ON CERTAIN NON-LINEAR OPERATORS AND PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction and summary. Consider a partial differential equation

$$\Phi\left(\frac{\partial^k u}{\partial t^k}, \frac{\partial^k u}{\partial t^{k-1}\partial y}, \cdots, u, y, t\right) = 0 \tag{1.1}$$

with boundary conditions of the type

$$\begin{aligned} &\left.\frac{\partial^{2i} u}{\partial y^{2i}}\right|_{y=0} = \left.\frac{\partial^{2i} u}{\partial y^{2i}}\right|_{y=\pi} = 0 \qquad (i = 0, 1, \cdots, j) ; \\ &\left.\frac{\partial^i u}{\partial t^i}\right|_{t=0} = f_i(y) , \qquad (i = 0, 1, \cdots, k) . \end{aligned} \tag{1.2}$$

By means of a Fourier sine-series expansion with respect to one of the independent variables, say y,

$$\begin{aligned} u(y, t) &= \sum_{n=1}^{\infty} X_n(t) \sin (ny) , \\ X_n(t) &= \frac{2}{\pi}\int_0^\pi u(y, t) \sin (ny) dy \end{aligned} \tag{1.3}$$

there corresponds to the system (1.1), (1.2) an infinite system of ordinary differential equations in the X_n's

$$\Phi_n\left(t, X_1(t), \frac{dX_1}{dt}, \cdots, \frac{d^k X_1}{dt^k}, X_2(t), \cdots\right) = 0 \tag{1.4}$$

with the boundary conditions

$$\left.\frac{d^i X_n}{dt^i}\right|_{t=0} = \frac{2}{\pi}\int_0^\pi f(y) \sin (ny) dy \tag{1.5}$$

where

$$\begin{aligned} \Phi_n(t, s_1^0, s_1^1, \cdots, s_1^k, s_2^0, \cdots) = \frac{2}{\pi}\int_0^\pi \Phi\Big(\sum_{i=1}^{\infty} s_i^k \sin (iy), \\ \sum_{i=1}^{\infty} i s_i^{k-1} \cos (iy), \cdots, \sum_{i=1}^{\infty} s_i^0 \sin (iy), y, t\Big) \sin (ny) dy . \end{aligned} \tag{1.6}$$

Disregarding for the moment all questions of convergence of the series and permissibility of term by term differentiation and integration, the two systems (1.1), (1.2) and (1.4), (1.5) are equivalent; so that a

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partial differential equation has thus been reduced to an (albeit infinite) system of ordinary differential equations.

D. C. Lewis [1], has put this method on a rigorous basis for second-order differential equations of the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial y^2} = \Phi\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}, u, y, t\right) \tag{1.7}$$

with boundary conditions of the type

$$\begin{gathered} u(0, t) = u(\pi, t) = 0 \\ u(y, 0) = f(y) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(y) \end{gathered} \tag{1.8}$$

where the functions f, g and Φ are assumed to satisfy certain conditions which are stated below. Lewis constructs a system of solutions $X_n(t)$ of the infinite system of the type (1.4), (1.5) corresponding to (1.7), (1.8) and proves that the function

$$u(y, t) = \sum_{n=1}^{\infty} X_n(t) \sin(ny)$$

is (in a certain generalized sense) a solution of (1.7), (1.8).

Following a suggestion of D. C. Lewis to generalize his result, the present paper does so by applying his method to operators in Hilbert space.

After introducing the notation and definitions of § 2, we establish in § 3 some results concerning solution of the equation $Tu = 0$ where T is a (non-linear) operator in Hilbert space. T is of the form $T = L - SN$ where L and N are linear, and S satisfies a Lipschitz condition—with respect to a "partial norm," which assigns to an element of the Hilbert space a real-valued function rather than a real number.

Sections 4, 5 present, as applications, Lewis' theorem and some existence theorems for non-linear higher order partial differential equations of the form

$$\frac{\partial^k}{\partial t^k}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)^m u$$
$$= \Phi\left(\frac{\partial^{m+k} u}{\partial t^{m+k}}, \frac{\partial^{m+k} u}{\partial t^{m+k-1}\partial y}, \cdots, \frac{\partial^{m+k} u}{\partial t^k \partial y^m}, \frac{\partial^{m+k-1} u}{\partial t^{m+k-1}}, \cdots, u, y, t\right).$$

We will conclude this introduction by restating Lewis' result for later reference:

THEOREM (D. C. Lewis). *Let $\Phi(p_1, p_2, u, y, t)$ be a real-valued function defined and uniformly continuous in y and t in a domain $\Omega = (p_1, p_2, u, y, t) \mid |u| \leqq h,\ 0 \leqq y \leqq \pi,\ 0 \leqq t \leqq \tau\}$. Suppose there exists a* *positive constant* θ *such that*

$$|\Phi(p_1, p_2, u, y, t) - \Phi(\bar{p}_1, \bar{p}_2, \bar{u}, y, t)| < \theta[|p_1 - \bar{p}_1| + |p_2 - \bar{p}_2| + |u - \bar{u}|] \quad \text{for } |u| \leqq h,\ |\bar{u}| \leqq h,\ 0 \leqq y \leqq \pi,\ 0 \leqq t \leqq \tau . \tag{1.9}$$

Let $f(y)$, $g(y)$ *be defined on* $0 \leqq y \leqq \pi$; *let* $f(y)$ *moreover be differentiable and let the Lebesque integrals*

$$\int_0^\pi [f'(y)]^2 dy , \qquad \int_0^\pi [g(y)]^2 dy$$

exist and be $\leqq 3h^2/4\pi^2$ *and suppose that* $f(0) = f(\pi) = 0$.

Then there exists a positive $\sigma \leqq \tau$ *such that the differential equation*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial y^2} = \Phi\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}, u, y, t\right) \tag{1.10}$$

has a unique solution $u(y, t)$ *in the generalized sense explained below, defined for* $0 \leqq y \leqq \pi$, $0 \leqq t \leqq \sigma$, *which satisfies the boundary conditions*

$$u(0, t) = u(\pi, t) = 0 , \qquad u(y, 0) = f(y) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(y) . \tag{1.11}$$

By solution in the generalized sense is meant that (in the domain $0 \leqq y \leqq \pi$, $0 \leqq t \leqq \sigma$) $u(y, t)$ is continuous, the first partial derivatives of u exist almost everywhere, and there exists a sequence of functions $u_n(y, t)$, each of class C'' for $0 \leqq y \leqq \pi$, $0 \leqq t \leqq \sigma$ such that

$$\lim_{n\to\infty} u_n(y, t) = u(y, t) \text{ uniformly in } y \text{ and } t \text{ on } (0 \leqq y \leqq \pi,\ 0 \leqq t \leqq \sigma)$$

$$\left.\begin{aligned} &\lim_{n\to\infty} \int_0^\pi \left(\frac{\partial u}{\partial y} - \frac{\partial u_n}{\partial y}\right)^2 dy = 0 \\ &\lim_{n\to\infty} \int_0^\pi \left(\frac{\partial u}{\partial t} - \frac{\partial u_n}{\partial t}\right)^2 dy = 0 \end{aligned}\right\} \text{uniformly in } t \text{ on } (0 \leqq t \leqq \sigma)$$

$$\lim_{n\to\infty} \int_0^\pi \int_0^\sigma \left[\frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^2 u_n}{\partial y^2} - \Phi\left(\frac{\partial u_n}{\partial t}, \frac{\partial u_n}{\partial y}, u_n, y, t\right)\right]^2 dy dt = 0 .$$

2. Notation and definitions. Greek and small latin letters denote: real numbers; real-valued functions; and elements of Hilbert space, function spaces, and measure spaces. Capital latin letters denote subsets of, and operators defined in, these spaces. The symbols $\in$, $\subset$, $\cup$, $\cap$, $\{x \mid \cdots\}$ resp. have the usual meanings: element of, subset of, union, intersection, the set of elements X for which $\cdots$ holds, resp.

The abstract space considered here is a *complete and separable Hilbert space* H over the field of real numbers. The *inner product* of two elements u, v of H will be denoted by (u, v); the *norm* of u by $\|u\|$.

If u_n is a sequence of elements of H, converging (in the norm) to an element u of H, that is, if $\lim_{n\to\infty} \| u - u_n \| = 0$, then we denote this by $u_n \to u$. If $u \in H$, and ρ is a positive real number, then $C_\rho(u)$ denotes the closed ρ-neighborhood of u: $C_\rho(u) = \{v \mid \| v - u \| \leqq \rho\}$.

Let I be any *measure space* (a set with a completely additive, nonnegative measure μ defined on some σ-ring of subsets of I), with finite total measure σ: $\mu(I) = \sigma < \infty$. Let $L^2(I)$ have the usual meaning of the set of real-valued functions defined and square-summable on I. Two such functions f, g which are equal almost everywhere (everywhere with the possible exception of a set of measure zero; in notation, p.p.) are identified and considered to represent the same element of $L^2(I)$; thus, strictly speaking, the elements of $L^2(I)$ are not functions but equivalence classes of functions.

The *scalar product* and *norm* in $L^2(I)$ are defined in the usual way:

$$(f, g) = \int_I fg d\mu \tag{2.1}$$

$$\| f \| = \left(\int_I f^2 d\mu\right)^{1/2} . \tag{2.2}$$

By $f_n \to f$ we shall mean $\lim_{n\to\infty} \| f - f_n \| = 0$; by $f_n \to f$ (*p.p.*) we shall mean, as usual, that for almost all $t \in I$, $\lim_{n\to\infty} f_n(t) = f(t)$; by $f_n \to f$ *uniformly p.p.* we shall mean that $f_n(t)$ converges to $f(t)$ uniformly on a subset of I whose complement has measure zero.

If $f \in L^2(I)$ and a is a real number such that $f(t) \leqq a$ (p.p.) then a is said to be an *essential upper bound* of f. The *essential maximum* of $f(t)$, denoted by e.m. f, is defined as the greatest lower bound of the set of all essential upper bounds of f. With this notation, $f_n \to f$ uniformly p.p. if and only if e.m. $| f - f_n | \to 0$.

DEFINITION 2.I. By a *partial norm* on H we shall mean a map Z from H to $L^2(I)$, with the following properties:

2.I.1. For every $u \in H$, Zu is a unique element of $L^2(I)$ which is nonnegative almost everywhere on I: $Zu \geqq 0$ (p.p.);

2.I.2. For any $u, v \in H$ we have $Z(u + v) \leqq Zu + Zv$ (p.p.);

2.I.3. For any $u \in H$ and real number a, $Z(au) = | a | Zu$ (p.p.);

2.I.4. Z is isometric: for any $u \in H$, $\| Zu \| = \| u \|$; i.e. $\int_I (Zu)^2 d\mu = \| u \|^2$.

Note that when I consists of a single point with measure one, then the partial norm Z reduces to the ordinary norm $\| \ \|$ of H.

Throughout §§ 2 and 3 we shall suppose a fixed partial norm Z.

DEFINITION 2.II. The *Z-norm*, denoted by $\| \ \|_Z$, is defined on H as follows. If $u \in H$, then $\| u \|_Z = $ e.m. (Zu). If a sequence of elements

$u_n \in H$ converges to $u \in H$ "in the Z-norm," i.e. if $\| u - u_n \|_Z \to 0$, we denote this by $u_n \xrightarrow{Z} u$.

Note that the Z-norm is not always finite, but otherwise satisfies all the conditions of a norm: $\| u \|_Z = 0$ if and only if $u = 0$, $\| u + v \|_Z \leqq \| u \|_Z + \| v \|_Z$, and if a is any real number then $\| au \|_Z = | a | \cdot \| u \|_Z$. This Z-norm, of course, defines a metric, and hence a topology, on H. The following remarks discuss the relation of the two topologies determined on H by its two norms.

REMARK 2.III.1. If $u_n \xrightarrow{Z} u$ then it is easily seen that $u_n \to u$.

REMARK 2.III.2. If $u_n \to u$, then there exists a subsequence u_{n_k} of u_n such that $Z(u - u_{n_k}) \to 0$ (p.p.). For, if $u_n \to u$ then $\| Z(u - u_n) \| = \| u - u_n \| \to 0$, and then by a well-known theorem there exists a subsequence u_{n_k} such that $Z(u - u_{n_k}) \to 0$ (p.p.)

REMARK 2.III.3. The Z-norm makes H a complete metric space. That is, if $\lim_{n,k\to\infty} \| u_n - u_k \|_Z = 0$, then there exists a unique element u such that $u_n \xrightarrow{Z} u$.

To see this, first note that $\lim_{n,k\to\infty} \| u_n - u_k \| = 0$ and hence u_n is a Cauchy sequence (with respect to the "ordinary" norm). Then, by the completeness of the space H (again with respect to the "ordinary" norm), there must exist a unique $u \in H$ such that $u_n \to u$.

Then, by Remark 2.III.2. there exists a subsequence u_{n_k} such that $Z(u - u_{n_k}) \to 0$ (p.p.) as $k \to \infty$. Clearly, by 2.I.2.,

$$Z(u - u_j) \leqq Z(u - u_{n_k}) + Z(u_{n_k} - u_j) \qquad \text{(p.p.)} .$$

Letting j and k go to ∞, we get $\lim_{j\to\infty} Z(u - u_j) = 0$ (p.p.). To show that the convergence is uniform p.p., note that

$$\begin{aligned} Z(u - u_j) &\leqq Z(u - u_{j+k}) + Z(u_{j+k} - u_j) \\ &\leqq Z(u - u_{j+k}) + \sup_{l=1,2,\cdots} Z(u_{j+l} - u_j) \qquad \text{(p.p.)} \end{aligned}$$

for $j = 1, 2, \cdots$, $k = 1, 2, \cdots$. Letting $k \to \infty$, we get

$$Z(u - u_j) \leqq \sup_{l=1,2,\cdots} Z(u_{j+l} - u_j) \qquad \text{(p.p.)}$$

and hence

$$\begin{aligned} \| u - u_j \|_Z &= \text{e.m. } Z(u - u_j) \leqq \sup_{l=1,2,\cdots} \text{e.m. } Z(u_{j+l} - u_j) \\ &= \sup_{l=1,2,\cdots} \| u_{j+l} - u_j \|_Z . \end{aligned}$$

But $\sup_{l=1,2,\cdots} \| u_{j+l} - u_j \|_Z \to 0$ as $j \to \infty$, so $\| u - u_j \|_Z \to 0$.

REMARK 2.III.4. $\|u\| \leqq \sqrt{\sigma}\|u\|_Z$ (where σ is the total measure of I); for

$$\|u\| = \|Zu\| = \left(\int_I (Zu)^2 d\mu\right)^{1/2} \leqq \left(\int_I \|u\|_Z^2 d\mu\right)^{1/2} = \sqrt{\sigma}\|u\|_Z .$$

REMARK 2.III.5. If c is a nonnegative real number and $u, v \in H$ are such that $Zu \leqq cZv$ (p.p.), then $\|u\| \leqq c\|v\|$; for $\|u\|^2 = \int_I (Zu)^2 d\mu \leqq c^2\int_I (Zv)^2 d\mu = c^2\|v\|^2$.

REMARK 2.III.6. If u, v are any two elements of H, then by 2.I.2., $Zu \leqq Z(u-v) + Zv$ (p.p.); or $Zu - Zv \leqq Z(u-v)$ (p.p.). Also, by 2.I.3., $Zv - Zu \leqq Z(v-u) = Z(u-v)$ (p.p.). Hence

$$|Zu - Zv| \leqq Z(u-v) \qquad \text{(p.p.)} .$$

We conclude this section with a brief review of some of the standard terminology in operators, linear manifolds, etc.

By an *operator* T in H is meant a mapping which assigns to each element u of a certain subset of H, a unique element Tu of H. The *domain* of definition of T is denoted by $D(T)$; the *range* of T, that is, the set of elements Tu, is denoted by $R(T)$; the *nullspace* of T, that is, the set of elements $u \in D(T)$ for which $Tu = 0$, is denoted by $\mathcal{N}(T)$.

If $T, T_1, T_2, \cdots$ are operators and a is a real number, then the operators aT, $T_1 + T_2$, T_1T_2, $\lim_{n\to\infty} T_n$ are defined as follows:

$$D(aT) = D(T) , \qquad (aT)u = a(Tu) ;$$

$$D(T_1 + T_2) = D(T_1) \cap D(T_2) , \qquad (T_1 + T_2)u = T_1u + T_2u$$

$$D(T_1T_2) = \{u \mid u \in D(T_2) \text{ and } T_2u \in D(T_1)\} , \qquad (T_1T_2)u = T_1(T_2u)$$

$$D\left(\lim_{n\to\infty} T_n\right) = \left\{u \,\middle|\, u \in \bigcap_{n=1}^{\infty} D(T_n) \text{ and } \lim_{n\to\infty} (T_nu) \text{ exists}\right\} ,$$

$$\left(\lim_{n\to\infty} T_n\right)u = \lim_{n\to\infty} (T_nu) .$$

The *graph* $G(T)$ of an operator T is the subset of $H \times H$ consisting of all ordered pairs of the form $\langle u, Tu\rangle$ with $u \in D(T)$. If the operator T_2 is an extension of the operator T_1 (that is, if $D(T_1) \subset D(T_2)$ and $T_1u = T_2u$ for $u \in D(T_1)$—in other words, if $G(T_1) \subset G(T_2)$), then we denote this by $T_1 \subset T_2$.

If T is an operator in H and A is a subset of $D(T)$, then TA denotes the set $\{Tu \mid u \in A\}$.

An operator T is called *linear* if $D(T)$ is a linear manifold (a set A is a linear manifold if $u, v \in A$ implies that $au + bv \in A$ for any real numbers a and b) and if, for any $u, v \in D(T)$ and real numbers a and b, $T(au + bv) = aTu + bTv$—in other words, T is linear if and only if

$G(T)$ is a linear manifold in $H \times H$.

A linear operator T is said to be *closed* if its graph $G(T)$ is a closed subset of $H \times H$; that is if given a sequence of elements $u_1, u_2, \cdots$ of $D(T)$ such that $u_n \to u$ and $Tu_n \to v$, it follows that $u \in D(T)$ and $Tu = v$.

Let T be a linear operator with the property that, whenever a sequence of elements $u_n \in D(T)$ converges to zero and Tu_n converges to some element v, then $v = 0$. It is easily seen that under these conditions the closure of $G(T)$ in $H \times H$ is again the graph of a linear operator, call it $\bar{T}$. It is clear that $\bar{T}$ is the *smallest closed linear extension* of T (smallest in the sense that any other closed linear extension of T is an extension of $\bar{T}$).

If A is a subset of H, then the intersection of all linear manifolds containing A is itself a linear manifold, this is called the *linear span* of A and is denoted by $[A]$. If A and B are two linear manifolds, then the linear span of their union, $[A \cup B]$, is easily seen to be simply the set $\{u + v \mid u \in A, v \in B\}$. If the two linear manifolds A and B are disjoint (i.e. their intersection $A \cap B$ contains only the zero element), then the decomposition of an element of $[A \cup B]$ as a sum $u + v$ with $u \in A$, $v \in B$ is unique; in that case the linear span $[A \cup B]$ is also called the *direct sum* of A and B, written as $A \oplus B$.

If A and B are disjoint linear manifolds, then the *projection of* $A \oplus B$ *onto* A *along* B is the linear continuous operator P defined by: $D(P) = A \oplus B$; if $u \in A$, $v \in B$, then $P(u + v) = u$. Note that P is idempotent: $P^2 = P$.

If A is a closed linear manifold, then the *orthogonal projection onto* A is the projection of H onto A along the orthogonal complement of A (that is the set $\{u \mid (u, v) = 0 \text{ for all } v \in A\}$).

A linear operator T is said to be *reduced* by a closed linear manifold G if $PT \subset TP$, where P denotes the orthogonal projection onto G.

3. On the solution of $Tu = o$ for certain non-linear operators T.

THEOREM 3.I.

3.I.1. *Let L be a linear operator, A a linear manifold $\subset H$ such that $D(L) = A \oplus \mathcal{N}(L)$.*

3.I.2. *Suppose there exists a constant γ such that for any $u \in A$ we have $(Zu)^2 \leqq \gamma \| Lu \|^2$* (p.p.)

3.I.3. *Let S be an operator in H. Suppose there exists a constant $\alpha < 1/\Gamma$, where $\Gamma = \max(\gamma, \sigma)$, such that for any $u, v \in D(S)$ we have $Z(Su - Sv) \leqq \alpha Z(u - v)$* (p.p.)

3.I.4. *Let ϕ be an element of $D(S) \cap \mathcal{N}(L)$.*

3.I.5. *Suppose there exists a set $B \subset H$ with the following properties:*

3.I.5(a). *$\phi \in B \subset D(S)$ and $S(B \cap C_\rho(\phi)) \subset R(L)$, where $\rho = \Gamma \| S\phi \| / (1 - \alpha\Gamma)$ (and $C_\rho(\phi) = \{u \mid \| u - \phi \| \leqq \rho\}$);*

3.I.5(b). *If $u \in H$, $u_n \in B \cap D(L) \cap C_\rho(\phi)$ for $n = 1, 2, \cdots$ and $u_n \xrightarrow{Z} u$, then $u \in B$;*

3.I.5(c). *If $u \in B \cap C_\rho(\phi)$, $Su = Lv$, and $v - \phi \in A$, then $v \in B$.*

Then there exists a unique solution u of the system

$$(3.1) \qquad (L - S)u = 0 \quad \text{and} \quad u - \phi \in A\,.$$

This solution u will belong to $C_\rho(\phi) \cap B$.

Proof. Let Q denote the projection of $D(L)$ onto A along $\mathscr{N}(L)$. That is, $D(Q) = D(L)$; if $u \in D(L)$, then we know there exist unique $v \in A$ and $w \in \mathscr{N}(L)$ such that $u = v + w$; we define $Qu = v$.

Let K denote that right inverse of L whose range $R(K) = A$. That is, $D(K) = R(L)$, $LK =$ the restriction of the identity operator I to $R(L)$, and $KL = Q$. Clearly, K is linear.

Now note that the equation

$$(3.2) \qquad u = \phi + KSu$$

has exactly the same solutions as the system (3.1). For, suppose first that u satisfies (3.2). Then $u - \phi = KSu \in R(K) = A$. Also, $\phi \in \mathscr{N}(L) \subset D(L)$; further $KSu \in R(K) = A \subset D(L)$; hence by the linearity of L, $u = \phi + KSu \in D(L)$ and $Lu = L\phi + LKSu$. But $L\phi = 0$; and since $LK \subset I$, $LKSu = Su$. Thus $(L - S)u = 0$ and u satisfies (3.1). Conversely, let u be a solution of (3.1). Then $Su = Lu \in R(L) = D(K)$ and $KSu = KLu = Qu = u - \phi$. Thus u satisfies (3.2).

We shall prove the theorem by showing the existence and uniqueness of solutions to equation (3.2). We shall prove the existence by the method of successive approximations.

Define a sequence of elements of $B \cap C_\rho(\phi) \cap D(L)$ as follows. Let $u^{(0)} = \phi$. Clearly $u^{(0)} \in B \cap C_\rho(\phi) \cap D(L)$. Supposing that for $n \leqq k$, $u^{(n)}$ is defined and is an element of $B \cap C_\rho(\phi) \cap D(L)$, let $u^{(k+1)} = \phi + KSu^{(k)}$.

Clearly, $u^{(k+1)}$ is well-defined, for $u^{(k)} \in B \cap C_\rho(\phi) \subset D(S)$ and $Su^{(k)} \in S(B \cap C_\rho(\phi)) \subset R(L) = D(K)$. Also $u^{(k+1)} - \phi \in R(K) = A$ and $Lu^{(k+1)} = L\phi + LKSu^{(k)} = Su^{(k)}$, hence by 3.I.5.(c) $u^{(k+1)} \in B$. Further, using 3.I.2, 3.I.3, Remarks 2.III.4 and 2.III.5, and the definition of ρ in 3.I.5.(a), as well as our inductive assumption,

$$\begin{aligned} \| u^{(k+1)} - \phi \| &= \| KSu^{(k)} \| \leqq \sqrt{\gamma\sigma} \| Su^{(k)} \| \leqq \Gamma \| Su^{(k)} \| \\ &\leqq \Gamma(\| Su^{(k)} - S\phi \| + \| S\phi \|) \leqq \Gamma(\alpha \| u^{(k)} - \phi \| + \| S\phi \|) \\ &\leqq \Gamma(\alpha\rho + \| S\phi \|) = \rho\,. \end{aligned}$$

Thus $u^{(k+1)}$ is an element of $B \cap C_\rho(\phi) \cap D(L)$.

So we have now a sequence of elements $u^{(n)}$ of $B \cap C_\rho(\phi) \cap D(L)$, satisfying

$$u^{(0)} = \phi\,, \qquad u^{(n+1)} = \phi + KSu^{(n)} \qquad n = 0, 1, 2, \cdots \tag{3.3}$$

Clearly, using again 3.I.2,3 and Remarks 2.III.4,5, as well as (3.3),

$$\begin{aligned} \| u^{(n+1)} - u^{(n)} \| &= \| KSu^{(n)} - KSu^{(n-1)} \| \\ &= \| K(Su^{(n)} - Su^{(n-1)}) \| \leqq \Gamma \| Su^{(n)} - Su^{(n-1)} \| \\ &\leqq \alpha\Gamma \| u^{(n)} - u^{(n-1)} \| \,. \end{aligned}$$

By induction on n this yields

$$\| u^{(n+1)} - u^{(n)} \| \leqq (\alpha\Gamma)^n \| u^{(1)} - u^{(0)} \| \leqq (\alpha\Gamma)^n \rho \,.$$

Since $\alpha\Gamma < 1$ it follows that $u^{(n)}$ is a Cauchy sequence. Further, since

$$\begin{aligned} Z(u^{(k)} - u^{(n)}) &= Z(KSu^{(k-1)} - KSu^{(n-1)}) \leqq \sqrt{\Gamma} \| Su^{(k-1)} - Su^{(n-1)} \| \\ &\leqq \alpha \sqrt{\Gamma} \| u^{(k-1)} - u^{(n-1)} \| \qquad \text{(p.p.)}\,, \end{aligned}$$

it is clear that $\lim_{n,k\to\infty} \| u^{(k)} - u^{(n)} \|_z = 0$ and hence by Remark 2.III.3 there exists a unique $u \in H$ such that $Z(u - u^{(n)}) \to 0$ uniformly p.p.

Then it follows from 3.I.5(b) that $u \in B$. Obviously, $u \in C_\rho(\phi)$. Hence by 3.I.5(a) $Su \in R(L) = D(K)$. Further,

$$\| KSu - KSu^{(n)} \| \leqq \Gamma \| Su - Su^{(n)} \| \leqq \alpha\Gamma \| u - u^{(n)} \| \,,$$

hence $KSu^{(n)} \to KSu$. Now taking the limit as $n \to \infty$ on both sides of the second equation in (3.3), we get $u = \phi + KSu$. Thus u satisfies (3.2).

We must still show that the solution of (3.2) is unique. Suppose, on the contrary, the existence of two solutions u, v. Then $u - v = K(Su - Sv)$, and hence

$$\| u - v \| = \| K(Su - Sv) \| \leqq \Gamma \| Su - Sv \| \leqq \alpha\Gamma \| u - v \| \,.$$

But $\alpha\Gamma < 1$; hence it follows that $\| u - v \| = 0$ and $u = v$.

This completes the proof of Theorem 3.I.

THEOREM 3.II.

3.II.1. *Let L, N be linear operators, A a linear manifold such that $D(L) = A \oplus \mathscr{N}(L)$. Let $D(L) \subset D(N)$.*

3.II.2(a). *Suppose there exists a constant γ such that for any $u \in A$ we have* $(ZNu)^2 \leqq \gamma \| Lu \|^2$ (p.p.).

3.II.2(b). *Suppose there exists a constant β such that for any $u \in D(N)$ we have $Zu \leqq \beta ZNu$* (p.p.).

3.II.3. *Let S be an operator in H. Suppose there exists a constant $\alpha < 1/\Gamma$, where $\Gamma = \max(\gamma, \sigma)$, such that for any $u, v \in D(S)$ we have $Z(Su - Sv) \leqq \alpha Z(u - v)$* (p.p.).

3.II.4. *Let ϕ be an element of $\mathscr{N}(L) \cap D(SN)$.*

3.II.5. *Suppose there exists a set $B \subset H$ with the following properties:*

3.II.5(a). *$\phi \in B \subset D(SN)$ and $S(NB \cap C_\rho(N\phi)) \subset R(L)$, were $\rho = \Gamma \| SN\phi \| / (1 - \alpha\Gamma)$;*

3.II.5(b). *If $u, v \in H$, $u_n \in D(L) \cap B$ and $Nu_n \in C_\rho(N\phi)$ for $n = 1, 2, \cdots$, $u_n \xrightarrow{Z} u$ and $Nu_n \xrightarrow{Z} v$, then $u \in B$ and $Nu = v$;*

3.II.5(c). *If $u \in B$, $Nu \in C_\rho(N\phi)$, $SNu = Lv$, and $v - \phi \in A$, then $v \in B$.*

Then there exists a unique solution u of the system

$$(3.4) \qquad (L - SN)u = 0 \quad \textit{and} \quad u - \phi \in A\,.$$

This solution u will belong to $N^{-1}C_\rho(N\phi) \cap B$ (i.e. $Nu \in C_\rho(N\phi)$ and $u \in B$).

Proof. From 3.II.2(b) and Remark 2.III.5 it follows that, for any $u \in D(N)$, $\| u \| \leqq \beta \| Nu \|$. Hence if, for some u, $Nu = 0$, then $u = 0$. Hence N has a unique inverse N^{-1}, defined on the range of N.

It is now easily seen that the system (3.4) is equivalent to the system

$$(3.5) \qquad (LN^{-1} - S)v = 0 \quad \text{and} \quad v - N\phi \in NA$$

in the sense that there is a one-to-one correspondence between the solutions of (3.4) and the solutions of (3.5), the correspondence being given by $v = Nu$.

We wish to prove that the system (3.4) has a unique solution. We shall do this by showing that $L^1 = LN^{-1}$, $A^1 = NA$, S, $\phi^1 = N\phi$, $B^1 = NB$ satisfy the conditions of Theorem 3.I. Then we can apply that theorem and this will give us the existence and uniqueness of solutions of (3.5), which we have seen to be equivalent to the existence and uniqueness of solutions to (3.4).

Now we must show that L^1, A^1, ϕ^1, B^1, as defined in the preceding paragraph, and S, satisfy the conditions of Theorem 3.I.

3.I.1. Clearly, L^1 is linear. Also, A^1 is a linear manifold contained in $D(L^1)$. Further, A^1 and $\mathscr{N}(L^1)$ are disjoint. For, suppose $u \in A^1 \cap \mathscr{N}(L^1)$. Then $u = Nv$ with $v \in A$, and $Lv = LN^{-1}Nv = L^1u = 0$. Thus $v \in A \cap \mathscr{N}(L)$; but by 3.II.1, A and $\mathscr{N}(L)$ are disjoint, and hence $v = 0$. But then $u = Nv = 0$. Thus A^1 and $\mathscr{N}(L^1)$ are indeed disjoint, and since they are both subsets of $D(L^1)$ it is clear that $A^1 \oplus \mathscr{N}(L^1) \subset D(L^1)$.

We must still show that $D(L^1) \subset A^1 \oplus \mathscr{N}(L^1)$. Let u be any element of $D(L^1)$. That is, $u \in D(LN^{-1})$. Then $N^{-1} u \in D(L) = A \oplus \mathscr{N}(L)$. Hence there exist $v \in A$, $w \in \mathscr{N}(L)$, such that $N^{-1}u = v + w$. Then $u = Nv + Nw$; clearly $Nv \in NA = A^1$, $L^1Nw = LN^{-1}Nw = 0$, or $Nw \in \mathscr{N}(L^1)$. Hence $u \in A^1 \oplus \mathscr{N}(L^1)$, as was to be proved.

3.I.2. If $u \in A^1$; then $N^{-1} u \in A$ and hence by 3.II.2(a)

$$(Zu)^2 = (ZNN^{-1}u)^2 \leqq \gamma \| LN^{-1}u \|^2 = \gamma \| L^1u \|^2 \qquad (\text{p.p.})\,.$$

3.I.3. is clearly identical with 3.II.3.

3.I.4. By 3.II.4., $L^1\phi^1 = LN^{-1}N\phi = L\phi = 0$, so $\phi^1 \in \mathscr{N}(L^1)$. Also, $\phi^1 = N\phi \in D(S)$.

3.I.5(a). By 3.II.5(a), $\phi \in B \subset (SN)$. Hence $\phi^1 = N\phi \in NB = B^1$; if $u \in B^1 = NB$, then $N^{-1}u \in B \subset D(SN)$ and hence $u \in D(SNN^{-1}) \subset D(S)$ and so $B^1 \subset D(S)$. Also, by 3.II.5(a) again, $S(B^1 \cap C_\rho(\phi^1)) = S(NB \cap C_\rho(N\phi)) \subset R(L)$. Further, $R(L) \subset R(L^1)$, for if $u \in R(L)$, then $u = Lv$ for some $v \in D(L) \subset D(N)$ (by 3.II.1.) and so $u = LN^{-1}Nv = L^1Nv \in R(L^1)$.

3.I.5(b). If $u \in H$, $u_n \in B^1 \cap D(L^1) \cap (C_\rho(\phi^1)$ for $n = 1, 2, \cdots$ and $u_n \xrightarrow{Z} u$, then $N^{-1}u_n \in D(L) \cap B$ and, by 3.II.2(b)

$$Z(N^{-1}u_n - N^{-1}u_k) \leqq \beta Z(u_n - u_k) \qquad \text{(p.p.)}$$

and hence $\lim_{n,k\to\infty} \| N^{-1}u_n - N^{-1}u_k \|_Z = 0$.

Hence by Remark 2.III.3. there exists a unique element $v \in H$ such that $N^{-1}u_n \xrightarrow{Z} v$. Then it follows from 3.II.5(b) that $v \in B$ and $Nv = u$; hence $u \in NB = B^1$ as desired.

3.I.5(c). If $u \in B^1 \cap C_\rho(\phi^1)$, $Su = L^1v$, and $v - \phi^1 \in A^1$, then $N^{-1}u \in B$, $NN^{-1}u \in C_\rho(N\phi)$, $SNN^{-1}u = LN^{-1}v$, and $v - N\phi \in NA$ or $N^{-1}v - \phi \in A$. Then by 3.II.5(c) $N^{-1}v \in B$ and hence $v \in NB = B^1$, as was to be proved.

This completes the proof of Theorem 3.II.

THEOREM 3.III.

3.III.1(a). *Let L, N be linear operators, A a linear manifold such that $D(L) = A \oplus \mathscr{N}(L)$. Let $D(L) \subset D(N)$.*

3.III.1(b). *Suppose that if $u_n \in D(L)$ for $n = 1, 2, \cdots$, $u_n \to 0$, and $Lu_n \to v$, then $v = 0$. Thus there exists a smallest closed extension $\bar{L}$ of L; let L' denote the restriction of $\bar{L}$ to $D(\bar{L}) \cap D(N)$.*

3.III.2(a). *Suppose there exists a constant γ such that for any $u \in A$ we have $(ZNu)^2 \leqq \gamma \| Lu \|^2$ (p.p.).*

3.III.2(b). *Suppose that there exists a constant β such that for any $u \in D(N)$ we have $Zu \leqq \beta ZNu$ (p.p.).*

3.III.3. *Let S be an operator in H, with $D(S) \supset R(N)$. Suppose there exists a constant $\alpha < 1/\Gamma$, where $\Gamma = \max(\gamma, \sigma)$, such that for any $u, v \in D(S)$ we have $Z(Su - Sv) \leqq \alpha Z(u - v)$ (p.p.).*

3.III.4. *Let ϕ be an element of $\mathscr{N}(L') \cap D(SN)$. Suppose there exists a sequence of elements $\phi_n \in \mathscr{N}(L)$ such that $N\phi_n \xrightarrow{Z} N\phi$.*

3.III.5. *Suppose there exists a set $B \subset H$ with the following properties:*

3.III.5(a). *$\phi \in B \subset D(SN)$ and $S(NB \cap C_\rho(N\phi)) \subset R(L')$, where $\rho = \Gamma \| SN\phi \| / (1 - \alpha\Gamma)$;*

3.III.5(b). *If $u, v \in H$, $u_n \in D(N)$ for $n = 1, 2, \cdots$, $u_n \xrightarrow{Z} u$ and $Nu_n \xrightarrow{Z} v$, then $u \in D(N)$ and $Nu = v$; if in addition $u_n \in D(L') \cap B$ and $Nu_n \in C_\rho(N\phi)$ for $n = 1, 2, \cdots$, then $u \in B$;*

3.III.5(c). *If $u \in B$, $Nu \in C_\rho(N\phi)$, $SNu = L'v$, and $v - \phi \in A' =$*

$\{w \mid$ there exists a sequence of elements $w_n \in A$ such that $w_n \to w$ and $\lim_{n\to\infty} Lw_n$ exists$\}$, then $v \in B$.

Then there exists a unique element u with the following property:

(3.6) *There exists a sequence of elements $u_n \in D(L - SN)$ such that $Nu_n \xrightarrow{Z} Nu$ and $(L - SN)u_n \to 0$; and $u - \phi \in A'$.*

This unique "solution" u will belong to $N^{-1}C_\rho(N\phi) \cap B$ (i.e. $Nu \in C_\rho(N\phi)$ and $u \in B$); if in addition $u \in D(L)$, then u is a solution of (3.4).

Proof. First we shall show that L', A', N, S, ϕ, B satisfy the hypotheses of Theorem 3.II.

3.II.1. Clearly L' and N are linear. It is also clear that $A' \subset D(L')$. Further A' and $\mathscr{N}(L')$ are disjoint. For, suppose $u \in A' \cap \mathscr{N}(L')$. Then, since $u \in A'$, there exists a sequence of elements $u_n \in A$ such that $u_n \to u$ and $\lim_{n\to\infty} Lu_n$ exists. But u is also in $\mathscr{N}(L')$, hence $\lim_{n\to\infty} Lu_n = L'u = 0$. Now by 3.III.2(a),(b), and Remarks 2.III.4,5,

$$\| u_n \| \leqq \beta \| Nu_n \| \leqq \beta\Gamma \| Lu_n \| .$$

Hence $u_n \to 0$ and so $u = 0$. Thus A' and $\mathscr{N}(L')$ are disjoint, and it is now clear that $A' \oplus \mathscr{N}(L') \subset D(L')$.

We must still show that $D(L') \subset A' \oplus \mathscr{N}(L')$; that is, that any element of $D(L')$ is also an element of $A' \oplus \mathscr{N}(L')$. Let $u \in D(L')$. Then there exists a sequence of elements $u_n \in D(L)$ such that $u_n \to u$ and $\lim_{n\to\infty} Lu_n$ exists. (In fact, $Lu_n \to L'u$). Let $v_n = Qu_n$, where Q again denotes the projection of $D(L)$ onto A along $\mathscr{N}(L)$. Since $v_n \in A$, we have (again from 3.III.2(a),(b) and Remarks 2.III.4, 5)

$$\| v_n - v_k \| \leqq \beta \| Nv_n - Nv_k \| \leqq \beta\Gamma \| Lv_n - Lv_k \| .$$

Thus v_n is a Cauchy sequence. Let $v = \lim_{n\to\infty} v_n$. Then $Lv_n = Lu_n \to L'u$; hence $v \in A'$, and $L'v = L'u$; this last means that $u - v \in \mathscr{N}(L')$. Hence $u \in A' \oplus \mathscr{N}(L')$, as was to be proved.

Obviously, $D(L') \subset D(N)$.

3.II.2(a). Let $u \in A'$. Then we know there exists a sequence of elements $u_n \in A$ such that $u_n \to u$ and $Lu_n \to L'u$. By 3.III.2(a),

$$ZN(u_n - u_k) \leqq \sqrt{\Gamma} \| L(u_n - u_k) \| \qquad \text{(p.p.)}$$

and hence $\lim_{n,k\to\infty} \| N(u_n - u_k) \|_Z = 0$. Also by 3.III.2(b), $Z(u_n - u_k) \leqq \beta ZN(u_n - u_k)$ (p.p.) and hence also $\lim_{n,k\to\infty} \| (u_n - u_k) \|_Z = 0$. Then by Remark 2.III.3. there exist two unique elements u_0, v such that $u_n \xrightarrow{Z} u_0$ and $Nu_n \xrightarrow{Z} v$. Then by Remark 2.III.1., $u_n \to u_0$. But $u_n \to u$; hence $u_0 = u$. Also, by 3.III.5(b), $v = Nu$, and so $Z(Nu - Nu_n) \to 0$ uniformly p.p. Then, by Remark 2.III.6, $| ZNu - ZNu_n | \to 0$ uniformly p.p., or, $ZNu_n \to ZNu$ uniformly p.p. But by 3.III.2(a),

$$ZNu_n \leqq \sqrt{\gamma}\,||Lu_n|| \qquad \text{(p.p.)}\,,$$

hence

$$ZNu \leqq \sqrt{\gamma}\lim_{n\to\infty}||Lu_n|| = \sqrt{\gamma}\,||L'u||\,.$$

The remaining assumptions of Theorem 3.II. are obviously satisfied. Hence there exists a unique solution u of

$$(3.7) \qquad (L' - SN)u = 0 \qquad \text{and} \qquad u - \phi \in A'\,.$$

We now want to show that u satisfies (3.6). Clearly there exists a sequence of elements $v_n \in A$ such that $v_n \to u - \phi$ and $Lv_n \to L'u$. Now let $u_n = v_n + \phi_n$ (see 3.III.4). Then

$$ZN(u - u_n) = ZN(u - v_n - \phi_n) \leqq ZN(u - \phi - v_n) + ZN(\phi - \phi_n)\,.$$

By 3.III.4., $N(\phi - \phi_n) \xrightarrow{Z} 0$. Also, $u - \phi - v_n \in A'$; and hence, as we have seen in the preceding paragraph,

$$ZN(u - \phi - v_n) \leqq \sqrt{\Gamma}\,||L'(u - \phi - v_n)|| = \sqrt{\Gamma}\,||L'u - Lv_n||\,;$$

thus also $N(u - \phi - v_n) \xrightarrow{Z} 0$. Hence it follows that $N(u - u_n) \xrightarrow{Z} 0$.

It now follows from Remark 2.III.1. that $Nu_n \to Nu$; and from 3.III.3. and Remark 2.III.5. that $SNu_n \to SNu$. Hence $(L - SN)u_n \to (L' - SN)u = 0$. Thus u does indeed satisfy (3.6). It is clear that if, in addition, $u \in D(L)$, then u satisfies (3.4).

We must still prove the uniqueness of "solutions" of (3.6). Let u, v be two such "solutions." Thus, $u - \phi \in A'$, $v - \phi \in A'$, and there exist sequences of elements $u_n, v_n \in D(L - SN)$ such that $Nu_n \xrightarrow{Z} Nu$, $Nv_n \xrightarrow{Z} Nv$, $(L - SN)u_n \to 0$, and $(L - SN)v_n \to 0$. By 3.III.3.,

$$Z(SNu_n - SNu_k) \leqq \alpha ZN(u_n - u_k)\,;$$

and hence by Remark 2.III.4., SNu_n is a Cauchy sequence. Thus $\lim_{n\to\infty} SNu_n$ exists. Clearly then, since $(L - SN)u_n \to 0$, $\lim_{n\to\infty} Lu_n$ exists. But from 3.III.1(b) and the definition of L' (see also the definition of $\bar{T}$ in § 2) it is then easily seen that $\lim_{n\to\infty} Lu_n = L'u$.

Let Q again denote the projection of $D(L)$ onto A along $\mathcal{N}(L)$. Then $u - \phi - Qu_n \in A'$ and hence by 3.III.2(b), Remarks 2.III.4, 5, and 3.II.2(a) for L' (proved above),

$$\begin{aligned}||u - \phi - Qu_n|| &\leqq \beta\,||N(u - \phi - Qu_n)|| \\ &\leqq \beta\Gamma\,||L'(u - \phi - Qu_n)|| = \beta\Gamma\,||L'u - Lu_n||\,.\end{aligned}$$

Thus, $Qu_n \to u - \phi$ and $NQu_n \to N(u - \phi)$. Similarly $Qv_n \to v - \phi$ and $NQv_n \to N(v - \phi)$. Hence $Qu_n - Qv_n \to u - v$. Also, $N[u_n - Qu_n - (v_n - Qv_n)] \to 0$. But by 3.III.2(a), 3.III.3, and Remarks 2.II.4, 5.

$$\begin{aligned}\| N(Qu_n - Qv_n) \| &\leqq \Gamma \| L(Qu_n - Qv_n) \| \\ &= \Gamma \| L(u_n - v_n) \| \leqq \Gamma [\| (L - SN)u_n \| \\ &\quad + \| (L - SN)v_n \| + \| SNu_n - SNv_n \|] \\ &\leqq \Gamma [\| (L - SN)u_n \| + \| (L - SN)v_n \|] \\ &\quad + \alpha\Gamma \| N(u_n - v_n) \| \\ &\leqq \Gamma[\| (L - SN)u_n \| + \| (L - SN)v_n \|] \\ &\quad + \alpha\Gamma \| N(u_n - Qu_n - (v_n - Qv_n)) \| \\ &\quad + \alpha\Gamma \| N(Qu_n - Qv_n) \| .\end{aligned}$$

Thus,

$$\begin{aligned}\| N(Qu_n - Qv_n) \| \leqq \frac{1}{1 - \alpha\Gamma} \{&\Gamma[\| (L - SN)u_n \| + \| (L - SN)v_n \|] \\ &+ \alpha\Gamma \| N(u_n - Qu_n - (v_n - Qv_n)) \|\} .\end{aligned}$$

Now the right hand side of the above inequality $\rightarrow 0$ as $n \rightarrow \infty$, hence $N(Qu_n - Qv_n) \rightarrow 0$; then by 3.III.2(b), $Qu_n - Qv_n \rightarrow 0$. But $u - v = \lim_{n\rightarrow\infty} (Qu_n - Qv_n)$, so $u = v$. This completes the proof of Theorem 3.III.

THEOREM 3.IV.

3.IV.0. *Let $G_1 G_2, \cdots$ be a sequence of pairwise orthogonal closed linear manifolds whose direct sum is H. Let P_n denote the orthogonal projection onto G_n. Suppose that the partial norm Z and the subspaces G_n are related as follows: for any $u \in H$,*

$$(Zu)^2 = \sum_{n=1}^{\infty} (ZP_n u)^2 \qquad \text{(p.p.)} .$$

3.IV.1(a). *Let L, N be linear operators, A a linear manifold such that $D(L) = A \oplus \mathcal{N}(L)$. Let $D(L) \subset D(N)$.*

3.IV.1(b). *Let L and N be reduced by each G_n, $n = 1, 2, \cdots$ ($P_n L \subset LP_n$, $P_n N \subset NP_n$). Suppose that $P_n A \subset A$, for $n = 1, 2, \cdots$. Suppose that $\mathcal{N}(L) \cap G_n$ is closed, for $n = 1, 2, \cdots$.*

3.IV.2(a). *Suppose there exists a constant γ such that for any $u \in A$ we have $(ZNu)^2 \leqq \gamma \| Lu \|^2$* (p.p.).

3.IV.2(b). *Suppose there exists a constant β such that for any $u \in D(N)$ we have $Zu \leqq \beta ZNu$* (p.p.).

3.IV.3. *Let S be an operator in H, with $D(S) \supset R(N)$. Suppose there exists a constant $\alpha < 1/\Gamma$, where $\Gamma = \max(\gamma, \sigma)$, such that for any $u, v \in D(S)$ we have $Z(Su - Sv) \leqq \alpha Z(u - v)$* (p.p.).

3.IV.4. *Let ϕ be an element of $\mathcal{N}(L') \cap D(SN)$, where L' denotes the restriction of $\sum_{n=1}^{\infty} LP_n$ to $D(\sum_{n=1}^{\infty} LP_n) \cap D(N)$; suppose that $\sum_{i=1}^{n} NP_i\phi \xrightarrow{Z} N\phi$ as $n \rightarrow \infty$.*

3.IV.5. *Suppose there exists a set $B \subset H$ with the following properties:*

3.IV.5(a). *$\phi \in B \subset D(SN)$ and $SNB \subset R(L')$ (where L' is defined in 3.IV.4);*

3.IV.5(b). *If $u, v \in H$, $u_n \in D(N)$ for $n = 1, 2, \cdots$, $u_n \xrightarrow{Z} u$ and $Nu_n \xrightarrow{Z} v$, then $u \in D(N)$ and $Nu = v$. If in addition $u_n \in D(L') \cap B$ and $Nu_n \in C_\rho(N\phi)$ for $n = 1, 2, \cdots$, where $\rho = \Gamma \| SN\phi \| / (1 - \alpha\Gamma)$, then $u \in B$.*

3.IV.5(c). *If $u \in B$, $Nu \in C_\rho(N\phi)$, $SNu = L'v$, and $v - \phi \in A' = \{u \mid u \in D(L'), \text{ and } P_n u \in A \text{ for } n = 1, 2, \cdots\}$, then $v \in B$.*

Then there exists a unique $u \in B$ satisfying

(3.8) *$u - \phi \in A'$ and there exists a sequence of elements $u_n \in D(L - SN)$ such that $Nu_n \xrightarrow{Z} Nu$ and $(L - SN)u_n \to 0$.*

Proof. We first show that L', A', N, S, ϕ, B satisfy the hypotheses of Theorem 3.II.

3.II.1. The operator L' (see 3.IV.4) is the following: $D(L') = \{u \mid u \in D(N),\ P_n u \in D(L) \text{ for } n = 1, 2, \cdots, \text{ and } \sum_{n=1}^\infty LP_n u \text{ exists}\}$, and for any $u \in D(L')$, $L'u = \sum_{n=1}^\infty LP_n u$. It is easily seen that L' is linear, reduced by each G_n, $n = 1, 2, \cdots$, and that $L \subset L'$. Also, $D(L') \subset D(N)$.

The set A' (see 3.IV.5(c)) is given by $A' = \{u \mid u \in D(N),\ P_n u \in A \text{ for } n = 1, 2, \cdots, \text{ and } \sum_{n=1}^\infty LP_n u \text{ exists}\}$. It is easily seen that A' is linear; and of course $A' \subset D(L')$.

A' and $\mathscr{N}(L')$ are disjoint, for suppose $u \in A' \cap \mathscr{N}(L')$. Then, since $u \in A'$, $P_n u \in A$ for $n = 1, 2, \cdots$. On the other hand, $L'u = 0$; hence $P_n L'u = LP_n u = 0$. Thus $P_n u \in A \cap \mathscr{N}(L)$ for $n = 1, 2, \cdots$ Hence $P_n u = 0$ for $n = 1, 2, \cdots$ and so $u = \sum_{n=1}^\infty P_n u = 0$.

From the last two paragraphs it follows that $A' \oplus \mathscr{N}(L') \subset D(L')$. We must still show $D(L') \subset A' \oplus \mathscr{N}(L')$. To see this, let u be any element of $D(L')$. Then for $n = 1, 2, \cdots$, $P_n u \in D(L) = A \oplus \mathscr{N}(L)$ (by 3.IV.1(a)). Hence there exist elements $v_n \in A$, $w_n \in \mathscr{N}(L)$, such that $P_n u = v_n + w_n$. Clearly, since any projection is idempotent, $P_n u = P_n^2 u = P_n v_n + P_n w_n$; $P_n v_n \in A$ by 3.IV.1(b), $LP_n w_n = P_n L w_n = 0$ or $P_n w_n \in \mathscr{N}(L)$. But the representation of an element of $A \oplus \mathscr{N}(L)$ as a sum of an element of A and an element of $\mathscr{N}(L)$ is unique, so $P_n v_n = v_n$ and $P_n w_n = w_n$. That is, $v_n, w_n \in R(P_n) = G_n$. Further, by 3.IV.2(a),(b),

$$Z\Big(\sum_{i=n}^{k} v_i\Big) \leqq \beta ZN\Big(\sum_{i=n}^{k} v_i\Big) \leqq \beta\sqrt{\gamma}\left\| L\Big(\sum_{i=n}^{k} v_i\Big)\right\| = \beta\sqrt{\gamma}\left\|\sum_{i=n}^{k} LP_i u\right\| .$$

But $u \in D(L')$, so $\sum_{n=1}^\infty LP_n u$ converges and hence $\lim_{n,k\to\infty} \| \sum_{i=n}^k LP_i u \| = 0$. Hence $\lim_{n,k\to\infty} \| \sum_{i=n}^k v_i \|_Z = \lim_{n,k\to\infty} \| N(\sum_{i=n}^k v_i) \|_Z = 0$. Then, by Remark 2.III.3., there exist unique elements v, v' such that $\sum_{i=1}^n v_i \xrightarrow{Z} v$ and $N \sum_{i=1}^n v_i \xrightarrow{Z} v'$. Clearly $v = \sum_{i=1}^\infty v_i$. By 3.IV.5(b), $v \in D(N)$ and $Nv = v'$. Also, $P_n v = P_n \sum_{i=1}^\infty v_i = P_n \sum_{i=1}^\infty P_i v_i = \sum_{i=1}^\infty P_n P_i v_i = P_n v_n = v_n \in A$, and $\sum_{n=1}^\infty LP_n v = \sum_{n=1}^\infty L v_n = \sum_{n=1}^\infty LP_n u = L'u$; thus, $v \in A'$ and $L'v = L'u$, or, $u - v \in \mathcal{N}(L')$. Hence $u \in A' \oplus \mathcal{N}(L')$, as was to be proved.

3.II.2(a). Let u be an element of A'. Then $P_n u \in A$ and hence

$$(ZNP_n u)^2 \leqq \gamma \| LP_n u \|^2 \qquad \text{(p.p.)} .$$

Applying 3.IV.0 and 3.IV.1(b) gives

$$(ZNu)^2 = \sum_{n=1}^{\infty} (ZP_n Nu)^2 = \sum_{n=1}^{\infty} (ZNP_n u)^2 \leqq \gamma \sum_{n=1}^{\infty} \| LP_n u \|^2 \qquad \text{(p.p.)} .$$

Now note that $P_n LP_n u = LP_n^2 u = LP_n u$, so $LP_n u \in R(P_n) = G_n$. Also $L'u = \sum LP_n u$; hence $LP_n u = P_n L'u$. Therefore

$$\sum \| LP_n u \|^2 = \sum \| P_n L'u \|^2 = \| L'u \|^2 .$$

Thus,

$$(ZNu)^2 \leqq \gamma \| L'u \|^2 \qquad \text{(p.p.)}$$

as was to be proved.

It is immediately seen that the remaining conditions of Theorem 3.II. are satisfied. Hence we can apply that Theorem and obtain the existence of a unique solution u of

$$(3.9) \qquad (L' - SN)u = 0 \qquad u - \phi \in A' .$$

We shall now prove the equivalence of (3.8) and (3.9). Suppose first that u satisfies (3.9). Let $u_n = \sum_{i=1}^{n} P_i u$. Clearly $u_n = \sum_{i=1}^{n} P_i \phi \in A$ and hence, by 2.I.2, 3.IV.0, 3.IV.1(b) and 3.IV.2(a),

$$\begin{aligned} ZN(u - u_n) &\leqq ZN\Big(u - \phi - \Big(u_n - \sum_{i=1}^{n} P_i\phi\Big)\Big) + ZN\Big(\phi - \sum_{i=1}^{n} P_i\phi\Big) \\ &\leqq \sqrt{\gamma}\Big\|L'\Big(u - \phi - \Big(u_n - \sum_{i=1}^{n} P_i\phi\Big)\Big)\Big\| + ZN\Big(\phi - \sum_{i=1}^{\infty} P_i\phi\Big) \\ &= \sqrt{\gamma}\Big\|\sum_{i=n+1}^{\infty} P_i L'u\Big\| + \Big[(ZN\phi)^2 - \sum_{i=1}^{n} (ZNP_i\phi)^2\Big]^{1/2} \qquad \text{(p.p.)} \end{aligned}$$

Thus from 3.IV.4. it follows that $Nu_n \xrightarrow{Z} Nu$. Also we know from Theorem 3.II. that the only solution of (3.9) belongs to $B \subset D(SN)$; hence by 3.IV.3. and Remark 2.III.5., it follows that $SNu_n \to SNu$. Therefore

$$\lim_{n\to\infty} (L' - SN)u_n = \lim_{n\to\infty} Lu_n - \lim_{n\to\infty} SNu_n = L'u - SNu = 0 .$$

Hence u satisfies (3.8).

Conversely suppose $u \in B$ to be a solution of (3.8). Thus, we suppose that $u - \phi \in A'$, and that there exists a sequence of elements $u_n \in D(L - SN)$ such that $Nu_n \xrightarrow{Z} Nu$ and $(L - SN)u_n \to 0$. From 3.IV.3. and Remark 2.III.5. it then immediately follows that $SNu_n \to$

SNu; hence also $Lu_n \to SNu$. Note that by 3.IV.5(a) $SNu \in SNB \subset R(L')$; hence there exists $v \in D(L')$ such that $SNu = L'v$, and, since we have seen that $D(L') = A' \oplus \mathcal{N}(L')$, it is clear that we can choose v so as to belong to A'. Then, letting Q denote, as usual, the projection of $D(L)$ onto A along $\mathcal{N}(L)$,

$$Z(v - Qu_n) \leqq \beta ZN(v - Qu_n) \leqq \beta\sqrt{\gamma}\,\|L'v - Lu_n\|$$

and hence by Remark 2.III.4., $Qu_n \to v$. Further

$$P_k(u - v) = \lim_{n\to\infty} P_k(u_n - Qu_n)\,.$$

Now $u_n - Qu_n \in \mathcal{N}(L)$, so $P_k(u_n - Qu_n) \in \mathcal{N}(L) \cap G_k$. Hence, since by 3.IV.1(b) $\mathcal{N}(L) \cap G_k$ is closed, $P_k(u - v) \in \mathcal{N}(L) \cap G_k$. Hence, since $u - v \in D(N)$, $u - v \in \mathcal{N}(L')$. Thus we now have two decompositions of the element u of $D(L')$ as a sum of an element of A' and an element of $\mathcal{N}(L')$: $u = (u - \phi) + \phi = v + (u - v)$. But we have seen that $D(L') = A' \oplus \mathcal{N}(L')$, so this decomposition is unique and hence $u - \phi = v$. Hence, $(L' - SN)u = L'(u - \phi) - SNu = L'v - SNu = 0$, as was to be proved.

This completes the proof of Theorem 3.IV.

4. Applications. In this section we shall apply the results of § 3, specifically Theorem 3.IV., to non-linear partial differential equations of the form

$$(4.1)\quad \frac{\partial^k}{\partial t^k}\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)^m u(y, t) = \Phi\Big(\frac{\partial^{m+k}u}{\partial t^{m+k}}, \frac{\partial^{m+k}u}{\partial t^{m+k-1}\partial y}, \cdots, \frac{\partial^{m+k}u}{\partial t^k \partial y^m}, \frac{\partial^{m+k-1}u}{\partial t^{m+k-1}}, \cdots, \frac{\partial^{m+k-1}u}{\partial t^{k-1}\partial y^m}, \frac{\partial^{m+k-2}u}{\partial t^{m+k-2}}, \cdots, \frac{\partial u}{\partial y}, u, y, t\Big)$$

(the partial derivatives of u that are permitted to occur on the right hand side are $\partial^{i+j}u/\partial t^i \partial y^j$ with $j \leqq m$ and $i + j \leqq m + k$), where Φ is a real valued function of $((m + 1)(m + 2))/2 + k(m + 1) + 2$ real variables, continuous in the last two (y and t), and satisfying a Lipschitz condition in all the other variables, in a domain defined by $|u| \leqq h$, $0 \leqq y \leqq \pi$, $0 \leqq t \leqq \tau$.

We are interested in solutions $u(y, t)$ of (4.1), valid for $0 \leqq y \leqq \pi$, $0 \leqq t \leqq \sigma \leqq \tau$, which satisfy the initial conditions

$$(4.2)\qquad \frac{\partial^j u}{\partial t^j}\Big|_{t=0} = f_j(y) \qquad (j = 0, 1, \cdots, 2m + k - 1)$$

and the boundary conditions

$$(4.3)\quad a\,\frac{\partial^{2i}u}{\partial y^{2i}}\bigg|_{y=0} + b\,\frac{\partial^{2i+1}u}{\partial y^{2i+1}}\bigg|_{y=0} = a\,\frac{\partial^{2i}u}{\partial y^{2i}}\bigg|_{y=\pi} + b\,\frac{\partial^{2i+1}u}{\partial y^{2i+1}}\bigg|_{y=\pi} = 0 \qquad (1 \leqq 2i+1 \leqq m)\,.$$

Under certain conditions, we shall obtain results on the existence and uniqueness of solutions, in a certain *generalized sense*, of the system (4.1), (4.2), (4.3). Before we give a precise definition of this, it will be useful to introduce the following notation.

DEFINITION 4.I. A real-valued function u of one or two variables, defined on some domain R, is said to be of *class* S^q *on* R if and only if

(a) u is of class C^{q-1} (all derivatives up to order $q-1$ exist and are continuous) on R;

(b) all derivatives of order q exist almost everywhere on R and are of class $L^2(R)$ (see § 2),

(c) all derivatives of order $q-1$ are absolutely continuous on R (if u is a function of two variables, say y and t, this will mean that these derivatives are absolutely continuous functions of y for almost all values of t).

DEFINITION 4.II. A real-valued function $u(y, t)$ of the two real variables y and t, defined on a domain $R_\sigma = \{(y, t) \mid 0 \leqq y \leqq \pi,\ 0 \leqq t \leqq \sigma\}$ is said to be of *class* T_q^p *on* R_σ if and only if

(a) u is of class S^q on R_σ;

(b) for almost all y on the interval $(0 \leqq y \leqq \pi)$ it is true that $\partial^i u/\partial y^i$ (considered as a function of t alone) is of class S^{p-i} on the interval $(0 \leqq t \leqq \sigma)$, for $i = 0, 1, \cdots, q$.

Note that for $p \leqq q$, $T_q^p = S^q$.

Now let us define what we mean by a "solution in the generalized sense."

DEFINITION 4.III. A real-valued function $u(y, t)$ of the two real variables y and t, defined on a domain $R_\sigma = \{(y, t) \mid 0 \leqq y \leqq \pi,\ 0 \leqq t \leqq \sigma\}$ for some positive $\sigma \leqq \tau$, is said to be a *solution in the generalized sense* (abbreviated as *G-solution*) of the system (4.1), (4.2), (4.3), if and only if the following conditions are satisfied:

4.III.1.

(a) u is of class T_m^{2m+k} on R_σ;

(b) u satisfies (4.3);

(c) u satisfies (4.2).

4.III.2. There exists a sequence of functions $u^1(y, t)$, $u^2(y, t), \cdots,$ such that

(a) each u^n is defined and of class T_{2m}^{2m+k} on R_σ;

(b) each u^n satisfies (4.3) for $(1 \leqq 2i+1 \leqq 2m+k)$;

(c) $\left.\dfrac{\partial^i u^n}{\partial t^i}\right|_{t=0} \in S^{3m+k-i-1}$ on the interval $(0 \leqq y \leqq \pi)$ for $i = 0, 1, \cdots, 2m + k - 1$;

(d) $\lim_{n\to\infty}\left[\dfrac{\partial^{i+j}u}{\partial t^i \partial y^j} - \dfrac{\partial^{i+j}u^n}{\partial t^i \partial y^j}\right] = 0$ uniformly on R_σ for $(j \leqq m - 1, i + j \leqq m + k - 1)$;

(e) $\lim_{n\to\infty}\displaystyle\int_0^\pi\left[\dfrac{\partial^{m+k}u}{\partial t^{m+k-i}\partial y^i} - \dfrac{\partial^{m+k}u^n}{\partial t^{m+k-i}\partial y^i}\right]^2 dy = 0$ uniformly in t on $(0 \leqq t \leqq \sigma)$, for $i \leqq m$;

(f) $\lim_{n\to\infty}\displaystyle\int_0^\pi\int_0^\sigma\left[\dfrac{\partial^k}{\partial t^k}\left(\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial y^2}\right)^m u^n - \Phi\left(\dfrac{\partial^{m+k}u^n}{\partial t^{m+k}}, \cdots, u^n, y, t\right)\right]^2 dydt = 0.$

THEOREM 4.I. *Let $m > 0$, $k \geqq 0$, be arbitrary but fixed integers. Let $\lambda = ((m + 1)(m + 2))/2 + k(m + 1)$.*

4.I.1. *Let $\Phi(p_1, p_2, \cdots, p_\lambda, y, t)$ be a real-valued function of $(\lambda + 2)$ real variables defined, continuous in y and t, and satisfying a Lipschitz condition with Lipschitz constant θ in the first λ variables, on the domain $\Omega = \{(p_1, \cdots, p_\lambda, y, t) \mid 0 \leqq y \leqq \pi,\ 0 \leqq t \leqq \tau\}$ where h and τ are positive constants (Lipschitz condition in the p_i's: if $0 \leqq y \leqq \pi$, $0 \leqq t \leqq \tau$, then $|\Phi(p_1, \cdots, p_\lambda, y, t) - \Phi(\bar{p}_1, \cdots, \bar{p}_\lambda, y, t)| \leqq \theta[|p_1 - \bar{p}_1| + \cdots + |p_\lambda - \bar{p}_\lambda|]$).*

4.I.2. *Let there be given $2m + k$ real-valued functions $f_0(y)$, $f_1(y), \cdots, f_{2m+k-1}(y)$, each $f_i(y)$ being defined and of class $S^{2m+k-1-i}$ on the interval $(0 \leqq y \leqq \pi)$. Let*

$$\delta^2 = \max_{i=0,1,\cdots,2m+k-1}\int_0^\pi\left[\frac{d^{2m+k-1-i}f_i}{dy^{2m+k-1-i}}\right]^2 dy\ .$$

4.I.3. *Let $h > \sqrt{\pi}(2m + k)\delta c$, where c is defined by Lemma 4.III.*

Then, for some positive real number $\sigma \leqq \tau$, there exists a unique G-solution of (4.1), (4.2), (4.3) with $b = 0$, on R_σ. This solution u will satisfy $|u(y, t)| \leqq h$ on $0 \leqq y \leqq \pi$, $0 \leqq t \leqq \sigma$.

COROLLARY 4.II. *Lewis' theorem, stated in* § 1.

Proof of Corollary. It is the special case $m = 1$, $k = 0$, of Theorem 4.I.

Proof of Theorem 4.I. We wish to obtain Theorem 4.I by an application of Theorem 3.IV.

We shall take for the Hilbert space of § 3 the space $[L^2(R_\sigma)]^\lambda$ (that is, the space of ordered λ-tuples $\langle u_1, \cdots, u_\lambda\rangle$ of functions u_i defined and of class L^2 on R_σ) where σ will be determined later. The subspace obtained by setting $u_i = 0$ for $i = 2, 3, \cdots, \lambda$ (which space is, of course, isomorphic to $L^2(R_\sigma)$ by the natural isomorphism $u \longleftrightarrow \langle u, 0, \cdots, 0\rangle$) will be denoted by H^1.

The sequence G_n of pairwise orthogonal subspaces of H is defined as follows: $G_n = \{\langle x_1(t)\sin(ny), x_2(t)\sin(ny), \cdots, x_\lambda(t)\sin(ny)\rangle \mid x_i(t)$ is of class L^2 on the interval $(0 \leqq t \leqq \sigma)$ for $i = 1, 2, \cdots, \lambda\}$. Since the functions $\sin(ny)$ are pairwise orthogonal on the interval $(0 \leqq y \leqq \pi)$, the spaces G_n are clearly pairwise orthogonal. We shall use ϕ_n to denote $\sin(ny)$, normalized to norm 1:

$$\phi_n = \sqrt{\frac{2}{\pi}}\sin(ny)\ . \tag{4.4}$$

The orthogonal projection of H onto G_n, denoted by P_n, is given by

$$\begin{aligned} &P_n\langle u_1, \cdots, u_\lambda\rangle \\ &= \left\langle \left(\int_0^\pi u_1(\eta, t)\phi_n(\eta)d\eta\right)\phi_n(y), \cdots, \left(\int_0^\pi u_\lambda(\eta, t)\phi_n(\eta)d\eta\right)\phi_n(y)\right\rangle . \end{aligned} \tag{4.5}$$

We define the partial norm Z as follows:

$$Z\langle u_1, \cdots, u_\lambda\rangle = \left[\int_0^\pi (u_1(y, t))^2 dy + \cdots + \int_0^\pi (u_\lambda(y, t))^2 dy\right]^{1/2} . \tag{4.6}$$

It is easily seen that Z is indeed a partial norm on H (see Definition 2.I), and that the G_n's and Z satisfy Condition 3.IV.0 of Theorem 3.IV.

We shall also consider the orthonormal basis for $L^2(R_\sigma)$ consisting of the normalized cosine functions:

$$\psi_0(y) = \frac{1}{\sqrt{\pi}}\ , \qquad \psi_n(y) = \sqrt{\frac{2}{\pi}}\cos(ny) \qquad n = 1, 2, \cdots . \tag{4.7}$$

If u is an element of $L^2(R_\sigma)$, then it is easily seen that there exists a unique element u^* of $L^2(R_\sigma)$ such that $\int_0^\pi u^*(y, t)\psi_n(y)dy = \int_0^\pi u(y, t)\phi_n(y)dy$ (p.p.) for $n = 1, 2, \cdots$, and $\int_0^\pi u^*(y, t)\psi_0(y)dy = 0$ (p.p.). We now define the operator U in $L^2(R_\sigma)$ by $Uu = u^*$. U is clearly linear and isometric; hence U has a unique inverse U^{-1} on the range of U, which is the class of functions v such that $\int_0^\pi v(y, t)dy = 0$ for almost all t.

Roughly speaking, the operators N, S, L of Theorem 3.IV will be the following:

$$N\langle u, 0, \cdots, 0\rangle = \left\langle \frac{\partial^{m+k}u}{\partial t^{m+k}},\ U^{-1}\frac{\partial^{m+k}u}{\partial t^{m+k-1}\partial y}, \cdots, u\right\rangle$$

$$S\langle u_1, \cdots, u_\lambda\rangle = \langle \Phi(u_1(y, t), (Uu_2)(y, t), \cdots, u_\lambda(y, t), y, t), 0, \cdots, 0\rangle$$

$$L\langle u, 0, \cdots, 0\rangle = \left\langle \frac{\partial^k}{\partial t^k}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)^m u, 0, \cdots, 0\right\rangle .$$

Let us now define L, S, N precisely.

DEFINITION 4.IV. $D(N) = \Big\{\langle u, 0, \cdots, 0\rangle \Big| u$ is of class T_m^{m+k} on R_σ;

u satisfies (4.3) with $b = 0$; and

$$\sum_{n=1}^{\infty}\left[n^i \frac{d^j x_n}{dt^j}\right]^2$$

converges, uniformly p.p. to a bounded function, for $i + j \leqq m + k$ and $j \leqq m$, where $x_n(t) = \int_0^\pi u(y, t)\phi_n(y)dy\}$.

In that domain N is defined by

$$N\langle u, 0, \cdots, 0\rangle = \left\langle \frac{\partial^{m+k}u}{\partial t^{m+k}}, U^{-1}\frac{\partial^{m+k}u}{\partial t^{m+k-1}\partial y}, \frac{\partial^{m+k}u}{\partial t^{m+k-2}\partial y^2}, \right.$$

$$\left. U^{-1}\frac{\partial^{m+k}u}{\partial t^{m+k-3}\partial y^3}, \cdots, \frac{\partial^2 u}{\partial t^2}, U^{-1}\frac{\partial^2 u}{\partial t \partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial t}, U^{-1}\frac{\partial u}{\partial y}, u \right\rangle .$$

It must still be verified that the above definition is meaningful; that is, that $(\partial^{i+2j+1i}u/\partial t^i \partial y^{2j+1})$ belongs to $D(U^{-1}) = R(U)$, for $2j + 1 \leqq m$, $i + 2j + 1 \leqq m + k$. That is, we must show that

$$\int_0^\pi \frac{\partial^{i+2j+1}u}{\partial t^i \partial y^{2j+1}}\, dy = 0 \qquad \text{(p.p.)} .$$

But this is easily seen, for by (4.3)

$$\int_0^\pi \frac{\partial^{i+2j+1}u}{\partial t^i \partial y^{2j+1}}\, dy = \frac{\partial^{i+2j}u}{\partial t^i \partial y^{2j}}\bigg|_{y=0}^{y=\pi} = \frac{d^i}{dt^i}\left[\frac{\partial^{2j}u}{\partial y^{2j}}\bigg|_{y=\pi} - \frac{\partial^{2j}u}{\partial y^{2j}}\bigg|_{y=0}\right] = 0$$

for $2j + 1 \leqq m$, $i + 2j + 1 \leqq m + k$.

DEFINITION 4.V. $D(L) = \{\langle u, 0, \cdots, 0\rangle \mid u$ is of class T_{2m}^{2m+k} on R_σ, $\langle u, 0, \cdots, 0\rangle \in D(N)$, u satisfies (4.3) for $2i < 2m$, and $\partial^i u/\partial t^i|_{t=0} \in S^{3m+k-i-1}$ on the interval $(0 \leqq y \leqq \pi)$ for $i = 0, 1, \cdots, 2m + k - 1\}$.

In that domain L is defined by

$$L\langle u, 0, \cdots, 0\rangle = \left\langle \frac{\partial^k}{\partial t^k}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)^m u, 0, \cdots, 0 \right\rangle .$$

DEFINITION 4.VI.

$$B = D(N) \cap \{\langle u, 0, \cdots, 0\rangle \mid |u(y, t)| \leqq h \qquad \text{(p.p.)}\} .$$

$$D(S) = H$$

$$S\langle v_{m+k,0}, \cdots, v_{00}\rangle = \langle \Phi(w_{m+k,0}(y, t) \cdots, w_{00}(y, t), y, t), 0, \cdots, 0\rangle$$

where $w_{i,2j} = v_{i,2j}$ and $w_{i,2j+1} = Uv_{i,2j+1}$.

DEFINITION 4.VII.

$A = \{\langle u, 0, \cdots, 0\rangle \mid \langle u, 0, \cdots, 0\rangle \in D(L)$ and $\partial^i u/\partial t^i|_{t=0} = 0$ for $i = 0, 1, \cdots, 2m + k - 1\}$.

DEFINITION 4.VIII. Let $r_i^n(t)$ denote the solution of the ordinary differential equation

$$\frac{d^k}{dt^k}\left(\frac{d^2}{dt^2}+n^2\right)^m r = 0 \tag{4.8}$$

which satisfies the initial conditions

$$\left.\frac{d^j r_i^n}{dt^j}\right|_{t=0} = \delta_{ij} \quad \text{for } j = 0, 1, \cdots, 2m+k-1 . \tag{4.9}$$

LEMMA 4.III. *There exists a constant c, depending only on m, σ, and k (and not on n) such that*

$$\left|\frac{d^\mu r_i^n}{dt^\mu}\right| \leqq cn^{m+\mu-1-i} \tag{4.10}$$

for $\mu = k, k+1, \cdots, 2q+k,\ i = 0, 1, \cdots, 2m+k-1,$

$$\left|\frac{d^\mu r_i^n}{dt^\mu}\right| \leqq cn^{m+k-1-i} \tag{4.11}$$

for $\mu = 0, 1, \cdots, k-1,\ i = 0, 1, \cdots, 2m+k-1.$

The proof of Lemma 4.III. is given in § 5.

DEFINITION 4.IX. $f(y, t) = \sum_{n=1}^{\infty} [\sum_{i=0}^{2m+k-1} a_{ni} r_i^n(t)]\phi_n(y)$ where $a_{ni} = \int_0^\pi f_i(y)\phi_n(y)dy;\ \phi = \langle f, 0, \cdots, 0\rangle$.

We now wish to show that L, S, N, A, ϕ, B as given by Definitions 4.IV–VII and 4.IX satisfy the assumptions of Theorem 3.IV.

3.IV.0. has already been verified.

3.IV.1. It is clear that L and N are linear and that A is a linear manifold. Also, by definition, $D(L) \subset D(N)$. We shall next show that L and N are reduced by each G_n.

If $\langle u, 0, \cdots, 0\rangle \in D(N)$, then it is clear that $P_n\langle u, 0, \cdots, 0\rangle \in D(N)$; and

$$\begin{aligned}
&P_n N\langle u, 0, \cdots, 0\rangle \\
&\quad = P_n\left\langle \frac{\partial^{m+k}u}{\partial t^{m+k}},\ U^{-1}\frac{\partial^{m+k}u}{\partial t^{m+k-1}\partial y}, \cdots, \frac{\partial u}{\partial t},\ U^{-1}\frac{\partial u}{\partial y},\ u\right\rangle \\
&\quad = \left\langle \left(\int_0^\pi \frac{\partial^{m+k}u(\eta, t)}{\partial t^{m+k}}\phi_n(\eta)d\eta\right)\phi_n(y),\ \left(\int_0^\pi \frac{\partial^{m+k}u(\eta, t)}{\partial t^{m+k-1}\partial\eta}\psi_n(\eta)d\eta\right)\phi_n(y), \cdots,\right. \\
&\qquad \left(\int_0^\pi \frac{\partial u(\eta, t)}{\partial t}\phi_n(\eta)d\eta\right)\phi_n(y),\ \left(\int_0^\pi \frac{\partial u(\eta, t)}{\partial \eta}\psi_n(\eta)d\eta\right)\phi_n(y), \cdots, \\
&\qquad \left.\left(\int_0^\pi u(\eta, t)\phi_n(\eta)d\eta\right)\phi_n(y)\right\rangle .
\end{aligned}$$

Recalling that $\partial^{2i}u/\partial y^{2i}$ is zero at $y = 0$ and at $y = \pi$ for $2i < m$ (see

Definition 4.IV), that ϕ_n is zero at $y = 0$ and at $y = \pi$, and that $d^2\phi_n/dy^2 = -n^2\phi_n$, we obtain by repeated partial integration

$$\int_0^\pi \frac{\partial^{i+2j}u(\eta, t)}{\partial t^i \partial \eta^{2i}} \phi_n(\eta)d\eta = (-n^2)^j \frac{d^i}{dt^i} \int_0^\pi u(\eta, t)\phi_n(\eta)d\eta \qquad \text{for } 2j \leqq m,\ i + 2j \leqq m + k$$

and

$$\int_0^\pi \frac{\partial^{i+2j+1}u(\eta, t)}{\partial t^i \partial \eta^{2j+1}} \psi_n(\eta)d\eta = n(-n^2)^j \frac{d^i}{dt^i} \int_0^\pi u(\eta, t)\phi_n(\eta)d\eta \qquad \text{for } 2j + 1 \leqq m,\ i + 2j + 1 \leqq m + k\ .$$

Hence, letting $x_n(t)$ denote $\int_0^\pi u(\eta, t)\phi_n(\eta)d\eta$,

$$\begin{aligned} P_nN\langle u, 0, \cdots, 0\rangle &= \Big\langle \frac{d^{m+k}x_n(t)}{dt^{m+k}} \phi_n(y),\ n\frac{d^{m+k-1}x_n(t)}{dt^{m+k-1}} \phi_n(y),\ \cdots, \\ &\qquad \frac{dx_n(t)}{dt}\phi_n(y),\ nx_n(t)\phi_n(y),\ x_n(t)\phi_n(y) \Big\rangle \\ &= \Big\langle \frac{\partial^{m+k}}{\partial t^{m+k}}(x_n(t)\phi_n(y)),\ U^{-1}\Big[\frac{\partial^{m+k}}{\partial t^{m+k-1}\partial y}(x_n(t)\phi_n(y))\Big],\ \cdots, \\ &\qquad \frac{\partial}{\partial t}(x_n(t)\phi_n(y)),\ U^{-1}\Big[\frac{\partial}{\partial y}(x_n(t)\phi_n(y))\Big],\ x_n(t)\phi_n(y)\Big\rangle \\ &= N\langle x_n(t)\phi_n(y), 0, \cdots, 0\rangle = NP_n\langle u, 0, \cdots, 0\rangle\ . \end{aligned}$$

Thus N is indeed reduced by each G_n.

In an entirely similar manner, we obtain that if $\langle u, 0, \cdots, 0\rangle \in D(L)$, then $P_n\langle u, 0, \cdots, 0\rangle \in D(L)$ and

$$\begin{aligned} &P_nL\langle u, 0, \cdots, 0\rangle \\ &= \Big\langle\Big(\int_0^\pi \Big[\sum_{i=0}^m \binom{m}{i}(-1)^{m-i} \frac{\partial^{2m+k}u(\eta, t)}{\partial t^{2i+k}\partial\eta^{2m-2i}}\Big]\phi_n(\eta)d\eta\Big)\phi_n(y), 0, \cdots, 0\Big\rangle \\ &= \Big\langle\Big[\sum_i \binom{m}{i} n^{2(m-i)} \frac{d^{2i+k}}{dt^{2i+k}} \int_0^\pi u(\eta, t)\phi_n(\eta)d\eta\Big]\phi_n(y), 0, \cdots, 0\Big\rangle \\ &= \Big\langle \sum_i \binom{m}{i}(-1)^{m-i} \frac{\partial^{2m+k}}{\partial t^{2i+k}\partial y^{2(m-i)}}\Big[\Big(\int_0^\pi u(\eta, t)\phi_n(\eta)d\eta\Big)\phi_n(y)\Big], 0, \cdots, 0\Big\rangle \\ &= \Big\langle \frac{\partial^k}{\partial t^k}\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)^m \Big[\Big(\int_0^\pi u(\eta, t)\phi_n(\eta)d\eta\Big)\phi_n(y)\Big], 0, \cdots, 0\Big\rangle \\ &= LP_n\langle u, 0, \cdots, 0\rangle\ . \end{aligned}$$

Thus, L and N are reduced by the G_n's. Obviously, $P_nA \subset A$. If $\langle u, 0, \cdots, 0\rangle \in \mathscr{N}(L) \cap G_n$, then it is easily seen that $u(y, t)$ is (almost everywhere) of the form $x(t)\phi_n(y)$, and that $x(t)$ is a solution of equation (4.8). Hence

$$x(t) = \sum_{i=0}^{2m+k-1} \left(\frac{d^i x}{dt^i} \Big|_{t=0} \right) r_i^n(t) .$$

Conversely it is clear that any element of the form

$$\left\langle \sum_{i=0}^{2m+k-1} c_i r_i^n(t) \phi_n(y), 0, \cdots, 0 \right\rangle ,$$

where the c_i's are real numbers, belongs to $\mathcal{N}(L) \cap G_n$. Thus $\mathcal{N}(L) \cap G_n$ is a finite dimensional linear manifold, and hence must be closed. This completes the proof of 3.IV.1(b). Of 3.IV.1(a) we must still show that $D(L) = A \oplus \mathcal{N}(L)$.

By definition, $A \subset D(L)$. A and $\mathcal{N}(L)$ are disjoint; for suppose $\langle u, 0, \cdots, 0 \rangle \in A \cap \mathcal{N}(L)$. Then $LP_n\langle u, 0, \cdots, 0 \rangle = P_n L\langle u, 0, \cdots, 0 \rangle = 0$, and hence $P_n\langle u, 0, \cdots, 0 \rangle \in D(L) \cap G_n$; therefore $x_n(t) = \int_0^\pi u(y, t)\phi_n(y)dy$ is of the form

$$x_n(t) = \sum_{i=0}^{2m+k-1} \left(\frac{d^i x_n}{dt^i} \Big|_{t=0} \right) r_i^n(t) .$$

On the other hand, $\langle u, 0, \cdots, 0 \rangle \in A$, so

$$\frac{d^i x_n}{dt^i} \Big|_{t=0} = \int_0^\pi \left(\frac{\partial^i u}{\partial t^i} \Big|_{t=0} \right) \phi_n(y) dy = 0 \qquad \text{for } i = 0, 1, \cdots, 2m + k - 1 .$$

Hence $x_n \equiv 0$ for $n = 1, 2, \cdots$, and therefore $u = 0$. Thus $A \oplus \mathcal{N}(L) \subset D(L)$.

We must still show that $D(L) \subset A \oplus \mathcal{N}(L)$. Let $\langle u, 0, \cdots, 0 \rangle \in D(L)$. Let again $x_n(t) = \int_0^\pi u(\eta, t)\phi_n(\eta)d\eta$. Define $w(y, t) = \sum_{n=1}^\infty w_n(t)\phi_n(y)$ where

$$w_n(t) = \sum_{i=0}^{2m+k-1} \left(\frac{d^i x_n}{dt^i} \Big|_{t=0} \right) r_i^n(t) .$$

By Lemma 4.III.,

$$\begin{aligned} \sum_{n=1}^\infty n^{2l} \left[\frac{d^j w_n}{dt^j} \right]^2 &= \sum_n n^{2l} \left[\sum_i \left(\frac{d^i x_n}{dt^i} \Big|_{t=0} \right) \frac{d^j r_i^n}{dt^j} \right]^2 \\ &\leqq \sum_n n^{2l} \left[\sum_i \left(\frac{d^i x_n}{dt^i} \Big|_{t=0} \right) c n^{m+\max(j,k)-1-i} \right]^2 \\ &\leqq \sum_{n=1}^\infty (2m+k) \sum_{i=0}^{2m+k-1} c^2 \left(\frac{d^i x_n}{dt^i} \Big|_{t=0} \right)^2 n^{2(l+m+\max(j,k)-1-i)} \end{aligned}$$

which last is finite for $l \leqq 2m$ and $l + j \leqq 2m + k$, since $\langle u, 0, \cdots, 0 \rangle \in D(L)$. It is now easily seen that $\langle w, 0, \cdots, 0 \rangle \in D(L)$, and of course $L\langle w, 0, \cdots, 0 \rangle = 0$. Further it is clear that $\langle u - w, 0, \cdots, 0 \rangle \in A$. Hence $\langle u, 0, \cdots, 0 \rangle \in A \oplus \mathcal{N}(L)$. This completes the proof that 3.IV.1. is satisfied.

3.IV.2(a). Let $\langle u, 0, \cdots, 0\rangle \in A$. Let

$$x_n(t) = \int_0^\pi u(y, t)\phi_n(y)dy \ , \qquad \xi_n(t) = \frac{d^k}{dt^k}\Big(\frac{d^2}{dt^2} + n^2\Big)^m x_n(t) \ ,$$

and

$$\eta_n(t) = \int_0^t \xi_n(s) r^n_{2m+k-1}(t-s)ds \ .$$

It is easily seen that

$$\frac{d^i\eta_n}{dt^i} = \int_0^t \xi_n(s)\,\frac{d^i r^n_{2m+k-1}(t-s)}{dt^i}\,ds \qquad \text{for } i = 0, 1, \cdots, 2m+k-1$$

$$\frac{d^{2m+k}\eta_n}{dt^{2m+k}} = \int_0^t \xi_n(s)\frac{d^{2m+k} r^n_{2m+k-1}(t-s)}{dt^{2m+k}}\,ds + \xi_n(t) \ .$$

Hence

$$\frac{d^k}{dt^k}\left(\frac{d^2}{dt^2} + n^2\right)^m \eta_n = \xi_n \tag{4.12}$$

and

$$\frac{d^i\eta_n}{dt^i}\bigg|_{t=0} = 0 \qquad (i = 0, 1, \cdots, 2m+k-1) \ . \tag{4.13}$$

But $x_n(t)$ also satisfies (4.12) and (4.13) and hence $x_n = \eta_n$. Thus using Lemma 4.III., we obtain, for $\mu = 0, 1, \cdots, 2m+k-1$,

$$\left|\frac{d^\mu x_n}{dt^\mu}\right| = \left|\int_0^t \xi_n(s)\,\frac{d^\mu r^n_{2m+k-1}(t-s)}{dt^\mu}\,ds\right| \leqq \left[\sigma\int_0^\sigma \xi_n^2(s)ds\right]^{1/2} c n^{\max(\mu-k-m,-m)} \ .$$

Hence

$$(ZN\langle u, 0, \cdots, 0\rangle)^2 = \sum_{n=1}^\infty (ZP_nN\langle u, 0, \cdots, 0\rangle)^2 = \sum_{n=1}^\infty \sum_{j=0}^m \sum_{i=0}^{m+k-j}\left(n^j \frac{d^i x_n}{dt^i}\right)^2$$

$$\leqq \sum_n \sum_j \sum_i \sigma \int_0^\sigma \xi_n^2(s)ds\, c^2 n^{2\max(i+j-k-m,\, j-m)}$$

$$\leqq \lambda\sigma c^2 \sum_n \int_0^\sigma \xi_n^2(s)ds = \lambda\sigma c^2 \,\|\, L\langle u, 0, \cdots, 0\rangle \,\|^2 \ .$$

Thus 3.IV.2(a) holds with

$$\gamma = \sigma c^2 \lambda \ . \tag{4.14}$$

Note that $c^2\lambda > 1$, so $\gamma > \sigma$; and hence $\Gamma = \max(\gamma, \sigma) = \gamma$.

3.IV.2(b). Let $\langle u, 0, \cdots, 0\rangle \in D(N)$. Then

$$(ZN\langle u, 0, \cdots, 0\rangle)^2 = \sum_{n=1}^\infty \sum_{j=0}^m \sum_{i=0}^{m+k-j}\left(n^j \frac{d^i x_n}{dt^i}\right)^2 \geqq \sum_{n=1}^\infty x_n^2$$

$$= (Z\langle u, 0, \cdots, 0\rangle)^2 \qquad \text{(p.p.)} \ .$$

Hence 3.IV.2(b) is satisfied with $\beta = 1$.

3.IV.3. Let $u = \langle u_{m+k,0}, \cdots, u_{00}\rangle$, $v = \langle v_{m+k,0}, \cdots, v_{00}\rangle$ be two elements of $D(S)$. Define u'_{ij}, v'_{ij} as follows:

$$\begin{aligned} u'_{i,2j} &= u_{i,2j} \qquad & v'_{i,2j} &= v_{i,2j} \qquad (2j \leqq m,\ i + 2j \leqq m + k) \\ u'_{i,2j+1} &= Uu_{i,2j+1} \qquad & v'_{i,2j+1} &= Uv_{i,2j+1} \\ & & & \qquad (2j + 1 \leqq m,\ i + 2j + 1 \leqq m + k)\,. \end{aligned}$$

Then, recalling Definition 4.VI., and our assumption of a Lipschitz condition in Φ

$$\begin{aligned} (Z(Su - Sv))^2 &= \int_0^\pi (Su - Sv)^2 dy \\ &= \int_0^\pi [\Phi(u'_{m+k,0}(y, t), \cdots, u'_{00}(y, t), y, t) \\ &\quad - \Phi(v'_{m+k,0}(y, t), \cdots, v'_{00}(y, t), y, t)]^2 dy \\ &\leqq \int_0^\pi \theta^2[|u'_{m+k,0} - v'_{m+k,0}| + \cdots + |u'_{00} - v'_{00}|]^2 dy \\ &\leqq \lambda\theta^2 \sum_{j=0}^{m} \sum_{i=0}^{m+k-j} \int_0^\pi [u'_{ij} - v'_{ij}]^2 dy \\ &= \lambda\theta^2 \sum_j \sum_i \int_0^\pi [u_{ij} - v_{ij}]^2 dy = \lambda\theta^2 (Z(u - v))^2 \qquad \text{(p.p.)}\,. \end{aligned}$$

Thus we have

$$Z(Su - Sv) \leqq \alpha Z(u - v) \qquad \text{(p.p.)}$$

with

$$\alpha = \theta\sqrt{\lambda}\,. \tag{4.15}$$

Note that σ has not yet been determined, and so far σ can be any real number in the interval $0 < \sigma \leqq \tau$. If we further restrict σ to

$$\sigma < \frac{1}{c^2\theta\sqrt{\lambda^3}}\,, \tag{4.16}$$

then we will have $\alpha\Gamma = \alpha\gamma < 1$, and so 3.IV.3 will be satisfied.

3.IV.4. From assumption 4.I.2. and Lemma 4.III. we obtain, for $j \leqq m$ and $j + l \leqq m + k$

$$\begin{aligned} \sum_{n=1}^{\infty} n^{2j}\left[\frac{d^l}{dt^l}\sum_{i=0}^{2m+k-1} a_{ni} r_i^n(t)\right]^2 &\leqq \sum_n n^{2j}\left[\sum_i a_{ni} c n^{m+\max(l,k)-i-1}\right]^2 \\ &\leqq (2m + k) \sum_i \sum_n c^2 a_{ni}^2 n^{2(2m+k-1-i)} \\ &= (2m + k)c^2 \sum_{i=0}^{2m+k-1} \int_0^\pi \left(\frac{d^{2m+k-i-1} f_i}{dy^{2m+k-i-1}}\right)^2 dy \leqq (2m + k)^2 c^2 \delta^2 < \infty\,. \end{aligned}$$

It is now easily seen that $\phi = \langle f, 0, \cdots, 0\rangle \in D(N)$. Further,

$$|f(y,t)| = \left|\int_0^y \frac{\partial f}{\partial \eta} d\eta\right| \leqq \left[\pi \int_0^\pi \left(\frac{\partial f}{\partial \eta}\right)^2 d\eta\right]^{1/2}$$
$$= \left[\pi \sum_n n^2 \left(\sum_i a_{ni} r_i^n(t)\right)^2\right]^{1/2} \leqq \sqrt{\pi}(2m+k)c\delta .$$

Thus, recalling 4.I.3.,

$$|f(y,t)| \leqq \sqrt{\pi}(2m+k)c\delta < h , \tag{4.17}$$

and hence $\phi \in B$. Also, ϕ obviously belongs to $\mathcal{N}(L')$. We have already seen that $ZN\phi$ is (essentially) bounded; it is easily seen that the convergence of $\sum_n (ZNP_n\phi)^2$ is uniform.

3.IV.5(a). By Definition 4.VI, $B \subset D(N) = D(SN)$. Thus, $\phi \in B \subset D(SN)$. To see that $SNB \subset R(L')$, we shall show that $R(L') = H^1$. Given any element $\langle u, 0, \cdots, 0\rangle$ of H^1, let $\xi_n(t) = \int_0^\pi u(y,t)\phi_n(y)dy$. Let $x_n(t) = \int_0^t \xi_n(s) r^n_{2m+k-1}(t-s)ds$ and let $v = \sum_{n=1}^\infty x_n(t)\phi_n(y)$. Then it is clear that $\langle v, 0, \cdots, 0\rangle \in D(\sum_{n=1}^\infty LP_n)$ and that $\sum_{n=1}^\infty LP_n\langle v, 0, \cdots, 0\rangle = \langle u, 0, \cdots, 0\rangle$. We must still show that $\langle v, 0, \cdots, 0\rangle \in D(N)$. Recalling that, for $j = 0, 1, \cdots, 2m+k-1$,

$$\left|\frac{d^j x_n}{dt^j}\right| = \left|\int_0^t \xi_n(s) \frac{d^j r^n_{2m+k-1}(t-s)}{dt^j} ds\right|$$
$$\leqq \left[\sigma \int_0^\sigma \xi_n^2(s)ds\right]^{1/2} cn^{\max(j-k-m,-m)} ,$$

we see that for $j \leqq m$ and $i + j \leqq m + k$ the series

$$\sum_{n=1}^\infty \left[n^j \frac{d^i x_n}{dt^i}\right]^2$$

is majorized, term by term, by the series

$$\sum_{n=1}^\infty c^2\sigma \int_0^\sigma \xi_n^2(s)ds$$

which of course has a finite sum, namely $c^2\sigma\int_0^\pi\int_0^\sigma [u(y,t)]^2 dydt$. It is now easily seen that $\langle v, 0, \cdots, 0\rangle \in D(N)$ and hence $\langle u, 0, \cdots, 0\rangle \in R(L')$.

3.IV.5(b). Let $\langle u^n, 0, \cdots, 0\rangle \in D(N)$ for $n = 1, 2, \cdots,$

$$\langle u^n, 0, \cdots, 0\rangle \xrightarrow{Z} u = \langle u_{m+k,0}, \cdots, u_{00}\rangle ,$$

and $N\langle u^n, 0, \cdots, 0\rangle \xrightarrow{Z} v = \langle v_{m+k,0}, \cdots, v_{00}\rangle$. That is,

$$\lim_{n\to\infty}\left[\int_0^\pi (u_{m+k,0} - u^n)^2 dy + \int_0^\pi (u_{m+k-1,1})^2 dy + \cdots + \int_0^\pi u_{00}^2 dy\right] = 0 \qquad \text{uniformly p.p.} \tag{4.18}$$

(4.19)
$$\lim_{n\to\infty}\Big[\int_0^\pi\Big(v_{m+k,0}-\frac{\partial^{m+k}u^n}{\partial t^{m+k}}\Big)^2dy + \int_0^\pi\Big(v_{m+k-1,1}-U^{-1}\frac{\partial^{m+k}u^n}{\partial t^{m+k-1}\partial y}\Big)^2dy+\cdots + \int_0^\pi(v_{00}-u^n)^2dy\Big]=0 \qquad \text{uniformly p.p.}$$

Two things are obvious: first, all u_{ij}'s are zero except for $u_{m+k,0}$; second,

(4.20)
$$u_{m+k,0}=v_{00}\,.$$

Let $w_{i,2j}=v_{i,2j}$ and $w_{i,2j+1}=Uv_{i,2j+1}$. Then (4.19) can be written as

(4.21)
$$\lim_{n\to\infty}\Big[\int_0^\pi\Big(w_{m+k,0}-\frac{\partial^{m+k}u^n}{\partial t^{m+k}}\Big)^2dy + \int_0^\pi\Big(w_{m+k-1,1}-\frac{\partial^{m+k}u^n}{\partial t^{m+k-1}\partial y}\Big)^2dy+\cdots + \int_0^\pi(w_{00}-u^n)^2dy\Big]=0 \qquad \text{uniformly p.p.}$$

For $j<m$ and $i+j<m+k$,

$$\begin{aligned}\Big|\int_0^y w_{i,j+1}(\eta,t)d\eta-\frac{\partial^{i+j}u^n(\eta,t)}{\partial t^i\partial\eta^j}\Big|_{\eta=0}^{\eta=y}\Big| &= \Big|\int_0^y\Big[w_{i,j+1}(\eta,t)-\frac{\partial^{i+j+1}u^n(\eta,t)}{\partial t^i\partial\eta^{j+1}}\Big]d\eta\Big| \\ &\leqq\Big[\pi\int_0^\pi\Big[w_{i,j+1}(\eta,t)-\frac{\partial^{i+j+1}u^n(\eta,t)}{\partial t^i\partial\eta^{j+1}}\Big]^2d\eta\Big]^{1/2},\end{aligned}$$

which last converges to zero uniformly almost everywhere in t, by (4.21). Hence

$$\lim_{n\to\infty}\frac{\partial^{i+j}u^n(\eta,t)}{\partial t^i\partial\eta^j}\Big|_{\eta=0}^{\eta=y}=\int_0^y w_{i,j+1}(\eta,t)d\eta$$

uniformly p.p. on R_σ. But then it follows from (4.21) that

$$w_{ij}(y,t)-w_{ij}(0,t)=\int_0^y w_{i,j+1}(\eta,t)d\eta \qquad \text{(p.p.)}\,,$$

or,

(4.22)
$$\frac{\partial w_{ij}}{\partial y}=w_{i,j+1} \qquad (j<m,\ i+j<k+m)\,.$$

Similarly, for $j\leqq m$, $i+j<m+k$,

$$\int_0^\pi\left[\int_0^t w_{i+1,j}(y,\tau)d\tau - \frac{\partial^{i+j}u^n(y,\tau)}{\partial\tau^i\partial y^j}\Big|_{\tau=0}^{\tau=t}\right]^2 dy$$
$$= \int_0^\pi\left[\int_0^t\left(w_{i+1,j}(y,\tau) - \frac{\partial^{i+j+1}u^n(y,\tau)}{\partial\tau^{i+1}\partial y^j}\right)d\tau\right]^2 dy$$
$$\leqq \sigma\int_0^\pi\int_0^\sigma\left[w_{i+1,j}(y,\tau) - \frac{\partial^{i+j+1}u^n(y,\tau)}{\partial\tau^{i+1}\partial y^j}\right]^2 d\tau dy$$

which converges to zero as $n\to\infty$, by (4.21) and Remark 2.III.1. Hence we obtain

$$\frac{\partial w_{ij}}{\partial t} = w_{i+1,j} \qquad (j\leqq m,\ i+j<m+k)\,. \tag{4.23}$$

Combining (4.20), (4.22), (4.23) yields

$$w_{ij} = \frac{\partial^{i+j}u_{m+k,0}}{\partial t^i\partial y^j}\,. \tag{4.24}$$

Let $x_n(t)=\int_0^\pi u_{m+k,0}(y,t)\phi_n(y)dy$, $x_n^l(t)=\int_0^\pi u^l(y,t)\phi_n(y)dy$. Note that, for $j\leqq m$, $i+j\leqq m+k$, $l=1,2,\cdots$, $\nu=1,2,\cdots$,

$$\sum_{n=\nu}^\infty\left(n^j\frac{d^ix_n}{dt^i}\right)^2 \leqq 2\sum_{n=\nu}^\infty\left[n^j\frac{d^i}{dt^i}(x_n-x_n^l)\right]^2 + 2\sum_{n=\nu}^\infty\left(n^j\frac{d^ix_n^l}{dt^i}\right)^2.$$

Also,

$$2\sum_{n=\nu}^\infty\left[n^j\frac{d^i}{dt^i}(x_n-x_n^l)\right]^2 \leqq 2\sum_{n=1}^\infty\left[n^j\frac{d^i}{dt^i}(x_n-x_n^l)\right]^2$$
$$= 2\int_0^\pi\left[\frac{\partial^{i+j}}{\partial t^i\partial y^j}(u_{m+k,0}-u^l)\right]^2 dy$$
$$= 2\int_0^\pi\left[w_{i,j}-\frac{\partial^{i+j}u^l}{\partial t^i\partial y^j}\right]^2 dy\,.$$

Thus

$$\sum_{n=\nu}^\infty\left(n^j\frac{d^ix_n}{dt^i}\right)^2 \leqq 2\int_0^\pi\left[w_{ij}-\frac{\partial^{i+j}u^l}{\partial t^i\partial y^j}\right]^2 dy + 2\sum_{n=\nu}^\infty\left(n^j\frac{d^ix_n^l}{dt^i}\right)^2. \tag{4.25}$$

It ε is any positive number, then by (4.21) there exists an integer l_ε such that

$$\int_0^\pi\left[w_{ij}-\frac{\partial^{i+j}u^{l_\varepsilon}}{\partial t^i\partial y^j}\right]^2 dy < \frac{\varepsilon}{4} \qquad \text{(p.p.)}\,.$$

Since $\langle u_{l_\varepsilon}, 0, \cdots, 0\rangle \in D(N)$, there exists an integer n_ε such that

$$\sum_{n=n_\varepsilon}^\infty\left(n^j\frac{d^ix_n^{l_\varepsilon}}{dt^i}\right)^2 < \frac{\varepsilon}{4} \qquad \text{(p.p.)}\,.$$

Hence

$$\sum_{n=n_2}^{\infty}\left(n^j \frac{d^i x_n}{dt^i}\right)^2 < \varepsilon$$

and so the convergence of the series

$$\sum_{n=1}^{\infty}\left(n^j \frac{d^i x_n}{dt^i}\right)^2$$

is uniform p.p., for $j \leqq m$ and $i + j \leqq m + k$. It now follows easily that $u = \langle u_{m+k,0}, 0, \cdots, 0\rangle \in D(N)$; and by (4.23) $Nu = v$.

If in addition $\langle u^n, 0, \cdots, 0\rangle \in B$, then $|u^n(y, t)| \leqq h$ and hence obviously $|u_{m+k,0}(y, t)| = |\lim_{n\to\infty} u^n(y, t)| \leqq h$; hence $u \in B$.

3.IV.5(c). Suppose $\langle u, 0, \cdots, 0\rangle \in B$, $N\langle u, 0, \cdots, 0\rangle \in C_\rho(N\phi)$, $SN\langle u, 0, \cdots, 0\rangle = L'\langle v, 0, \cdots, 0\rangle$, and $P_n(\langle v, 0, \cdots, 0\rangle - \phi) \in A$ for $n = 1, 2, \cdots$. First, note that $\langle v, 0, \cdots, 0) \in D(L') \subset D(N)$; thus in order to prove that $\langle v, 0, \cdots, 0\rangle \in B$ all we must show is that $|v(y, t)| \leqq h$. Clearly, $|v(y, t)| \leqq |v(y, t) - f(y, t)| + |f(y, t)|$. Now

$$\begin{aligned}
|v(y, t) - f(y, t)| &= \left|\int_0^y \frac{\partial}{\partial \eta}[v(\eta, t) - f(\eta, t)]\,d\eta\right| \\
&\leqq \left[\pi\int_0^\pi\left(\frac{\partial}{\partial y}(v - f)\right)^2 dy\right]^{1/2} \\
&\leqq \sqrt{\pi}\, ZN(\langle v, 0, \cdots, 0\rangle - \phi) \\
&\leqq \sqrt{\pi\Gamma}\, \| L'\langle v, 0, \cdots, 0\rangle \| = \sqrt{\pi\Gamma}\, \| SN\langle u, 0, \cdots, 0\rangle \| \\
&\leqq \sqrt{\pi\Gamma}\,[\| SN\langle u, 0, \cdots, 0\rangle - SN\phi \| + \| SN\phi \|] \\
&\leqq \sqrt{\pi\Gamma}\,[\alpha \| N\langle u, 0, \cdots, 0\rangle - N\phi \| + \| SN\phi \|] \\
&\leqq \sqrt{\pi\Gamma}\,[\alpha\rho + \| SN\phi \|] = \sqrt{\pi\Gamma}\, \frac{\| SN\phi \|}{1 - \alpha\Gamma} .
\end{aligned}$$

Also we have shown in (4.17) that

$$|f(y, t)| \leqq \sqrt{\pi}(2m + k)c\delta .$$

Hence

$$|v(y, t)| \leqq \sqrt{\pi\Gamma}\, \frac{\| SN\phi \|}{1 - \alpha\Gamma} + \sqrt{\pi}(2m + k)c\delta .$$

We know that $\sqrt{\pi}(2m + k)c\delta < h$; hence the problem now is, can we choose a positive $\sigma \leqq \tau$, satisfying (4.16) and

$$\text{(4.26)} \qquad \sqrt{\pi\Gamma}\, \frac{\| SN\phi \|}{1 - \alpha\Gamma} \leqq h - \sqrt{\pi}(2m + k)c\delta .$$

Now, γ is a constant multiple of σ (see 4.14); α does not depend on σ

(see 4.15); while $\|SN\phi\|$ is a continuous function of σ, zero at $\sigma = 0$. Thus the function

$$g(\sigma) = \sqrt{\pi\Gamma}\,\frac{\|SN\phi\|}{1-\alpha\Gamma}$$

is continuous on $0 \leqq \sigma \leqq \tau$, and $g(0) = 0$. Hence there certainly exists a positive $\sigma \leqq \tau$, satisfying (4.16) and (4.26).

This completes the proof of Theorem 4.I.

In Theorem 4.I. we considered the system (4.1), (4.2), (4.3) with the restriction $b = 0$. This restriction can be omitted if we restrict the derivatives occurring on the right hand side of (4.1) to derivatives of the form $\partial^{i+2j}u/\partial t^i \partial y^{2j}$ with $2j \leqq m$, $i + 2j \leqq m + k$. Thus if (4.1) is replaced by

$$\text{(4.1a)}\quad \frac{\partial^k u}{\partial t^k}\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)^m u$$
$$= \Phi\left(\frac{\partial^{m+k}u}{\partial t^{m+k}}, \frac{\partial^{m+k}u}{\partial t^{m+k-2}\partial y^2}, \cdots, \frac{\partial^3 u}{\partial t^3}, \frac{\partial^3 u}{\partial t \partial y^2}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial t}, u, y, t\right)$$

we obtain

THEOREM 4.IV. *Let the conditions of Theorem* 4.I *be satisfied, except that* $\lambda = (m/2 + 1)(m/2 + k + 1)$ *if* m *is even, and* $\lambda = ((m + 1)/2 + k + 1)(m + 1)/2$ *if* m *is odd.*

Then, for some positive real number $\sigma \leqq \tau$ *there exists a unique G-solution of* (4.1a), (4.2), (4.3) *with arbitrary* a *and* b, *on* R_σ.

Proof. The proof runs along the same lines as that of Theorem 4.I, and we shall here merely indicate the major modification necessary in that proof. This modification consists of replacing (4.4) by

$$\text{(4.4a)}\qquad \phi_n = \sqrt{\frac{2}{\pi}}\sin n(y + \alpha_n)$$

where

$$\text{(4.27)}\qquad \alpha_n = \frac{1}{n}\operatorname{arc} tg\left(-\frac{nb}{a}\right).$$

5. **Proof of Lemma 4.III.** Throughout this section we shall use the standard abbreviations $f', f'', \cdots, f^{(\mu)}$, for $df/dt, d^2f/dt^2, \cdots, d^\mu f/dt^\mu$.

LEMMA 5.I. *Let* $f(t)$ *be a real-valued function of class* C^∞ *on the interval* $(0 \leqq t \leqq a)$. *Let* n *be a positive integer. Let* $x(t)$ *be a solution of the equation*

$$x''(t) + n^2x(t) = f(t) \tag{5.1}$$

valid for $0 \leqq t \leqq a$, *and satisfying the initial conditions*

$$x(0) = x'(0) = 0 \; . \tag{5.2}$$

Then, for $0 \leqq t \leqq a$ *and* $\mu = 0, 1, 2, \cdots$

$$|x^{(\mu)}(t)| \leqq \sum_{j=0}^{\mu-2} n^{\mu-2-j}|f^{(j)}(t)| + an^{\mu-1}\sup_{0\leqq s\leqq a}|f(s)| \; . \tag{5.3}$$

Proof. The proof will be by induction on μ. We first consider the cases $\mu = 0$ and $\mu = 1$. It is easily seen that

$$x(t) = \int_0^t f(s)\frac{1}{n}\sin n(t-s)ds \qquad \text{and} \qquad x'(t) = \int_0^t f(s)\cos n(t-s)ds.$$

Hence, for $0 \leqq t \leqq a$

$$|x(t)| \leqq a\frac{1}{n}\sup_{0\leqq s\leqq a}|f(s)| \qquad \text{and} \qquad |x'(t)| \leqq a\sup_{0\leqq s\leqq a}|f(s)| \; ,$$

which is exactly (5.3) for $\mu = 0$ and $\mu = 1$.

Now suppose that (5.3) has been proved for $\mu \leqq \nu \geqq 1$. Then,

$$\begin{aligned} |x^{(\nu+1)}(t)| &= \left|\left(\frac{d^2}{dt^2} + n^2\right)x^{(\nu-1)}(t) - n^2x^{(\nu-1)}(t)\right| \\ &= |f^{(\nu-1)}(t) - n^2x^{(\nu-1)}(t)| \leqq |f^{(\nu-1)}(t)| + n^2|x^{(\nu-1)}(t)| \\ &\leqq |f^{(\nu-1)}(t)| + n^2\left[\sum_{j=0}^{\nu-3} n^{\nu-3-j}|f^{(j)}(t)| + an^{\nu-2}\sup_{0\leqq s\leqq a}|f(s)|\right] \\ &\leqq \sum_{j=0}^{\nu-1} n^{\nu-1-j}|f^{(j)}(t)| + an^{\nu}\sup_{0\leqq s\leqq a}|f(s)| \end{aligned}$$

which is (5.3) with $\mu = \nu + 1$. This completes the proof of Lemma 5.I.

LEMMA 5.II. *Let f and x be as in Lemma* 5.I., *except that $x(t)$ need not satisfy the initial conditions* (5.2). *Then, for* $0 \leqq t \leqq a$ *and* $\mu = 0, 1, \cdots$,

$$\begin{aligned} |x^{(\mu)}(t)| \leqq\; & n^{\mu}|x(0)| + n^{\mu-1}|x'(0)| \\ & + \sum_{j=0}^{\mu-2} n^{\mu-2-j}|f^{(j)}(t)| + an^{\mu-1}\sup_{0\leqq s\leqq a}|f(s)| \; . \end{aligned} \tag{5.4}$$

Proof. Merely note that the function $x^*(t) = x(t) - x(0)\cos(nt) - x'(0)1/n\sin(nt)$ satisfies all the assumptions of Lemma 5.I.

LEMMA 5.III. *For every positive integer q and positive real number a there exist two real numbers $c(q, a)$, $d(q, a)$ (depending only on q and*

a, and not on n or f) such that, if $f(t)$ is a function of class C^∞ on $(0 \leqq t \leqq a)$ and if $x(t)$ is a solution of

$$\left(\frac{d^2}{dt^2} + n^2\right)^q x(t) = f(t) \tag{5.5}$$

valid for $0 \leqq t \leqq a$, then

$$|x^{(\mu)}(t)| \leqq n^{\mu-q}\left[c(q, a)\sum_{i=0}^{2q-1} n^{2q-1-i}|x^{(i)}(0)| + d(q, a)\sup_{0\leqq s\leqq a}|f(s)|\right] \tag{5.6}$$

for $0 \leqq t \leqq a$ and $\mu = 0, 1, \cdots, 2q$.

Proof. The proof will be by induction on q. If $q = 1$, then from Lemma 5.II., (5.6) follows immediately with $c(1, a) = 1$, $d(1, a) = 1 + a$.

Now suppose (5.6) proved for $q \leqq p$. Let $x(t)$ be a solution of (5.5) with $q = p + 1$. Then $g(t) = x''(t) + n^2x(t)$ satisfies (5.5) with $q = p$, and hence, by our inductive assumption we have, for $j = 0, 1, \cdots, 2p$,

$$\begin{aligned} |g^{(j)}| &\leqq n^{j-p}\left[c(p, a)\sum_{i=0}^{2p-1} n^{2p-1-i}|g^{(i)}(0)| + d(p, a)\sup_{0\leqq s\leqq a}|f(s)|\right] \\ &= n^{j-p}\left[c(p, a)\sum_{i=0}^{2p-1} n^{2p-1-i}|x^{(i+2)}(0) + n^2x^{(i)}(0)| + d(p, a)\sup_{0\leqq s\leqq a}|f(s)|\right] \\ &\leqq n^{j-p}\Bigg[c(p, a)\left(\sum_{i=2}^{2p+1} n^{2p+1-i}|x^{(i)}(0)| + \sum_{i=0}^{2p-1} n^{2p+1-i}|x^{(i)}(0)|\right) \\ &\qquad + d(p, a)\sup_{0\leqq s\leqq a}|f(s)|\Bigg] \\ &\leqq n^{j-p}\left[2c(p, a)\sum_{i=0}^{2p+1} n^{2p+1-i}|x^{(i)}(0)| + d(p, a)\sup_{0\leqq s\leqq a}|f(s)|\right]. \end{aligned}$$

Thus we have seen that, for $0 \leqq t \leqq a$, and $j = 0, 1, \cdots, 2p$

$$|g^{(j)}(t)| \leqq n^{j-p}\left[2c(p, a)\sum_{i=0}^{2p+1} n^{2p+1-i}|x^{(i)}(0)| + d(p, a)\sup_{0\leqq s\leqq a}|f(s)|\right]. \tag{5.7}$$

On the other hand, by Lemma 5.II.,

$$\begin{aligned} |x^{(\mu)}(t)| &\leqq n^\mu|x(0)| + n^{\mu-1}|x'(0)| \\ &\quad + \sum_{j=0}^{\mu-2} n^{\mu-2-j}|g^{(j)}(t)| + an^{\mu-1}\sup_{0\leqq s\leqq a}|g(s)| \end{aligned} \tag{5.8}$$

for $0 \leqq t \leqq a$, $\mu = 0, 1, 2, \cdots$. When $\mu \leqq 2p + 2$, then the highest order derivative of g occurring on the right hand side of (5.8) is at most $2p$, so that we can apply (5.7); thus, for $\mu = 0, 1, \cdots, 2p + 2$,

$$\begin{aligned} |x^{(\mu)}(t)| &\leqq n^\mu|x(0)| + n^{\mu-1}|x'(0)| \\ &\quad + \sum_{j=0}^{\mu-2} n^{\mu-2-j}n^{j-p}\left[2c(p, a)\sum_{i=0}^{2p+1} n^{2p+1-i}|x^{(i)}(0)| + d(p, a)\sup_{0\leqq s\leqq a}|f(s)|\right] \\ &\quad + an^{\mu-1}n^{-p}\left[2c(p, a)\sum_{i=0}^{2p+1} n^{2p+1-i}|x^{(i)}(0)| + d(p, a)\sup_{0\leqq s\leqq a}|f(s)|\right] \end{aligned}$$

$$\leqq n^{\mu-p-1}\sum_{i=0}^{2p+1} n^{2p+1-i}\,|\,x^{(i)}(0)\,|$$
$$+\,[(\mu-1)+a]n^{\mu-p-1}\Big[2c(p,a)\sum_{i=0}^{2p+1} n^{2p+1-i}\,|\,x^{(i)}(0)\,|$$
$$+\,d(p,a)\sup_{0\leqq s\leqq a}|\,f(s)\,|\Big]$$
$$\leqq n^{\mu-(p+1)}\Big[(1+2(2p+1+a)c(p,a))\sum_{i=0}^{2(p+1)-1} n^{2(p+1)-1-i}\,|\,x^{(i)}(0)\,|$$
$$+\,(2p+1+a)d(p,a)\sup_{0\leqq s\leqq a}|\,f(s)\,|\Big]\,.$$

Which is (5.6) for $q = p + 1$ and $c(p + 1, a) = 1 + 2(2p + 1 + a)c(p, a)$ and $d(p + 1, a) = (2p + 1 + a)d(p, a)$.

This completes the proof of Lemma 5.III.

LEMMA 5.IV. *For every positive integer q, nonnegative integer k, and positive real number a there exists a real number $e(q, a, k)$ (depending only on q, a, k, and not on n) such that, if $x(t)$ is a solution of*

$$\frac{d^k}{dt^k}\left(\frac{d^2}{dt^2}+n^2\right)^q x(t)=0 \tag{5.9}$$

valid on $(0 \leqq t \leqq a)$, then on $(0 \leqq t \leqq a)$,

$$|\,x^{(\mu)}(t)\,| \leqq n^{\mu-k-q}e(q,a,k)\sum_{i=k}^{2q+k-1} n^{2q+k-1-i}\,|\,x^{(i)}(0)\,| \tag{5.10}$$

for $\mu = k, k + 1, \cdots, 2q + k$

and

$$|\,x^{(\mu)}(t)\,| \leqq n^{-q}e(q,a,k)\sum_{i=\mu}^{2q+k-1} n^{2q+k-1-i}\,|\,x^{(i)}(0)\,| \tag{5.11}$$

for $\mu = 0, 1, \cdots, k - 1$.

Proof. Applying Lemma 5.III. to $x^{(k)}(t)$ we get, for $\mu = k, k + 1, \cdots, 2q + k$, and $(0 \leqq t \leqq a)$,

$$|\,x^{(\mu)}(t)\,| = |\,(x^{(k)})^{(\mu-k)}\,| \leqq n^{\mu-k-q}c(q,a)\sum_{i=0}^{2q-1} n^{2q-1-i}\,|\,x^{(k+i)}(0)\,|$$
$$= n^{\mu-k-q}c(q,a)\sum_{i=k}^{2q+k-1} n^{2q+k-1-i}\,|\,x^{(i)}(0)\,|\,.$$

Also, for $\mu < k$, we have, on $(0 \leqq t \leqq a)$

$$|\,x^{(\mu)}\,| \leqq |\,x^{(\mu)}(0)\,| + a\sup_{0\leqq t\leqq a}|\,x^{(\mu+1)}(t)\,| \leqq \cdots$$
$$\leqq |\,x^{(\mu)}(0)\,| + a\,|\,x^{(\mu+1)}(0)\,| + \cdots + a^{k-1-\mu}\,|\,x^{(k-1)}(0)\,|$$
$$+\,a^{k-\mu}\sup_{0\leqq t\leqq a}|\,x^{(k)}(t)\,|$$
$$\leqq \max(1,a^k)\cdot c(q,a)n^{-q}\sum_{i=\mu}^{2q+k-1} n^{2q+k-1-i}\,|\,x^{(i)}(0)\,|\,.$$

Thus (5.10) and (5.11) hold with

$$e(q, a, k) = \max(1, a^k) \cdot c(q, a) .$$

Proof of Lemma 4.III. Apply Lemma 5.IV. with $q = m$ and $a = \sigma$ to the functions r_i^n of Definition 4.V.

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