Now $fe \neq 0$, since

$$0 \neq f = L_fL_xa = L_fR_ax = \lim f(c_na) = \lim fec_na \;.$$

Thus, denoting by $(eae)^{-1}$ the inverse of eae in eAe, we have

$$R_{(eae)^{-1}}R_{eae}L_fx = L_fR_{(eae)^{-1}}R_{eae}x = L_fR_ex = L_fx = fe(eae)^{-1} \neq 0$$

so that the left ideal $R_xA \cap A \neq (0)$. By the corollary to Theorem 3.2 there exists a left projection g in R_xA: hence for some b in A, $g = R_xb = L_bx = L_bR_ex = R_eL_bx = R_eR_xb = ge \neq 0$. Since $g \leqq e$, we conclude from the minimality of e that $g = e$. Thus

$$e = R_xb = L_eR_xb = L_eL_bx = L_eL_bL_ex = L_{ebe}x \;,$$

and $x = (ebe)^{-1} \in eAe$. Therefore $L_eR_eH \subset eAe$.

THEOREM 3.3. *If e is a minimal left projection, then eAe is isomorphic to the complex number field.*

Proof. Since eAe is a complete metric ring whose product is continuous in each factor, it follows from a theorem of Arens [3, Theorem 5] that multiplication in eAe is continuous in both factors simultaneously. Then for any a, b in eAe, $\|ab\| \leqq M\|a\|\|b\|$: this holds by a variation of a theorem of Banach [4, pp. 40–41], as remarked in the introduction to [1]. The conclusion now follows from the Mazur-Gelfand theorem.

4. Discrete *SP*-algebras. We shall now consider the class of *SP*-

algebras that are discrete in the sense of Nakano [7]. For these algebras we may prove an analog of the first Wedderburn structure theorem.

DEFINITION 4.1. An SP-algebra is *discrete* if for any left projection e there exists a minimal left projection $f \leqq e$.

The following simple example of a commutative discrete SP-algebra illustrates all the concepts we have used up to the present.

EXAMPLE 4.1. Let (S, m) be a totally atomic measure space, and let A be the maximal extension in $L^2(S, m)$ of L, the algebra of all simple complex-valued functions on S, with pointwise multiplication and the usual scalar-product. Since every nonzero ideal of A contains an idempotent (which cannot be topologically right quasi-regular), A is topologically semi-simple; A is also readily seen to be right complemented and discrete.

DEFINITION 4.2. An SP-algebra A will be called (*topologically*) *simple* if A is topologically semi-simple and if there exists no proper closed two-sided ideal of A.

We shall need the following lemma, which here is not as immediate as in the case of Banach algebras.

LEMMA 4.1. *The left annihilator $l(R)$ of a closed right ideal R is a closed left ideal. The left annihilator $l(I)$ of a closed two-sided ideal I is a closed two-sided ideal.*

Proof. Let R be a closed right ideal. Then if $x \in l(R)$, $L_x a = 0$ for every a in $R \cap A$. If $b \in A$, then $L_{L_b x} a = R_a L_b x = L_b R_a x = L_b L_x a = 0$. Now for any y in R, consider $\{a_n\}$, where $a_n \rightarrow y$, $a_n \in R \cap A$ (by Lemma 3.2). We have $L_{L_b x} a_n = 0$ for each n, and since $L_{L_b x}$ is a closed operator, $L_{L_b x} y = 0$. Thus $L_b x \in l(R)$, and $l(R)$ is a left ideal. By a similar method it may be shown that $l(R)$ is closed and that if I is a closed two-sided ideal, $l(I)$ is a right ideal.

LEMMA 4.2. *If I is a two-sided ideal of A, then I^p is a two-sided ideal and $I^p = l(I) = r(I)$.*

LEMMA 4.3. *If I is the smallest closed two-sided ideal containing the minimal left projection e, then I contains no properly smaller closed nonzero two-sided ideal.*

The proofs are similar to those used when A is a Banach algebra.

LEMMA 4.4. *Every minimal closed two-sided ideal I is the completion of a simple right complemented SP-algebra $I \cap A$.*

Proof. $I \cap A$ is dense in I, by Lemma 3.2. That $I \cap A$ is simple and right complemented follows from the fact that a right (two-sided) ideal of $I \cap A$ is a right (two-sided) ideal of A. If L_x (R_x) is bounded on $I \cap A$, where $x \in I$, then L_x (R_x) is bounded on A and $x \in A$. $I \cap A$ is thus symmetric and maximal; the remaining requirements are readily verified.

THEOREM 4.1. *Every topologically semi-simple discrete right complemented SP-algebra A is a direct sum of simple right complemented SP-algebras, each of which is of the form $I \cap A$, where I is a closed two-sided ideal of A.*

Proof. Let e be a minimal left projection in A, and let I be the smallest closed two-sided ideal of A containing e. By Lemmas 4.3 and 4.4, $I \cap A$ is a simple right complemented *SP*-algebra. Furthermore, $P(A) = I \cap A$, where P denotes the projection operator with range I. $I^p \cap A$ is also a topologically semi-simple discrete right complemented *SP*-algebra. The proof is completed by the use of Zorn's lemma.

We may now introduce the notion of left adjoints and state some results on their existence; there will then follow a weak type of left complementation.

DEFINITION 4.3. The element x^l is a left adjoint of x in H if for any a, b in A we have $(L_x a, b) = (a, L_{x^l} b)$.

THEOREM 4.2. *Let e be a minimal left projection in A. Then every element of eA has a left adjoint in Ae. If A is discrete, the set of elements of A that have a left adjoint is dense in H.*

COROLLARY. *If A is discrete and L is a left ideal of A, then $L^p \cap A$ is a left ideal.* (*Note*: $L^p \cap A$ is not known to be dense in L^p.)

These results are established by exactly the same proofs as those given for Theorems 1 and 2 of [11]. For the second part of the theorem we must use the fact that an arbitrary left projection e in a discrete algebra belongs to the closed right ideal generated by the family of all minimal left projections in A. This is done in the following lemma.

DEFINITION 4.4. Two left projections e and f are *strongly orthogonal* if $ef = 0$.

LEMMA 4.5. *In a discrete SP-algebra A, every left projection is the limit of a countable sum of strongly orthogonal minimal left projections.*

Proof. If e is a left projection and f is a minimal left projection, where $f \leqq e$, then $g = e - f$ is a left projection. Since $f = fe = f + fg$, $fg = 0$, so that f and g are strongly orthogonal. Let K be any set of strongly orthogonal left projections k_i such that $k_i \leqq e$ for every k_i in K. Let $K_0 = \{k_i \mid 1 < \| k_i \|^2\}$, and let $K_n = \{k_i \mid 1/2^n < \| k_i \|^2 \leqq 1/2^{n-1}\}$. Then each left projection k_i belongs to exactly one set K_n, and it is easily seen that each K_n contains a finite number of k_i: hence K is countable. The proof is completed by a routine use of Zorn's lemma.

5. Annihilator SP-algebras.

DEFINITION 5.1. Let A be discrete. A will be called an *annihilator SP-algebra* if $l(R) \cap A \neq (0)$ for every proper closed right ideal R, and $r(L) \cap A \neq (0)$ for every proper closed left ideal L. A will then be said to have the *annihilation property.*

A Hilbert algebra may be shown to have the annihilation property.

THEOREM 5.1. *Every closed nonzero right ideal R of a right complemented annihilator SP-algebra A contains a left projection.*

Proof. We assume that R is proper, or the theorem already holds for R. Since R^p is a proper closed right ideal, $l(R^p) \cap A \neq (0)$. By the argument used in Theorem 3.2, $l(R^p) \cap A$ then contains an element a that is not topologically right quasi-regular. Let $Q = \{ab - b \mid b \in A\}$, for which a is a relative identity. Since $L_a(R^p) = (0)$, we have $R^p \subset \bar{Q}$, and hence $Q^p \subset R$. Letting $a = e + u$, $e \in Q^p$, $u \in \bar{Q}$, we obtain as in Theorem 3.1 a left projection e in R.

THEOREM 5.2. *If e is a minimal left projection in a right complemented annihilator SP-algebra A, then every element of R_eH has a left adjoint in eA.*

Proof. Suppose that $x \in R_eH$; then $x = R_ex = L_xe$. Consider the right ideal $\overline{L_xA} = \overline{L_{L_xe}A}$. We assume that $L_ex \neq 0$; otherwise x may be replaced by $x + e$ and we show that $x + e$ has a left adjoint. If $\{L_{L_xe}a_n\}$ is any sequence in L_xA such that $L_{L_xe}a_n \to y$, then $L_eL_{L_xe}a_n = L_eL_{R_ex}a_n = L_eR_{a_n}R_ex = R_{a_n}L_eR_ex = \lambda ea_n \to L_ey$ by Theorem 3.3, and $ea_n \to 1/\lambda L_ey$. Since L_{L_xe} is a closed operator, $y = L_{L_xe}(1/\lambda L_ey)$, so that every element in $\overline{L_xA}$ is of the form $L_{L_xe}z = L_xz$, where $z \in H$. By the preceding theorem $\overline{L_xA}$ contains a left projection $f = L_{L_xe}u$. Then $fe \neq 0$; otherwise, consider a sequence $\{c_n\}$, where $c_n \in A$, such that $c_n \to u$, $L_{R_ex}c_n \to L_{R_ex}u$. (This is possible, because L_{R_ex} is the closure of the graph of the operator L'_{R_ex} with domain A.) We would then have

$$0 = ef = L_e L_{L_x e} u = L_e L_{R_e x} u = \lim L_e L_{R_e x} c_n = \lim L_e R_{c_n} R_e x$$
$$= \lim R_{c_n} L_e R_e x = \lim \lambda e c_n = \lambda L_e u \ .$$

But

$$0 \neq f = L_{L_x e} u = \lim L_{R_e x} c_n = \lim R_{c_n} R_e x = \lim R_{e c_n} x = \lim L_x e c_n \ ,$$

and $ec_n \rightarrow L_e u$; this shows that $fe \neq 0$, and that $f = L_x L_e u$, since L_x is a closed operator. Thus

$$fe = R_e L_x L_e u = \lim R_e L_x e c_n = \lim R_e R_{e c_n} x = \lim R_{e c_n e} x$$
$$= \lim L_x e c_n e = \lim \mu_n L_x e = \mu x \ ,$$

so that $x^l = 1/\bar{\mu}(ef)$.

COROLLARY. $R_e H = Ae$.

Proof. For any x in $R_e H$, L_x is a closed operator, so that $L_x^{**} = L_x$. But $L_x^* = L_{x^l}$ is bounded and defined everywhere in H; hence the same is true of its adjoint. Thus $x \in R_e H \cap A = Ae$.

THEOREM 5.3. *Let A be a topologically semi-simple right complemented annihilator SP-algebra. Then A contains a left ideal L with the following properties:*

(1) *L is dense in A.*

(2) *L is isomorphic to an algebra M of matrices which are functions on a certain set $J \times J$. Every matrix X of M has a left adjoint $X^l = X^*$ in M.*

(3) *If $x, y \in L$, then $(x, y) = \operatorname{tr} XTY^*$, where X and Y are the matrices of M corresponding to x and y, and T is a bounded, self-adjoint, positive definite matrix operator on $L^2(J)$.*

Proof. (1) Let $F = \{e_i\}_{i \in J}$ be a maximal family of strongly orthogonal minimal left projections in A, where J is a suitable index set. Consider $R = \sum_{i \in J} e_i A$. We first prove that R is dense in H; using this fact we may then draw the same conclusion for L, the left ideal consisting of finite sums $\sum_{i \in J} A e_i$. The argument is exactly the one used in Theorem 3 of [10].

(2) Letting $e_i A e_j = A_{ij}$, we have $L = \sum_{i,j} A_{ij}$, where the sums over j are always finite. From Theorem 4.3 of [1], we know that each A_{ij} is one-dimensional and that we may choose matrix units e_{ij} in A_{ij} so that $e_{ii} = e_i$, $e_{ij} e_{jk} = e_{ik}$, and $e_{ij}{}^l = e_{ji}$. (We have here used the fact that every element of A_{ij} has a left adjoint in A_{ji}.) For any two elements of L we now have $x = \sum_{i,j} x_{ij} e_{ij}$ and $y = \sum_{i,j} y_{ij} e_{ij}$, where x_{ij} and y_{ij} are complex numbers. Moreover, the product xy is given by

$$(\sum x_{ij}e_{ij})(\sum y_{kl}e_{kl}) = (\sum x_{ij}e_{ij})(\sum y_{jl}e_{jl}) = \sum x_{ij}y_{jl}e_{il} ,$$

with the sums taken over all the indices shown in each case. Finally, if $x \in L$, then $x \in \sum Ae_j$ (where again the sum is finite), so that x^l exists and $\mathrm{x}^l = \bar{x}_{ij}e_{ji}$.

(3) Setting $t_{ij} = (e_{ii}, e_{ij})$, we define a matrix $T = (t_{ij})$. It is easy to see that T is well defined and selfadjoint. For any two elements $x = \sum_{i,j}x_{ij}e_{ij}$ and $y = \sum_{i,j}y_{ij}e_{ij}$ in L, we have

$$(x, y) = \sum_{i,j,k}x_{ik}t_{kj}y_{ij} = \mathrm{tr}\, XTY^* .$$

Next we shall show that T is a bounded, positive definite operator on $L^2(J)$. In order to do so, we first prove that W, the restriction to Ae_r of the conjugate-linear mapping $x \rightarrow x^l$, is bounded, where r is some fixed index in J. By Theorem 5.2 and its corollary, W is a mapping of the complete metric space Ae_r into e_rA; moreover, W is a closed operator. It follows from the closed graph theorem that W is bounded, and for every x in Ae_r,

$$(1) \qquad (x^l, x^l) \leqq M(x, x) .$$

Now each element $x = \sum_i x(i)e_{ir}$ of Ae_r corresponds to the element $X = x(i)$ of $L^2(J)$, and conversely, where each point of J is taken to have unit measure. For any finite sequences $x(i)$ and $y(i)$ in $L^2(J)$ and the corresponding x and y in Ae_r, we have

$$(2) \qquad (x^l, y^l) = \sum_{i,j}\bar{x}(i)t_{ij}y(j) .$$

By the continuity of W, the expression (2) holds for all X, Y in $L^2(J)$ and the corresponding x, y in Ae_r. Applying (1) we then have for any X in $L^2(J)$, $[XT, X] = \sum_{i,j}x(i)t_{ij}\bar{x}(j) = (\bar{x}^l, \bar{x}^l) \leqq M(\bar{x}, \bar{x}) = \| e_r \|^2 M[X, X]$, where $x = \sum_i \bar{x}(i)e_{ir}$ and $[\,,]$ is the scalar-product of $L^2(J)$. This shows that T is a bounded operator on $L^2(J)$, and that T is positive definite, since $(\bar{x}^l, \bar{x}^l) > 0$ if $X \neq 0$.

We may now remark that as in the case of Hilbert algebras a necessary and sufficient condition for H to be a Banach algebra under the given norm (or an equivalent one) is that $\inf \| e_i \| = m > 0$, where $\{e_i\}$ is the set of all left projections of A. The necessity is obvious; to prove the sufficiency it may be shown by the method of [13] that T has the lower bound m, from which it follows that T has an inverse and that consequently L is complete, as well as dense in H. Thus $L = H$, and one may further prove that every element of H has a left and right adjoint: H is, in fact, a two-sided H^*-algebra.

If A is discrete and commutative, then A is necessarily an annihilator SP-algebra, by Lemma 4.2. We therefore conclude with a theorem for algebras of this type.

THEOREM 5.4. *Let A be a commutative SP-algebra which is complemented and discrete. Then A is a Hilbert algebra isomorphic to that described in Example* 4.1.

Proof. Since for any two distinct minimal projections e and f, ef is a projection and $ef \leqq e$, $ef \leqq f$, we conclude that $ef = 0$; that is, distinct minimal projections are strongly orthogonal. As in the case of Theorem 4.3, we use the method of [11] to show that $H = \sum e_i A$, where $\{e_i\}_{i \in J}$ is the family of all minimal projections, and each $e_i A$ $(= e_i A e_i)$ is isomorphic to the complex number field. Thus, if $x \in H$, $x = \sum \lambda_i e_i$, its adjoint $x^* = \sum \bar{\lambda}_i e_i$ and $(x, x) = \sum |\lambda_i|^2 \| e_i \|^2$. It is now clear that H is isomorphic to $L^2(J, m)$, where $m(E) = \sum_{i \in E} \| e_i \|^2$ for every countable subset E of J. A is then isomorphic to the maximal extension of L, the set of all simple functions on J, which is a Hilbert algebra.

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