LEBESGUE DENSITY AS A SET FUNCTION

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Lebesgue (or metric) density is usually considered as a point function in the sense that a fixed subset of a space X is given and then the value of the density of this set is obtained at various points of the space. Suppose the density is considered in another sense. That is, let a point x of the space be fixed and consider the class $\mathscr{D}(x)$ of all sets whose density exists at this point. Then to each set E in $\mathscr{D}(x)$ we assign the value of its density at x, and denote this number by $D_x(E)$. Thus from this point of view the density is a finite set function. It was shown in [2] that if the space X is the real line then the image of $\mathscr{D}(x)$ under D_x is the closed unit interval.

It is evident from the definition of density of sets of real numbers, which we give below, that D_x is a finitely additive, subtractive, monotone, nonnegative set function and the class $\mathscr{D}(x)$ is closed under the formation of complements, proper differences, and disjoint unions. Therefore, if $\mathscr{D}(x)$ were closed under the formation of intersections, D_x would be a finitely additive measure. This however is not the case for if

$$R_n = \left\{x\colon \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < \frac{1}{n}\right\},$$

$$L_n = \left\{x\colon -\frac{1}{n} < x < -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)\right\}$$

and

$$L_n^* = \left\{x\colon -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < -\frac{1}{n+1}\right\},$$

the sets $\bigcup_n (R_n \cup L_n) = E$ and $\bigcup_n (R_n \cup L_n^*) = F$ are members of $D(0)$ but $E \cap F$ is not. In fact $D_0(E) = D_0(F) = \frac{1}{2}$ and the upper density of $E \cap F$ at zero is not less than $\frac{1}{2}$ while the lower density of $E \cap F$ at zero is zero.

In part 1 of this note we prove a theorem which is somewhat of an analogue of the Lebesgue density theorem [3] in the following respect. As noted above D_x is not a finitely additive measure, but we show that the upper density at x, $\bar{D}_x$, is a finitely subadditive outer measure defined on the class of all Lebesgue measurable subsets of X and the class of $\bar{D}_x$-measurable sets is the class of all sets whose density exists at x and has the value zero or one. In part 2 a Lebesgue density of a measurable set E on a fixed F_σ set of measure zero is defined and a similar result

Received May 24, 1960. Presented to American Mathematical Society. Part 1 of this note constitutes a portion of the author's doctoral dissertation written at Iowa State College under the direction of Professor H. P. Thielman.

proven for this function.

1. If E is a measurable subset of the real line X and I is any interval we shall denote the relative Lebesgue measure of E in I, $m(E \cap I)/m(I)$, by $\rho(E:I)$.

The upper Lebesgue density of a measurable subset E of X at a point $x \in X$, $\bar{D}_x(E)$, is defined by

$\bar{D}_x(E) = \limsup_{I \to x} \rho(E:I) = \sup \{\limsup_k \rho(E:I_k)\colon I_k \to x\}$ and the lower Lebesgue density of a measurable set $E \subset X$ at a point $x \in X$, $\underline{D}_x(E)$, is defined by

$$\underline{D}_x(E) = \liminf_{I \to x} \rho(E:I) = \inf \{\lim_k \rho(E:I_k)\colon I_k \to x\} ,$$

where $I_k \to x$ means the sequence $\{I_k\}$ of intervals converges to x in the sense that $x \in \bar{I}_k$ for all k and $m(I_k) \to 0$ as $k \to \infty$. In the case $\underline{D}_x(E) = \bar{D}_x(E)$ the common value is the Lebesgue density of E at x and will be denoted by $D_x(E)$.

LEMMA 1. *A necessary and sufficient condition that a set E be a member of $\mathscr{D}(x)$ is that*

$$\bar{D}_x(E) + \bar{D}_x(X - E) = 1 .$$

Proof. The necessity is immediate. To obtain the sufficiency we note that for any interval I containing x, $\rho(E:I) + \rho(X - E:I) = 1$ so that $\underline{D}_x(E) + \bar{D}_x(X - E) \geqq 1$. Therefore

$$\bar{D}_x(X - E) \geqq 1 - \underline{D}_x(E) = \bar{D}_x(X - E) + \bar{D}_x(E) - \underline{D}_x(E)$$

and it follows that $\bar{D}_x(E) \leqq \underline{D}_x(E)$.

LEMMA 2. *The set function $\bar{D}_x$ is a finitely subadditive outer measure defined on the class $\mathscr{M}$ of all Lebesgue measurable subsets of the real line.*

Proof. It is clear that $\bar{D}_x(\phi) = 0$ and $\bar{D}_x \geqq 0$. Let $E \subset F$ be two sets from M. Then since $\rho(E:I) \leqq \rho(F:I)$ for all intervals containing x, $\bar{D}_x$ is monotone. Let $E_1, E_2, \cdots, E_n$ be any finite collection of sets from $\mathscr{M}$. Since $\rho(\bigcup_{i=1}^n E_i : I) \leqq \sum_{i=1}^n \rho(E_i : I)$ for all intervals I containing x, we have

$$\bar{D}_x\Big(\bigcup_{i=1}^n E_i\Big) \leqq \sum_{i=1}^n \limsup_{I \to x} \rho(E_i : I) = \sum_{i=1}^n \bar{D}_x(E_i) .$$

Thus $\bar{D}_x$ is a finitely subadditive outer measure.

Let $\mathscr{M}(x)$ denote the class of all sets E such that for every $A \in \mathscr{M}$,

$\bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E)$. Since $\mathscr{M}(x)$ contains X and $\phi \mathscr{M}(x)$ is an algebra (in the sense of Halmos [1]) and the restriction of $\bar{D}_x$ to $\mathscr{M}(x)$ is a finitely additive measure.

LEMMA 3. *$\mathscr{M}(x)$ is a subset of $\mathscr{D}(x)$.*

Proof. Let $E \in \mathscr{M}(x)$. Since the real line X is a member of $\mathscr{M}$ and $\bar{D}_x(X) = 1$, we have

$$1 = \bar{D}_x(X) = \bar{D}_x(X \cap E) + \bar{D}_x(X - E) = \bar{D}_x(E) + \bar{D}_x(X - E)$$

which by Lemma 1 gives $E \in \mathscr{D}(x)$.

LEMMA 4. *If $E \in \mathscr{D}(x)$ and J is any interval with x as one end point then $\bar{D}_x(E \cap J) = D_x(E)$.*

Proof. Let $D_x(E) = d$. Since $\bar{D}_x$ is monotone, $d \geqq \bar{D}_x(E \cap J)$ and if $\{I_k\}$ is any sequence of intervals converging to x, $\limsup_k \rho((E \cap J) : I_k) \leqq d$.

Suppose first that J is a bounded interval. If x is the left end point of J, denote the right end point by y and let

$$I_n^* = \left\{z\colon\ x \leqq z \leqq x + \frac{1}{n}(y - x)\right\};$$

if x is the right end point of J, denote the left end point of J by y and let

$$I_n^* = \left\{z\colon\ x - \frac{1}{n}(x - y) \leqq z \leqq x\right\}.$$

In either case $I_n^* \to x$ and $\rho(E : I_n^*) = \rho((E \cap J) : I_n^*)$ for all n. Therefore, $\lim_n \rho((E \cap J) : I_n^*) = d$ and we have $\bar{D}_x(E \cap J) = D_x(E)$.

Suppose next that J is unbounded. If x is the left end point of J let $I_n^* = \{z\colon\ x \leqq z \leqq z + (1/n)\}$ and if x is the right end point of J let $I_n^* = \{z\colon\ x - (1/n) \leqq z \leqq x\}$. Again we have $I_n^* \to x$ and $\rho(E : I_n^*) = \rho((E \cap J) : I_n^*)$ for all n so that $\bar{D}_x(E \cap J) = D_x(E)$.

LEMMA 5. *Let $E \in \mathscr{D}(x)$ and let J be an interval open on the right with right end point at x and K be an interval closed on the left with left end point at x. Define the set A by $A = (E \cap K) \cup (J - E)$. Then $\bar{D}_x(A) = \max\{D_x(E), D_x(X - E)\}$.*

Proof. Suppose $D_x(X - E) \leqq D_x(E) = d$. By Lemma 4, $\bar{D}_x(J - E) = 1 - d \leqq d$ and since $\bar{D}_x$ is monotone, $\bar{D}_x(A) \geqq \bar{D}_x(E \cap K) = d$.

Let $\varepsilon > 0$ be given. Then there exists a sequence $\{I_k^*\}$ converging to x such that

$$\bar{D}_x(A) < \limsup_k \rho(A : I_k^*) + \frac{\varepsilon}{2} .$$

For each k, let $J_k = I_k^* \cap (J \cup K)$. Since $I_k^* \to x$, $J_k^* \to x$ and $\rho(A : I_k^*) = \rho(A : J_k)$ for all but a finite number of k. Therefore

$$(1) \qquad \bar{D}_x(A) < \limsup_k \rho(A : J_k) + \varepsilon/2 .$$

For each interval J_k we have

$$\rho(A : J_k) - d = \rho(K : J_k)[\rho(E : (K \cap J_k)) - d] + \rho(J : J_k)[\rho(X - E : (J \cap J_k)) - d] .$$

Since $E \in \mathscr{D}(x)$ and $K \cap J_k \to x$, $\lim_k \rho(E : (K \cap J_k)) = d$. Since $J \cap J_k \to x$, $\lim_k \rho(X - E : (J \cap J_k)) = 1 - d \leqq d$. Therefore there exist integers N_1 and N_2 such that for all $k > N_1$, $\rho(E : (K \cap J_k)) - d < \varepsilon/2$ and for all $k > N_2$, $\rho(X - E : (J \cap J_k)) - d < \varepsilon/2$. Thus for all $k > \max\{N_1, N_2\}$

$$\rho(A : J_k) - d < \frac{\varepsilon}{2}\rho(K : J_k) + \frac{\varepsilon}{2}\rho(J : J_k) = \frac{\varepsilon}{2} .$$

Therefore $\limsup_k \rho(A : J_k) < d + \varepsilon/2$ and we have by way of equation (1) that $\bar{D}_x(A) < d + \varepsilon$. Since ε was arbitrary, $\bar{D}_x(A) \leqq d$ which completes the proof of the lemma.

THEOREM 1. *The class $\mathscr{M}(x)$ of $\bar{D}_x$-measurable sets is the class of all sets whose density exists at x and has the value* 0 *or* 1.

Proof. First suppose $E \in \mathscr{M}(x)$ and $D_x(E) = d$. Let $J = \{z : x - 1 \leqq z < x\}$, $K = \{z : x \leqq z \leqq x + 1\}$. Define the set A by $A = (E \cap K) \cup (J - E)$. By Lemma 5, $\bar{D}_x(A) = \max\{1 - d, d\}$ and by Lemma 4, $\bar{D}_x(A \cap E) = \bar{D}_x(E \cap K) = d$ and $\bar{D}_x(A - E) = \bar{D}_x(J - E) = 1 - d$. Since $E \in \mathscr{M}(x)$

$$1 = d + 1 - d = \bar{D}_x(A \cap E) + \bar{D}_x(A - E) = \bar{D}_x(A) = \max\{1 - d, d\} .$$

Therefore $d = 0$ or 1.

Next let E be a set whose density at x is zero or one. Let A be any Lebesgue measurable set and suppose $D_x(E) = 0$. Since $\bar{D}_x$ is monotone, $\bar{D}_x(A \cap E) \leqq D_x(E) = 0$ and hence $\bar{D}_x(A \cap E) = 0$. Since $\bar{D}_x$ is an outer measure

$$\bar{D}_x(A - E) \geqq \bar{D}_x(A) - \bar{D}_x(E) = \bar{D}_x(A) ,$$

and since $\bar{D}_x$ is monotone $\bar{D}_x(A) \geqq \bar{D}_x(A - E)$. Therefore $\bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E)$ and E is in $\mathscr{M}(x)$. In case $D_x(E) = 1$ the above argument with E replaced by $X - E$ gives the desired result.

2. Suppose that Z represents an F_σ set of measure zero. Define

the upper Lebesgue density of a measurable set E or Z by

$$\bar{D}_Z(E) = \sup\{\bar{D}_x(E) : x \in Z\}$$

and the lower Lebesgue density of E or Z by

$$\underline{D}_Z(E) = \inf\{\underline{D}(E) : x \in Z\} .$$

If $\bar{D}_Z(E) = \underline{D}_Z(E)$ we will say that the Lebesgue density of E on Z, denoted by $D_Z(E)$, exists and has the common value of $\bar{D}_Z(E)$ and $\underline{D}_Z(E)$. It is clear that if the density of E exists on Z then the density exists at every point of Z and has the same value at each point. In [2] it was shown that for any number d such that $0 < d < 1$, there exists a set E such that $D_z(E) = d$. Thus if $\mathscr{D}(Z)$ denotes the class of all sets whose density on Z exists, D_Z is a set function which maps $\mathscr{D}(Z)$ onto the closed unit interval. It is clear that D_Z will have the same properties as D_x where x is any point in Z.

LEMMA 7. *$\bar{D}_Z$ is a finitely subadditive outer measure defined on the class $\mathscr{M}$.*

Proof. The lemma follows immediately from the monotoniety and subadditivity of $\bar{D}_x$ and the definition of $\bar{D}_Z$.

Let $\mathscr{M}(Z)$ denote the class of all sets E such that $E \in \mathscr{M}$ and for every $A \in \mathscr{M}$, $\bar{D}_Z(A) = \bar{D}_Z(A \cap E) + \bar{D}_Z(A - E)$. Then $\mathscr{M}(Z)$ is an algebra and the restriction of $\bar{D}_Z$ to $\mathscr{M}(Z)$ is a finitely additive measure.

LEMMA 8. *$\mathscr{M}(Z)$ is a subset of $\mathscr{D}(Z)$.*

Proof. Let $E \in \mathscr{M}(Z)$. The real line X is in $\mathscr{M}$ so we have

$$1 = \bar{D}_Z(X) = \bar{D}_Z(E) + \bar{D}_Z(X - E) \geqq \sup\{\bar{D}_x(E) + \bar{D}_x(X - E) : x \in Z\}$$

and

$$\bar{D}_x(E) + \bar{D}_x(X - E) \leqq 1$$

for all $x \in Z$. But for any $x \in Z$, $\bar{D}_x$ is subadditive so that $\bar{D}_x((E) + \bar{D}_x(X - E) \geqq 1$. Therefore $\bar{D}_x(E) + \bar{D}_x(X - E) = 1$ for all $x \in Z$ and by Lemma 1, the density of E exists at every point of Z. Hence $\underline{D}_x(E) + \bar{D}_x(X - E) = 1$ for all x in Z and

$$\begin{aligned}\underline{D}_Z(E) + \bar{D}_Z(X - E) &\geqq \inf\{\underline{D}(E) + \bar{D}_x(E) : x \in Z\} \\ &= 1 = \bar{D}_Z(E) + \bar{D}_Z(X - E) .\end{aligned}$$

Since $\bar{D}_Z$ if finite, $\underline{D}_Z(E) \geqq \bar{D}_Z(E)$ and it follows that $E \in \mathscr{D}(Z)$.

THEOREM 2. *The class of all $\bar{D}_Z$-measurable sets is the class of* *all sets from* $\mathscr{D}(Z)$ *which are mapped onto* 0 *or* 1 *by* D_Z.

Proof. Let $\mathscr{K} = \{E : E \in D(Z) \text{ and } D_Z(E) = 0 \text{ or } 1\}$. If $E \in \mathscr{K}$ we may show that $E \in \mathscr{M}(Z)$ exactly as was done in Theorem 1.

Suppose $E \in \mathscr{M}(Z)$. By Lemma 8, $E \in \mathscr{D}(Z)$ and hence $D_Z(E) = D_x(E) = d$ for all $x \in Z$. Let x_1 be any point in Z and let $J = \{z : z < x_1\}$, $K = \{z : z \geqq x_1\}$. Define the set A by $A = (J - E) \cup (E \cap K)$. Then by Lemmas 4 and 5, $\bar{D}_{x_1}(A) = \max\{d, 1-d\}$, $\bar{D}_{x_1}(A \cap E) = d$, and $\bar{D}_{x_1}(A - E) = 1 - d$. Since $A \in \mathscr{M}$ and $E \in \mathscr{M}(Z)$,

$$\sup\{\bar{D}_x(A) : x \in Z\} = \sup\{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\}\ .$$

Let $\varepsilon > 0$ be given. Then there exists an $x_2 \in Z$ such that

$$\begin{aligned} \bar{D}_{x_2}(A) + \varepsilon &> \sup\{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\} \\ &\geqq \bar{D}_{x_1}(A \cap E) + \bar{D}_{x_1}(A - E) = 1\ . \end{aligned}$$

Suppose $x_2 < x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(X - E)$ and $1 - d + \varepsilon > 1$. Since ε was arbitrary and $1 - d \leqq 1$ we have $1 - d = 1$ and $d = 0$.

Suppose $x_2 > x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(E)$ and $d + \varepsilon > 1$. Since ε was arbitrary and $d \leqq 1$ we have $d = 1$.

Suppose $x_2 = x_1$. Then $\bar{D}_{x_2}(A) = \max\{d, 1-d\}$, and $\max\{d, 1-d\} + \varepsilon > 1$. Since ε was arbitrary $\max\{d, 1-d\} \geqq 1$. But both d and $1 - d$ do not exceed 1 so that $d = 0$ or 1.

Therefore E is in $\mathscr{K}$ and we have $\mathscr{M}(Z) = \mathscr{K}$.

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