AUTOMORPHISMS OF MONOMIAL GROUPS

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If H be an arbitrary group and S a set, then one obtains a monomial group after the manner described in [2]. Ore has in [2] determined the automorphisms of the monomial group when the set S has finite order. Here we obtain all automorphisms of a large class of monomial groups when the order of the set S is infinite. A monomial substitution over H is a linear transformation mapping each element x of S in a one-to-one manner onto some element of S multiplied formally by an element h of H. The element h is termed a factor of the substitution. A substitution all of whose factors are the identity e of H is called a permutation, while a substitution which maps each element of S into itself multiplied by an element of H is called a multiplication. A multiplication all of whose factors are equal is termed a scalar. The monomial substitutions restricted by the definitions of C and D as given below are elements of the monomial group denoted by $\Sigma(H;\ B, C, D)$, where the symbols in the name are to be interpreted as follows, H the given group, B the order of the given set S, C a cardinal number such that the number of non-identity factors of any substitution is less than C, D a cardinal number such that the number of elements of S being mapped into elements of S distinct from themselves by a substitution is less than D. $S(B, D)$ will denote the subgroup consisting of all permutations, while $V(B, C)$ will denote the subgroup consisting of all multiplications. Any substitution u may be written as the product of a multiplication v and a permutation s. Hence we may write $\Sigma(H;\ B, C, D) = V(B, C) \cup S(B, D)$, where $\cup$ here and throughout will mean group generated by the set.

The main result of this paper is to determine all automorphisms of the monomial group $\Sigma(H;\ B, d, C)$, $d \leqq C < B^{+}$, where B^{+} is the successor of B, $d = \aleph_0$ and to determine the automorhism group of this group.

LEMMA 1. *The basis group $V(B, d)$ is a characteristic subgroup of $\Sigma(H\colon\ B, d, C)$, $d \leqq C \leqq B^{+}$.*

Crouch has shown in [1] that $V(B, d)$ is a characteristic subgroup of $\Sigma(H;\ B, d, d)$. It is easy to show that if N is a subgroup of $V(B, d)$, then N is normal in $\Sigma(H;\ B, d, d)$ if and only if N is normal in $\Sigma(H;\ B, C, D)$. With this result Lemma 1 is an easy generalization of the result of Crouch.

LEMMA 2. *If T is an endomorphism of $V(B, d)$, then there exist*

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endomorphisms T^i_j *of* H *such that*

(1) $(e, \cdots, e, h_i, e, \cdots)T = (h_iT^i_1, \cdots, h_iT^i_j, \cdots)$, *for all* h_i *in* H.

(2) *For all* h *in* H, *and all* i, $hT^i_j = e$, *for all but a finite number of* j.

(3) $h_iT^i_mh_jT^j_m = h_jT^j_mh_iT^i_m$, *for all* m *and all* i, j *such that* $i \neq j$. *Conversely if* $\{T^i_j\}$ *is a collection of endomorphisms of* H, *such that* (2) *and* (3) *are true, then there exists one and only one endomorphism* T *of* $V(B, d)$ *such that* (1) *is true.*

The proof of Lemma 2 follows from direct computation.

LEMMA 3. *If* $G = N \cup M$, $N \cap M = e$, N *a characteristic subgroup of* G, μ *an automorphism of* G, $m\mu = n'm'$, *then the correspondence* $m\lambda = m'$ *is an automorphism of* M.

Proof. If $m \in M$; then $m\mu^{-1} = n'm'$, $(n'm')\mu = m = (n')\mu(m')\mu$, $m'\mu = (n')^{-1}\mu m$, hence $m'\lambda = m$, i.e. λ is onto. To see that λ is multiplication preserving we observe that

$$(m_1m_2)\mu = m_1\mu m_2\mu = n'_1m'_1n'_2m'_2 = (n'_1m'_1n'_2m'^{-1}_1)(m'_1m'_2) = n'_3m'_1m'_2 ,$$

and hence

$$m_1\lambda = m'_1,\ m_2\lambda = m'_2,\ (m_1m_2)\lambda = m'_1m'_2,\ m_1\lambda m_2\lambda = (m_1m_2)\lambda .$$

The endomorphism λ of M has kernel e since N is a characteristic subgroup of G, and hence, $m\mu = n'm'$, $m' = e$, if and only if $m = e$.

LEMMA 4. *Let* μ *be an automorphism of* $\Sigma(H; B, d, C)$, $d \leqq C \leqq B^+$, *and let* $s \in S(B, C)$, $s\mu = v's'$, $s\lambda = s'$; *then* λ *is an automorphism of* $S(B, C)$.

Proof. $\Sigma(H; B, d, C)$ splits over the basic group $V(B, d)$. Lemma 1 asserts that the basis group is a characteristic subgroup of the monomial group. Lemma 4 is then a consequence of Lemma 3.

LEMMA 5. $\Sigma(H; B, d, d)$ *is a characteristic subgroup of* $\Sigma(H; B, d, C)$, $d \leqq C \leqq B^+$.

Proof. Let μ be an automorphism of $\Sigma(H; B, d, C)$ and $u = vs \in \Sigma(H; B, d, d)$. Then $u\mu = (vs)\mu = v\mu s\mu$. Since $V(B, d)$ is a characteristic subgroup of $\Sigma(H; B, d, C)$, $v\mu \in V(B, d) \subset \Sigma(H; B, d, d)$. Now $s\mu$ is some element $v's'$ of $\Sigma(H; B, d, C)$, and by reason of Lemma 4 the correspondence s to s' induced by μ defines an automorphism of $S(B, C)$. But the automorphisms of $S(B, C)$ are the restriction of some automorphism of $S(B, B^+)$ to $S(B, C)$, and every automorphism of $S(B, B^+)$ is inner. See [4]. We may then write $s\mu = (v')(sI_{s^+})$, where $s^+ \in S(B, B^+)$ and I_{s^+} is the inner automorphism generated by s^+. Now $s \in S(B, d)$, and since $S(B, d)$ is normal in $S(B, C)$, $(sI_{s^+}) \in S(B, d)$. Then $u\mu = vs\mu = (v\mu)(v')(sI_{s^+})$, and each member of this product is an element of $\Sigma(H; B, d, d)$, which completes the proof of Lemma 5.

LEMMA 6. *If*

(1) *N is a normal subgroup of a group G,*

(2) *G splits over N, $G = N \cup M$, $N \cap M = e$,*

(3) *M' and N' are groups isomorphic to M and N respectively, α the isomorphism of M to M', β the isomorphism of N to N', N' normal in G', and $G' = M' \cup N'$, $M' \cap N' = e$,*

then the correspondence μ, $(mn)\mu = m\alpha n\beta$ defines an isomorphism between G and G' if and only if $m\alpha n\beta m^{-1}\alpha = (mnm^{-1})\beta$ for all m in M and all n in N.

The proof of Lemma 6 is contained in [2].

We will first find the automorphism group of $\Sigma(H, B, d, d)$ and then the automorphism group of $\Sigma(H; B, d, C)$, $d < C < B^+$. By reason of Lemma 5 the problem of finding automorphisms of $\Sigma(H; B, d, C)$ is made easy once the automorphisms of $\Sigma(H; B, d, d)$ are known.

Before proceeding to the problem of determining the automorphism group of $\Sigma(H; B, d, d)$ we make the following considerations. If T is any automorphism of the group H, we define an automorphism T' of $V(B, C)$, $d \leqq C \leqq B^+$, by the correspondence

$$(h_1, h_2, h_3, \cdots)T' = (h_1T, h_2T, h_3T, \cdots) .$$

Let I denote the identity automorphism of $S(B, D)$, $d \leqq D \leqq B^+$; then according to Lemma 6 the correspondence T^+, $(vs)T^+ = (v)T'(s)I$, for all $v \in V(B, C)$ and all $s \in S(B, D)$, is an automorphism of the group $\Sigma(H; B, C, D)$ if and only if

$$(s)I(v)T'(s^{-1})I = (svs^{-1})T' .$$

Since $V(B, C)$ is a normal subgroup of $\Sigma(H; B, C, D)$, this is an equality between multiplications, and it is easy to see that the corresponding factors of the two multiplications are equal. Hence T^+ is an automorphism of $\Sigma(H; B, C, D)$.

In a similar manner we may associate with any endomorphism K of the group H an endomorphism K^+ of $V(B, C)$.

THEOREM 1. *μ is an automorphism of $\Sigma(H; B, d, d)$ if and only if there exist*

(1) *s^+ an element of $S(B, B^+)$,*

(2) *v^+ an element of $V(B, B^+)$,*

(3) *T an automorphism of H ,*

such that

$$(u)\mu = (u)T^{+}I_{s^+}I_{v^+}, \text{ for all } u \in \Sigma(H; B, d, d) .$$

Proof. Suppose μ is an automorphism of $\Sigma(H; B, d, d)$. Then $\Sigma(H; B, d, d) = V(B, d)\mu \cup S(B, d)\mu$. But $V(B, d)$, by reason of Lemma 1 is a characteristic subgroup of $\Sigma(H; B, d, d)$, hence $\Sigma(H; B, d, d) = V(B, d) \cup S(B, d)\mu$, and $V(B, d) \cap S(B, d)\mu = E$.

There exists an isomorphism between $S(B, d)$ and $S(B, d)\mu$, whose form we now seek to discover. Since $S(B, d)\mu$ is contained in $\Sigma(H; B, d, d)$, the image of any element $s \in S(B, d)$ must have the form $v' s'$, where $v' \in V(B, d)$, $s' \in S(B, d)$. We have seen in Lemma 4 that the correspondence s to s' is an automorphism of $S(B, d)$, and hence there must exist an element $s^+ \in S(B, B^+)$ such that $s' = (s)I_{s^+}$, since all automorphisms of $S(B, d)$ have this form. The element s^+ is the element whose existence was asserted in (1) of the theorem.

Any element of $S(B, d)$ may be written as the product of a finite number of elements of the form $(1, i)$. Hence to discover the image of $(1, i)$ under μ, is to know the image of all permutations. We therefore reduce our study of $s\mu$ to that of $(1, i)\mu$. $(1, i)\mu = v_i s'$, where $s' = (1, i)I_{s^+}$.

We next proceed to the characterization of v_i and the calculation v^+ of $V(B, B^+)$.

Since the order of any transposition is two, we have

$$[(1, i)\mu]^2 = [v_i(1s^+, is^+)]^2 = E .$$

This equality can exist if and only if each factor of v_i has order two except possibly the $1s^+$ and is^+ factors, and moreover the $1s^+$ and is^+ factors must be inverses of one another.

We have in Lemma 2 discovered the form which all endomorphisms of $V(B, d)$ must have, and hence the form of all automorphisms of this group. For an arbitrary element v of $V(B, d)$

$$v = (\cdots, e, h_{i_1}, \cdots, h_{i_n}, e, \cdots) ,$$

we have,

$$(v)\mu = (h_{i_1}T_1^{i_1}h_{i_2}T_1^{i_2} \cdots h_{i_n}T_1^{i_n}, \cdots, h_{i_1}T_j^{i_1}h_{i_2}T_j^{i_2} \cdots h_{i_n}T_j^{i_n}, \cdots) ,$$

where the T_j^i are endomorphisms of the group H, and only a finite number of the factors of the multiplication are different from the identity.

In the calculations which follow the subscript of an element h will always indicate the position of h in a multiplication, that is h_j will be the jth factor of some multiplication v. Whenever we require two factors of an element which is a multiplication to be distinct we will indicate this by employing superscripts; distinct superscripts indicate that the two factors are distinct elements of H. Whenever a multiplication has undergone a transformation by a permutation we will employ superscripts to indicate, after the shuffling of factors, the equality existing between the factors of the original and resulting multiplication, like superscripts indicating the same group element.

Let us consider generating elements,

$$s = (1, i) \text{ of } S(B, d) ,$$
$$v = (\cdots, e, h_j, e, \cdots) \text{ of } V(B, d) .$$

Since μ is an automorphism of $\Sigma(H; B, d, d)$ we have

$$(s)\mu(v)\mu(s^{-1})\mu = (svs^{-1})\mu ,$$

where

$$(1, i)\mu = (\cdots, e, k_{i_1}, \cdots, k_{i_n}, e, \cdots)(1s^+, is^+) ,$$
$$(\cdots, e, h_j, e, \cdots)\mu = (h_jT_1^j, h_jT_2^j, h_jT_3^j, \cdots) ,$$

where only finitely many of the factors are different from the identity. We compute this equality considering two cases.

Case 1. Suppose $j \neq 1$, $j \neq i$. Then since $(svs^{-1}) = v$, the equality reduces to

$$(s)\mu(v)\mu(s^{-1})\mu = (v)\mu ,$$

or

$$[(\cdots, e, k_{i_1}, \cdots, k_{i_n}, e, \cdots)(1s^+, is^+)(h_jT_1^j, h_jT_2^j, h_jT_3^j, \cdots)$$
$$\times (1s^+, is^+)(\cdots, e, k_{i_1}^{-1}, \cdots, k_{i_n}^{-1}, e, \cdots)] = (h_jT_1^j, h_jT_2^j, h_jT_3^j, \cdots) .$$

Direct computation on the left side of the equality yields the following multiplication,

$$(\cdots, h_jT_m^j, \cdots, k_{i_1}h_jT_{i_1}^jk_{i_1}^{-1}, \cdots, k_{1s^+}h_jT_{is^+}^jk_{1s^+}^{-1}, \cdots$$
$$\cdots, k_{is^+}h_jT_{1s^+}^jk_{is^+}^{-1}, \cdots, k_{i_t}h_jT_{i_t}^jk_{i_t}^{-1}, \cdots) .$$

Then the resulting equality between multiplications demands equality between corresponding factors. Hence we have

$$\text{(i)} \qquad hT_{is^+}^j = k_{is^+}hT_{1s^+}^jk_{is^+}^{-1} ,$$
$$\text{(ii)} \qquad hT_{i_m}^j = k_{i_m}hT_{i_m}^jk_{i_m}^{-1} ,$$

for $m = 1, \cdots, n$, $i_m \neq 1s^+$, $i_m \neq is^+$, and $j \neq 1s^+$, $j \neq is^+$. Since in equalities (i) and (ii) h represents the same group element, we have dropped the subscript.

Case 2. Suppose $j = 1$ or $j = i$. Either equality will yield the same result, and hence both cases are included in one consideration. The calculations recorded are for $j = 1$.

$$v = (h_1^1, e, \cdots), s = (1, i) ,$$
$$(svs^{-1}) = (\cdots, e, h_i^1, e, \cdots) , \quad \text{and}$$
$$(s)\mu(v)\mu(s^{-1})\mu = (\cdots, e, h_i^1, e, \cdots)\mu , \quad \text{or}$$
$$[(\cdots, e, k_{i_1}, \cdots, k_{i_n}, e, \cdots)(1s^+, is^+)(h_1^1T_1^1, h_1^1T_2^1, h_1^1T_3^1, \cdots)$$
$$\times (1s^+, is^+)(\cdots, e, k_{i_1}^{-1}, \cdots, k_{i_n}^{-1}, e, \cdots)] = (h_i^1T_1^i, h_i^1T_2^i, h_i^1T_3^i, \cdots) .$$

Direct computation on the left side of the equality yields the following multiplication,

$$(\cdots, h_1^1T_m^1, \cdots, k_{i_1}h_1^1T_{i_1}^1k_{i_1}^{-1}, \cdots, k_{1s^+}h_1^1T_{is^+}^1k_{1s^+}^{-1}, \cdots$$
$$\cdots, k_{1s^+}h_1^1T_{1s^+}^1k_{is^+}^{-1}, \cdots, k_{i_t}h_1^1T_{i_t}^1k_{i_t}^{-1}, \cdots) .$$

Then the resulting equality between multiplications demands the following equality between factors.

(iii) $$hT_{i_m}^i = k_{i_m}hT_{i_m}^1k_{i_m}^{-1} ,$$
$$m = 1, \cdots, n , \quad i_m \neq 1s^+ , \quad i_m \neq is^+ .$$
(iv) $$hT_{is^+}^i = k_{is^+}hT_{1s^+}^1k_{is^+}^{-1} ,$$
(v) $$hT_{1s^+}^i = k_{1s^+}hT_{is^+}^1k_{1s^+}^{-1} .$$

The equalities (i) through (v) are restrictions on the endomorphisms T_j^i of H. We may now further our study of images of multiplications under μ in view of these restrictions.

Suppose $j \neq 1$ and consider,

$$(\cdots, e, h_j, e, \cdots)\mu = (h_jT_1^j, h_jT_2^j, h_jT_3^j, h_jT_4^j, \cdots) .$$

According to restriction (i) each factor in the image multiplication is conjugate to $h_iT_{1s^+}^j$ except the factor $h_jT_{js^+}^j$. But since μ is an automorphism of $V(B, d)$, the image multiplication must be an element of $V(B, d)$, hence only finitely many of the factors may be different from the identity. It then follows that every factor save the factor $h_jT_{js^+}^j$ must be the identity and in this case the factor $h_jT_{js^+}^j$ must be different from the identity. That is, for j different from 1,

$$(\cdots, e, h_j, e, \cdots)\mu = (\cdots, e, h_jT_{js^+}^j, e, \cdots) .$$

We next consider the case where $j = 1$.

$$(h_1, e, \cdots)\mu = (h_1T_1^1, h_1T_2^1, h_1T_3^1, \cdots) .$$

If we rewrite (v) in the form $hT_{is^+}^1 = k_{is^+}hT_{1s^+}^ik_{is^+}^{-1}$, we see that every factor of the above recorded image multiplication is conjugate to some element $h_iT_{1s^+}^i$. But we have observed in the previous consideration that for $j \neq 1, h_jT_{1s^+}^j$ is the identity element, and hence all factors of the image multiplication are the identity except the $1s^+$ factor. That is,

$$(h_1, e, \cdots)\mu = (\cdots, e, h_1 T^1_{1s^+}, e, \cdots) .$$

In the beginning we assumed the most general representation of an automorphism of $V(B, d)$ for μ, and for the correspondence assigned we have only an endomorphism of $V(B, d)$. We must now determine what further restrictions are necessary to insure that the correspondence is an automorphism of $V(B, d)$. Suppose we are given an arbitray multiplication of $V(B, d)$,

$$(\cdots, e, h'_{i_1}, \cdots, h'_{i_n}, e, \cdots) .$$

We ask if this multiplication arose from the image of some other multiplication under μ. This is equivalent to asking under what conditions will the set of equations,

$$h_{i_m} T^{i_m}_{i_m s^+} = h'_{i_m} , \quad m = 1, \cdots, n ,$$

have unique solutions h_{i_m}, m $= 1, \cdots, n$, in H. Such a unique set of solutions can exist if and only if the $T^i_{is^+}$ are automorphisms of the group H. With this added restriction we have completed the characterization of the images of multiplications, but will employ (iv) to change the representation later.

Let us refer to equality (ii) restricting the endomorphisms whose subscripts are different from $1s^+$ and is^+. We have seen that if i_m be different from js^+ then $hT^j_{i_m}$ is the identity. In Case 1, which produced equality (ii) we have restricted j to be different from 1 and i, so that j may be so chosen that $js^+ = i_m$, and the following equality results,

$$k_{is^+} h T^j_{js^+} k^{-1}_{js^+} = h T^j_{js^+} .$$

Inasmuch as we have required that $T^j_{js^+}$ be an automorphism of H, we can only conclude that k_{js^+} belongs to the center of the group H. That is, the multiplication component of the image of $(1, i)$ under μ must have every factor except possibly the $1s^+$ and the is^+ factors belonging to the center of the group H.

We will now show that the factors of this multiplication which do not occupy the $1s^+$ and is^+ positions are the identity element.

Since $(1, i)(1, j)$ has order three, we have

$$[(1, i)(1, j)\mu]^3 = [(k_1, k_2, k_3, \cdots)(1s^+, is^+)(h_1, h_2, h_3, \cdots)(1s^+, js^+)]^3 = E .$$

By direct calculation we see that if n be different from $1s^+$, is^+, and js^+, then the nth factor is $k_n h_n k_n h_n k_n h_n = e$. We have previously seen that both h_n and k_n belong to the center of the group H, and moreover each has order two. It then follows that h_n and k_n are inverses of one another. The $1s^+$ factor of the above product is

$$k_{1s^+}h_{is^+}k_{is^+}h_{1s^+}k_{js^+}h_{js^+} = e ,$$

which, in view of the centrality of the elements k_{js^+} and h_{is^+} together with the equality

$$k_{1s^+}k_{is^+} = h_{1s^+}h_{js^+} = e ,$$

reduces to $h_{is^+}k_{js^+} = e$. Since h_{is^+} has order two, $h_{is^+} = k_{js^+}$. Thus the factors of the image multiplications $(1, i)$ and $(1, j)$ are the same if we exclude the $1s^+$, is^+, and js^+ factors, and further the js^+ factor of the multiplication component of $(1, i)\mu$ is equal to the is^+ factor of the multiplication component of $(1, j)\mu$.

In a similar manner by considering $(1, j)\mu$ and $(1, t)\mu$ where $t \neq i$, $t \neq j$, we find that the ts^+ factor of the multiplication component of $(1, j)\mu$ is equal to the js^+ factor of the multiplication component of $(1, t)\mu$.

But the ts^+ factors of the multiplication component of $(1, i)\mu$ and $(1, j)\mu$ are equal, and the js^+ factor of the multiplication component of $(1, i)\mu$ and $(1, t)\mu$ are equal. That is, the ts^+ and js^+ factors of the multiplication component of $(1, i)\mu$ are equal, and hence all factors of the multiplication component of $(1, i)\mu$, except possibly the $1s^+$ and is^+ factors. But this multiplication is an element of $V(B, d)$ and hence all factors except possibly the $1s^+$ and is^+ factors must be e. Then

$$(1, i)\mu = (\cdots, e, k_{1s^+} e, \cdots, e, k_{is^+}, e, \cdots)(1s^+, is^+) .$$

Let v^+ be the multiplication of $V(B, B^+)$ whose $1s^+$ factor is e, and whose is^+ factor is the is^+ factor k_{is^+} of the multiplication component of $(1, i)u$. This multiplication v^+ is the element of $V(B, B^+)$ whose existence we asserted in (2) of the theorem.

We have seen that

$$(h_1, e, \cdots)\mu = (\cdots, e, h_1T^1_{1s^+}, e, \cdots)$$

where $T^1_{1s^+}$ is an automorphism of H. Let $T^1_{1s^+}$ generate, in a manner described in the discussion preceding this theorem, an automorphism T^+ of $\Sigma(H; B, C, D)$, $d \leqq C, D \leqq B^+$, which is moreover an automorphism of $\Sigma(H; B, d, d)$ since $\Sigma(H; B, d, d)$ is a characteristic subgroup of $\Sigma(H; B, D, C)$. This is the automorphism which forms the first component of μ, and $T^1_{1s^+}$ is the automorphism of H whose existence we asserted in (3) of the theorem.

If we now refer to restriction (iv) on the automorphisms $T^i_{is^+}$, $hT^i_{is^+} = k_{is^+}hT^1_{1s}k^{-1}_{is^+}$, we observe that we may write

$$(\cdots, e, h_j, e, \cdots)\mu = (\cdots, e, k_{js^+}h_jT^1_{1s^+}k^{-1}_{js^+}, e, \cdots) ,$$
$$(1, i)\mu = (\cdots, e, k_{1s^+}, e, \cdots, e, k_{is^+}, e, \cdots) \times (1s^+, is^+) ,$$

which we may now record in simplified form as

$$(\cdots, e, h_h, e, \cdots)\mu = (\cdots, e, h_j, e, \cdots)T^+I_{s^+}I_{v^+} ,$$

$(1, i)\mu = (1, i)T^+I_sI_{v^+}$, and hence for an arbitrary element u of $\Sigma(H; B, d, d)$,

$$(u)\mu = (u)T^+I_{s^+}I_{v^+} .$$

Conversely suppose we are given an element s^+ of $S(B, B^+)$, $v^+ \in V(B, B^+)$, T an automorphism of H. Then I_{s^+}, I_{v^+}, and T^+ are each automorphisms of $\Sigma(H; B, C, D)$ $d \leqq C$, $D \leqq B^+$, and hence the product $T^+I_{s^+}I_{v^+}$ is an automorphism of the group. Then the groups $\Sigma(H; B, d, d)$ and $\Sigma(H; B, d, d)T^+I_{s^+}I_{v^+}$ are isomorphic. But each of the automorphisms T^+, I_{s^+}, and I_{v^+} take elements of $\Sigma(H; B, d, d)$ into $\Sigma(H; B, d, d)$. Hence the restriction of the automorphism $T^+I_{s^+}I_{v^+}$ of $\Sigma(H; B, C, D)$ to $\Sigma(H; B, d, d)$ is an automorphism of the latter group. This is the automorphism μ, and the proof of the theorem is complete.

COROLLARY 1. *$\mu = T^+I_{s^+}I_{v^+}$ is an inner automorphism of $\Sigma(H; B, d, d)$ if and only if T^+ is generated by an inner automorphism $T = I_{h^{-1}}$ of H, $s^+ \in S(B, d)$, v^+ is the product of an element of $V(B, d)$ and the scalar $[h]$ of $\Sigma(H; B, B^+, B^+)$.*

Proof. If T^+ is generated by the automorphism $I_{h^{-1}}$, $s^+ \in S(B, d)$, $v^+ = v_1^+[h]$, $v_1^+ \in V(B, d)$, $[h] \in V(B, B^+)$, then

$$\mu = T^+I_{s^+}I_{v_1^+[h]} = T^+I_{s^+}I_{[h]}I_{v_1^+} = T^+I_{[h]}I_{s^+}I_{v_1^+} = I_{s^+}I_{v_1^+} = I_{v_1^+s^+} ,$$

and hence μ is an inner automorphism of $\Sigma(H; B, d, d)$.

Conversely suppose μ is an inner automorphism of $\Sigma(H; B, d, d)$; then

$$\mu = I_u = I_{v's'} = I_{s'v'} .$$

Hence if $h = e$, and $T = I_{h^{-1}}$, $\mu = T^+I_{s'}I_{v^+}$, where $s' \in S(B, d)$, $v^+ \in V(B, B^+)$, $v^+ = v'[h] = v' \in V(B, d)$.

THEOREM 2. *The group of three-tuples (T, s^+, v^+), where T is an automorphism of the group H, $s^+ \in S(B, B^+)$, $v^+ \in V(B, B^+)$, with the operation*

$$(T_1, s_1^+, v_1^+)(T_2, s_2^+, v_2^+) = (T_1T_2, s_2^+s_1^+, v_2^+s_2^+(v_1^+T_1^+)s_2^{+-1}) ,$$

is homomorphic to the automorphism group of $\Sigma(H; B, d, d)$ under the correspondence λ, $(T, s^+, v^+)\lambda = \mu, \mu = T^+I_{s^+}I_{v^+}$, and the kernel K of λ is the set of all three-tuples (T, s^+, v^+), where s^+ is the identity permutation of $S(B, B^+)$, v^+ is a scalar $[h]$ of $V(B, B^+)$, and T is the inner *automorphism* $I_{h^{-1}}$ *of the group* H.

Proof. That the set of three-tuples (T, s^+, v^+) with the above defined operation form a group follows by direct computation.

Let λ be the correspondence between the group of three-tuples and the automorphism group of $\Sigma(H; B, d, d)$ as defined in the theorem. We will show λ is a homomorphism.

The correspondence λ is onto, for given any automorphism $\mu = T^+ I_{s^+} I_{v^+}$, there exists a three-tuple, namely (T, s^+, v^+), such that $(T, s^+, v^+)\lambda = \mu$. Direct computation reveals that λ is multiplication preserving.

Therefore λ is a homomorphism from the group of three-tuples onto the automorphism group of $\Sigma(H; B, d, d)$.

We compute the kernel K of λ. Let $\mu_0 \in K$.

$$(1, i)\mu_0 = (1, i)T^+ I_{s^+} I_{v^+} = (1, i) .$$

But T^+ acts as the identity automorphism on permutations, and therefore the equality reduces to

$$(1, i)I_{s^+} I_{v^+} = (1, i) ,$$

which can exist for all i if and only if s^+ leaves all i fixed and therefore is the identity permutation. Then $(1, i)I_{+} = (1, i)$, for all i, if and only if v^+ is a scalar $[h]$.

Consider

$$(h, e\text{-} \cdots)\mu_0 = (h, e, \cdots)T^+ I_{s^+} I_{v^+} = (\overline{khT}k^{-1}, e, \cdots) = (h, e, \cdots) .$$

This equality can exist if and only if $T = I_{k^{-1}}$.

Thus we have shown that the kernel K of λ is the set of three-tuples (T, s^+, v^+), where s^+ is the identity permutation of $S(B, B^+)$, v^+ a scalar $[k]$ of $V(B, B^+)$, T the inner automorphism $I_{k^{-1}}$ of H.

COROLLARY 1. *Let A denote the automorphism group of $\Sigma(H; B, d, d)$, A_s those elements of A which leave $S(B, d)$ fixed elementwise. Then*

(1) *A_s is a subgroup of A, such that any automorphism μ in A_s has the form $\mu = T^+ I_{[h]}$, $[h]$ a scalar of $V(B, B^+)$.*

(2) *The set of two-tuples (T, h), T an automorphism of H, h an element of H, form a group with the operation, $(T_1, h_1)(TT_2, h_2) = (T_1T_2, h_2(h_1T_2))$.*

(3) *The group of two-tuples are homomorphic to A_s under the homomorphism λ, $(T, h)\lambda = \mu$, $\mu = T^+ I_{[h]}$.*

(4) *The kernel K of λ is the set of two-tuples $(I_{h^{-1}}, h)$.*

Proof. The assertions (1) through (4) are immediate consequences of the theorem, since the set of two-tuples form a group isomorphic to a subgroup of the group of three-tuples under the correspondence

$$(T, h) \longleftrightarrow (T, s_0^+, [h]).$$

THEOREM 3. *μ is an automorphism of $\Sigma(H; B, d, C)$, $d < C < B^+$, if and only if there exist*

(1) $s^+ \in S(B, B^+)$,

(2) $v^+ \in V(B, d)$,

(3) *T an automorphism of H ,*

such that $(u)\mu = (u)T^+I_{s^+}I_{v^+}$ for all $u \in \Sigma(H; B, d, C)$.

Proof. We have seen in Lemma 5 that $\Sigma(H; B, d, d)$ is a characteristic subgroup of $\Sigma(H; B, d, C)$; hence if μ is an automormphism of $\Sigma(H; B, d, C)$, its restriction to $\Sigma(H; B, d, d)$ is an automorphism of that group. We have in Theorem 1 determined all automorphisms of $\Sigma(H; B, d, d)$; hence we will be concerned with extending the automorphisms of $\Sigma(H; B, d, d)$ to automorphisms of $\Sigma(H; B, d, C)$. As is evident from the statement of the theorem not all automorphisms of $\Sigma(H; B, d, d)$ may be extended to an automorphism of $\Sigma(H; B, d, C)$.

There is determined by μ an element s^+ of $S(B, B^+)$ such that

$$(s)\mu = (v')(sI_{s^+}),\ s \in S(B, d)\ .$$

If $s \in S(B, C)$ then

$$(s)\mu = v's',\ v' \in V(B, d),\ s' \in S(B, C).$$

According to Lemma 4 the correspondence λ, $s\lambda = s'$, is an automorphism of $S(B, C)$. The automorphism induced on $S(B, d)$ by μ extends to $S(B, C)$ in one and only one way, hence $\lambda = I_{s^+}$, and the elements $s^+ \in S(B, B^+)$ is the element whose existence was asserted in (1) of the theorem.

Any element $s \in S(B, C)$, may be decomposed into the product of two elements s_1s_2 such that the order of each s_1 and s_2 is two. See [4]. We will therefore reduce our study of $s\mu$ to that of $s_1\mu$.

We then have

$$(s_1)\mu = v_1(s_1I_{s^+}),\ \ v_1 \in V(B, d)\ ,$$

and since s_1 has order two, $[v_1(s_1I_{s^+})]^2 = E$. We observe the factors of v_1 considering two cases.

Suppose n is an index such that x_n does not belong to the set of elements moved by $s_1I_{s^+}$; then it follows from the above equality that the nth factor of v_1 has order two. On the other hand if i is an index such that x_i is moved by $(s_1)I_{s^+}$, then there is an index j such that (x_i, x_j) is a transposition of $(s_1)I_{s^+}$. Then the above equality demands that the ith and jth factors of v_1 must be inverses of one another.

If n is an index such that x_n does not belong to the set of elements moved by s_1, we will show that k_{ns^+} belongs to the center of the group H. Let

$$v = (\cdots, e, h_n, e, \cdots)$$

and consider

$$\begin{aligned}(s_1 v s_1^{-1})\mu &= (v)\mu = (\cdots, e, h_n T^n_{ns^+}, e, \cdots) = (s_1)\mu(v)\mu(s_1^{-1})\mu \\ &= v_1(s_1 I_{s^+})(\cdots, e, h_n T^n_{ns^+}, e, \cdots)(s_1^{-1})I_{s^+}(v_1^{-1}) \\ &= (\cdots, e, k_{ns} h_n T^u_{ns^+} k^{-1}_{ns^+}, e, \cdots) .\end{aligned}$$

This equality of multiplications demands the following equality of factors: $hT^n_{ns^+} = k_{ns^+} h T^n_{ns^+} k^{-1}_{ns^+}$.

Since $T^n_{ns^+}$ is an automorphism of the group H, it follows that k_{ns^+} belongs to the center of H. That is, all factors of v_1 belongs to the center of H except possibly those factors j such that x_{js^+} belongs to the set of elements moved by $(s_1)I_{1^+}$.

We next show that each of these factors which belong to the center of H is moreover the identity element of H. Let

$$s_1 = (x_1, x_2)(x_3, x_4) \cdots$$

and define an element $s_t \in S(B, C)$ as follows,

$$s_t = (x_1, x_{t_2})(x_3, x_{t_4}) \cdots$$

where the x_{t_i} do not belong to the set of elements moved by s_1, and hence s_t has order two. The existence of such an element s_t is insured since we have required that $c < B^+$, and hence s_1 must move fewer than B elements. Since $s_t s_1$ has order three, we have

$$\begin{aligned}&[(s_1 s_t)\mu]^3 = E , \\ &(s_1)\mu = v_1(s_1)I_{s^+} , \qquad v_1 = (h_1, h_2, h_3, \cdots) , \\ &(s_t)\mu = v_t(s_t)I_{s^+} , \qquad v_t = (k_1, k_2, k_3, \cdots) .\end{aligned}$$

By direct calculation of the above equality we discover that we have in the $1s^+$ position the factor

$$h_{1s^+} k_{2s^+} h_{2s^+} k_{1s^+} h_{t_2 s^+} k_{t_2 s^+} = e .$$

But x_2 does not belong to the set of elements moved by s_t and x_{t_2} does not belong to the set of elements moved by s_1, hence k_{2s^+} and $h_{t_2s^+}$ belong to the center of H, and since $h_{1s^+} h_{2s^+} = k_{1s^+} k_{t_2s^+} = e$, the factor reduces to $k_{2s^+} h_{t_2s^+} = e$. Then $k_{2s^+} = h_{t_2s^+}$ since each of the elements has order two.

Consider a third permutation of $S(B, C)$,

$$s_w = (x_1, x_{w_2})(x_3, x_{w_4}) \cdots$$

where the x_{w_i} do not belong to the set of elements moved by s_1 or s_t.

$$(s_w)\mu = v_w(s_w)I_{s^+} , \quad v_w = (f_1, f_2, f_3, \cdots) .$$

Then calculations similar to those just performed with the elements s_t and s_w yield

$$k_{w_i s^+} = f_{t_i s^+} , \qquad i = 2, 4, 6, \cdots ,$$

but $h_{w_i s^+} = k_{w_i s^+}$, and hence,

$$h_{w_i s^+} = f_{t_i s^+} = h_{t_i s^+} .$$

Therefore all factors of v_1 are equal except possibly those factors h_j such that $x_j(s_1)I_{s^+} \neq x_j$. But $v_1 \in V(B, d)$, and hence all factors of v_1 are e except possibly the factors h_j, j an index such that $x_j(s_1)I_{s^+} \neq x_j$.

We have then the following information regarding v_1, $s_1\mu = v_1(s_1)I_{s^+}$. If (x_i, x_j) is a transposition of s_1 then $h_{is^+}h_{js^+} = e$. If x_m does not belong to the set of elements moved by s_1 then h_{ms^+} is the identity.

Let us consider $(s_1)\mu(x_i, x_j)\mu$, where (x_i, x_j) is a transposition of s_1. Since (x_i, x_j) is an element of $\Sigma(H; B, d, d)$, a characteristc subgroup of $\Sigma(H; B, d, C)$, we have

$$\begin{aligned}(x_i, x_j)\mu &= (\cdots, e, h_{is^+}, e, \cdots, e, h_{js^+}, e, \cdots)(x_{is^+}, x_{js^+}) \\ &= v_1'(s_1')I_{s^+} , \quad \text{where} \quad s_1' = s(x_i, x_j) .\end{aligned}$$

Since (x_i, x_j) is not a transposition of s_1' the is^+th and js^+th factors of v_1' are e, but the is^+th factor of v_1' is the product of the is^+th factor of v_1 and the js^+th factor of the multiplication component of $(x_i, x_j)\mu$. Hence the is^+ and js^+ factors of v_1 are identical with the factors in the corresponding positions of the multiplication component of $(x_i, x_j)\mu$. The multiplication component of $(x_i, x_j)\mu$ was formed by conjugating $(x_i, x_j)I_{s^+}$ with an element $v_1^+ \in V(B, B^+)$. It is evident that the is^+ and js^+ factors of v_1 can be formed in the same manner.

In the event that s_1 moves an infinite number of elements, it is not possible that all h_{js^+} be different from the identity; yet we have seen that all h_{js^+} are formed by conjugation by the element v_1^+ determined by the restriction of μ to $\Sigma(H; B, d, d)$. If (x_i, x_j) be a transformation of s_1, and if the is^+ and js^+ factors of v_1^+ are distinct, then the is^+ and js^+ factors of v_1 will be distinct. We must then restrict v_1^+ in such a manner that this situation can happen only a finite number of times. Hence we must require that no two factors of v_1^+ be repeated infinitely often, and there must not occur in v_1^+ an infinite number of distinct factors. Under these restrictions v_1 will always be an element of $V(B, d)$.

The v_1^+ so restricted may then be written as a product $v^+[k]$, where $v^+ \in V(B, d)$ and $[k] \in V(B, B^+)$, k being that factor of v_1^+ which was repeated infinitely often. Then

$$\mu = T_1^+ I_{s^+} I_{v^+} = T_1^+ I_{s^+} I_{v^+[k]} = T_1^+ I_{s^+} I_{[k]} I_{v^+} = T_1^+ I_{[k]} I_{s^+} I_{v^+} = T^+ I_{s^+} I_{v^+} ,$$

where T^+ is generated by the automorphism TI_k of H.

Conversely, given an element $s^+ \in S(B, B^+)$, $v^+ \in V(B, B^+)$, and T an automorphism of H, then I_{s^+}, I_{v^+}, and T^+ are automorphisms of $\Sigma(H; B, B^+, B^+)$. Hence the groups $\Sigma(H; B, d, C)$ and $\Sigma(H; B, d, C)T^+ I_{s^+} I_{v^+}$ are isomorphic. But each of the automorphisms T^+, I_{s^+}, and I_{v^+} of $\Sigma(H; B, B^+, B^+)$ takes elements of $\Sigma(H; B, d, C)$ into elements of $\Sigma(H; B, d, C)$. Hence the restriction of the automorphism $T^+ I_{s^+} I_{v^+}$ of $\Sigma(H; B, B^+, B^+)$ to $\Sigma(H; B, d, C)$ is an automorphism of the latter group. This is the automorphism μ, and this completes the proof of the theorem.

COROLLARY 1. *μ is an inner automorphism of $\Sigma(H; B, d\, C)$, $d < C < B^+$, if and only if T^+ is generated by the identity automorphism of H, and s^+ is an element of $S(B, C)$.*

Proof. If T^+ is generated by the identity automorphism of H, and $s^+ \in S(B, C)$, then

$$\mu = T^+ I_{s^+} I_{v^+} = I_{s^+ v^+},\quad v^+ s^+ \in \Sigma(H; B, d, C) ,$$

and hence μ is an inner automorphism.

Conversely suppose μ is inner,

$$\mu = I_u ,\quad u \in \Sigma(H; B, d, C);\quad \text{then}$$
$$\mu = I_u = T^+ I_{s^+} I_{v^+} ,\quad \text{and}\quad T^+ I_{s^+} = I_u I_{(v^+)^{-1}} .$$

Moreover, $(s)I_{s^+} = (s)I_u I_{v^+}$ for all $s \in S(B, C)$; therefore $s^+ \in S(B, C)$. Then finally $T^+ = I_u I_{(v^+ s^+)^{-1}}$ is an inner automorphism. Since T^+ leaves $S(B, C)$ fixed elementwise, $T^+ = I_{[h]}$, but $[e]$ is the only scalar of $\Sigma(H; B, d\, C)$, hence T^+ is generated by the identity automorphism.

THEOREM 4. *The group of three-tuples (T, s^+, v^+), T an automorphism of H, $s^+ \in S(B, B^+)$, $v^+ \in V(B, d)$, with the operation,*

$$(T_1, s_1^+, v_1^+)(T_1, s_1^+, v_1^+) = (T_1 T_2, s_2^+ s_1^+, v_2^+ s_2^+ (v_1 T_2) s_2^{+-1}) ,$$

is isomorphic to the automorphism group of

$$\Sigma(H; B, d, C) ,\quad d < C < B^+ .$$

Proof. The set of three-tuples form a sub-group of the set of three-tuples of Theorem 3 and hence the mapping defined there is a homomorphic mapping of the set of three-tuples named above onto the automorphism group of $\Sigma(H; B, d, C)$. Call this restriction of the homomorphism λ of Theorem 3, λ'. Then the kernel K' of λ' is contained in the kernel K of λ. But the only scalar contained in $V(B, d)$ is the identity multiplication; hence K' has order one, and λ' is the desired isomorphism.

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