TAME CANTOR SETS IN E^3

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1. Introduction. Let M_1 be the straight line interval $[0, 1]$, M_2 be the sum of $[0, 1/3]$ and $[2/3, 1]$ obtained by deleting the open middle third of M_1, M_3 be the sum of the four intervals obtained by deleting from M_2 the open middle thirds of the components of $M_2, \cdots$. The intersection M of $M_1, M_2, \cdots$ is the familiar Cantor set. Any homeomorphic image of M is called a *Cantor set*.

It may be shown that if C_1 is a Cantor set on the real line, there is a homeomorphism of the line onto itself taking C_1 onto M. Also, if C_2 is a Cantor set in a plane, there is a homeomorphism of the plane onto itself taking C_2 into a line. For these reasons, we say that each Cantor set in a line or a plane is tame. A Cantor set C_3 in E^3 is *tame* if there is a homeomorphism of E^3 onto itself taking C_3 into a line. Not each Cantor set in E^3 is tame. Antoine gives an example of such a wild Cantor set in [1]. A diagram of one is found in [3, 8]. Blankenship has described [7] wild Cantor sets in Euclidean spaces of all dimensions greater than 2.

Characterizations of tame Cantor sets are provided by Theorems 1.1, 3.1, 4.1 and 5.1. In § 6 we prove theorems about the sums of tame Cantor sets and apply these results in § 7 to show that for each closed 2-dimensional set X in E^3, there is a homeomorphism h of E^3 onto itself that is close to the identity and such that $h(X)$ contains no straight line interval. An example is given of a disk containing intervals pointing in all directions showing that such a homeomorphism h may need to be something more than a rigid motion.

A *simple neighborhood* in E^3 is an open set topologically equivalent to the interior of a sphere. By using simple neighborhoods, we give the following characterization of tame Cantor sets. The proof is obtained in a straight forward fashion.

THEOREM 1.1. *A necessary and sufficient condition that a Cantor set C in E^3 be tame is that for each positive number ε, C can be covered by a finite collection of mutually exclusive simple neighborhoods of mesh less than ε.*

Proof. That the condition is necessary follows from the facts that a homeomorphism of a closed and bounded set in E^3 is uniformly continuous and if the homeomorphic image of C lies on a line, this image

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can be covered with a finite collection of mutually exclusive small interiors of spheres.

The sufficiency condition is a bit more involved but its proof may be obtained as outlined below.

1. Using results on piecewise linear approximations to homeomorphisms [12, 5] show that for each $\varepsilon > 0$, there is a finite collection of mutually exclusive polyhedral cubes of mesh less than ε such that the interiors of these polyhedral cubes cover C.

2. Let $X_1, X_2, \cdots$ be a sequence such that each X_i is a collection of mutually exclusive polyhedral cubes of mesh less than $1/i$, each element of X_{i+1} lies on the interior of an element of X_i, and the interiors of the elements of X_i cover C.

3. Using Moise's extension of Alexander's theorem on polyhedral 2-spheres [11], show that there is a piecewise linear homeomorphism h_1 of E^3 onto itself that takes each element of X_1 onto a cube of diameter less than 1 and with center on the x-axis.

4. Reapplying Moise's result, show that there is a piecewise linear homeomorphism h_2 of E^3 onto itself such that $h_1 = h_2$ except in elements of X_1, h_2 takes each element of X_2 onto a cube of diameter less than 1/2 with center on the x-axis. Continuing in this fashion, we obtain $h_3, h_4, \cdots$. The limit of $h_1, h_2, h_3, \cdots$ is a homeomorphism of E^3 onto itself that takes C into the x-axis.

COROLLARY 1.2. *A Cantor set in E^3 is tame if its complement is topologically equivalent to the complement of a tame Cantor set.*

Theorem 1.1 reveals how to find a tame Cantor set in any Cantor set in E^3. Hence, we have the following theorem.

THEOREM 1.3. *Each Cantor set in E^3 contains a tame Cantor set.*

2. Making 2-spheres mutually exclusive. The Cantor set C mentioned in the statement of Theorem 1.1 did not intersect the boundaries of the simple neighborhoods mentioned there. We shall see in Theorem 3.1 of the next section that we can dispense with the condition that the simple neighborhoods be mutually exclusive if we retain this property. As a start in this direction, we consider the following two properties of a finite collection $(N_1, N_2, \cdots, N_n)$ of simple neighborhoods that cover a closed set X.

Property 1. X misses the boundary of each N_i.

Property 2. The N_i's are mutually exclusive.

In this section we see how to replace a collection satisfying Property 1 with a collection satisfying Property 2.

As noted in the proof of Theorem 1.1, a simple neighborhood can be reduced slightly so that the boundary of the resulting neighborhood is a polyhedral 2-sphere. We now prove some theorems about polyhedral 2-spheres. We use $V(A, \varepsilon)$ to denote the set of all points whose distances from A is less than ε.

THEOREM 2.1. *Suppose* $(K_1, K_2, \cdots, K_n)$ *is a collection of mutually exclusive polyhedral* 2-*spheres*; X *is a closed set that lies in each* $\text{Ext}\, K_i$; $(S_1, S_2, \cdots, S_m)$ *is a collection of mutually exclusive polyhedral* 2-*spheres such that* $X \subset (\text{Int}\, S_1 + \text{Int}\, S_2 + \cdots + \text{Int}\, S_m)$; *and* $\varepsilon > 0$. *Then there is a collection of mutually exclusive polyhedral* 2-*spheres* $(S_1', S_2', \cdots, S_r')$ *such that*

$$X \subset \text{Int}\, S_1' + \text{Int}\, S_2' + \cdots + \text{Int}\, S_r' ,$$
$$\Sigma S_j' \cdot \Sigma K_i = 0 , \quad \text{and}$$
$$S_j' \subset V(\Sigma S_i + \Sigma K_i, \varepsilon) .$$

Proof. It would be convenient to have each S_i and each K_j in general position in the sense that if p is a point of $S_i \cdot K_j$, there is a neighborhood N of p and a homeomorphism h of N onto E^3 such that $h(N \cdot S_i)$ and $h(N \cdot K_j)$ are perpendicular planes. Such a convenient position would insure that each component of $S_i \cdot K_j$ is a simple closed curve.

We show how a slight adjustment of S_i brings about such a convenient position. Suppose S_i and each K_j is triangulated. Let $v_1, v_2, \cdots, v_n$ be the vertices of the triangulation of S_i and X be the collection of the vertices of the triangulation of ΣK_j. Adjust v_1 slightly to v_1' so that there results a 2-sphere close to S_i but v_1' is not coplanar with any three noncollinear points of X. Next adjust v_2 to v_2' so that v_2' is not coplanar with any three noncollinear points of $X + v_1'$. If we continue this adjusting of the vertices of S_i one at a time, we obtain a 2-sphere S_i' close to S_i so that no vertex of S_i' lies on any K_j, no vertex of an K_j lies on S_i', and no edge of S_i' intersects any edge of any K_j.

In view of remarks in the preceding paragraph we suppose with no loss of generality that each S_i and each K_j are in general position. (If the S's are not conveniently placed, we replace them by some that are.) The proof of Theorem 2.1 is by induction on the number of components of $\Sigma K_i \cdot \Sigma S_i$.

Let J be such a component that bounds a disk D in K_i such that $D \cdot \Sigma S_j = J$. Assume $J \subset S_m$ and E_1, E_2 are the two disks in S_m bounded by J. Let S_{m_1}, S_{m_2} be two polyhedral 2-spheres obtained by separate deformations of $E_1 + D$ and $E_2 + D$ in a neighborhood of D such that

$$X \cdot \operatorname{Int} S_m \subset \operatorname{Int} S_{m_1} + \operatorname{Int} S_{m_2} ,$$
$$(S_{m_1} + S_{m_2}) \cdot \Sigma K_i = S_m \cdot \Sigma K_i - J ,$$
$$S_{m_1} \cdot S_j \quad (j \neq m) = 0 .$$

The truth of the theorem now follows by induction because $(S_1 + S_2 + \cdots + S_{m-1} + S_{m_1} + S_{m_2}) \cdot \Sigma K_i$ has one less component than $\Sigma S_i \cdot \Sigma K_i$.

THEOREM 2.2. *Suppose $(K_1, K_2, \cdots, K_n)$ is a collection of mutually exclusive polyhedral 2-spheres, X is a closed set which lies in each* Ext K_i, *S is a polyhedral 2-sphere such that $X \subset \operatorname{Int} S$, and $\varepsilon > 0$. Then there is a polyhedral 2-sphere S' such that*

$$X \subset \operatorname{Int} S' ,$$
$$S' \cdot \Sigma K_i = 0 ,$$
$$S' \subset V(S + \Sigma K_i, \varepsilon) .$$

Proof. It follows from the preceding theorem that there is a collection of mutually exclusive polyhedral 2-spheres $(S_1, S_2, \cdots, S_m)$ such that

$$X \subset \operatorname{Int} S_1 + \operatorname{Int} S_2 + \cdots + \operatorname{Int} S_m ,$$
$$S_j \cdot \Sigma K_i = 0 , \quad \text{and}$$
$$S_i \subset V(S + \Sigma K_, , \varepsilon) .$$

The polyhedral 2-sphere S' required by the theorem is obtained by cutting holes in some of the S_j's and joining the boundaries of these holes with tubes near $S + \Sigma K_i$. The expression "some, of" is to suggest that we ignore the S's on the interior of any K_j or any other S_i.

THEOREM 2.3. *Suppose X is a closed, bounded subset of E^3, $(K_1, K_2, \cdots, K_n)$ is a finite collection of topological 2-spheres such that $X \subset \Sigma \operatorname{Int} K_i$ but $X \cdot \Sigma K_i = 0$; and $\varepsilon > 0$. Then there is a collection of mutually exclusive polyhedral 2-spheres $(K_1', K_2', \cdots, K_n')$ such that*

$$X \subset \Sigma \operatorname{Int} K_i'$$
$$\Sigma K_i' \subset V(\Sigma K_i, \varepsilon) .$$

Proof. Using the approximation theorem for surfaces [4] we suppose with no loss of generality that the 2-spheres $(K_1, K_2, \cdots, K_n)$ are polyhedral. We then let $K_1' = K_1$.

Next we use the preceding theorem to pick K_2' so that

$$X\cdot(\operatorname{Int} K_1 + \operatorname{Int} K_2) \subset X\cdot(\operatorname{Int} K_1' + \operatorname{Int} K_2') ,$$
$$K_2' \text{ is very close to } K_1 + K_2 \text{ and misses } X + K_1 .$$

Repeated applications of Theorem 2.2 gives the collection $(K_1', K_2', \cdots, K_n')$ promised by the theorem.

The following result is an immediate consequence of Theorem 2.3.

THEOREM 2.4. *Suppose X is a closed bounded subset of E^3; $(N_1, N_2, \cdots, N_n)$ is a finite collection of simple neighborhoods covering X and satisfying Property* 1; *and $\varepsilon > 0$. Then there is a collection of mutually exclusive simple neighborhoods $(N_1', N_2', \cdots, N_n')$ covering X such that*

$$\Sigma \operatorname{Bd} N_i' \subset V(\Sigma Bd N_i, \varepsilon)$$

and each Bd N_i' *is a polyhedral* 2-*sphere.*

3. **Separating Cantor sets with small 2-spheres.** We use the results of the preceding section to refine the characterization of a tame Cantor set given by Theorem 1.1.

An example is known [2] of a wild Cantor set C such that if p, q are two points of C, there is a 2-sphere in $E^3 - C$ that separates p from q. However, in this particular example, there is no such 2-sphere of small size. This example along with Theorem 1.1 suggest the following theorem. I am indebted to O. G. Harrold, Jr. for suggesting its proof.

THEOREM 3.1. *A Cantor set C in E^3 is tame if for each positive number ε and each pair of points p, q of C there is an ε 2-sphere in $E^3 - C$ separating p from q.*

Proof. The theorem is established by way of Theorem 1.1 since we show that for each positive number ε there is an ε collection of mutually exclusive simple neighborhoods covering C.

Let U be an open set containing C such that $E^3 - U$ is connected and each component of U is of diameter less than ε.

The hypothesis of the theorem tells us that for each point p of C with the possible exception of one point p_0, there is a 2-sphere in U that misses C and contains p on its interior. The approximation theorem for surfaces [4] reveals that there is a polyhedral cube C_p in U such that $p \in C_p$, and $C\cdot \operatorname{Bd} C_p = 0$.

Let $(C_{p_0}, C_1, C_2, \cdots, C_n)$ be a finite collection of polyhedral cubes in U whose interiors cover C and such that $C\cdot(\operatorname{Bd} C_1 + \operatorname{Bd} C_2 + \cdots + \operatorname{Bd} C_n) = 0$.

It follows from Theorem 2.4 that the collection of C_i's may be replaced by mutually exclusive C_i''s in U so that

$$C\cdot(C_1 + C_2 + \cdots + C_2) \subset C\cdot(C'_1 + C'_2 + \cdots + C'_n)$$

and

$$C\cdot(\operatorname{Bd} C'_1 + \operatorname{Bd} C'_2 + \cdots + \operatorname{Bd} C'_n) = 0 .$$

Letting X be the closed set consisting of the part of C not in $(C'_1 + C'_2 + \cdots + C'_n)$ we find from Theorem 2.2 that there is a polyhedral cube C'_0 in U such that

$$C'_0, C'_1, \cdots, C'_2 \text{ are mutually exclusive,}$$
$$X \subset \operatorname{Int} C'_0 .$$

Then $(C'_0, C'_1, \cdots, C'_2)$ is an ε collection of mutually exclusive polyhedral cubes whose interiors covers C and it follows from Theorem 1.1 that C is tame.

COROLLARY 3.2. *A Cantor set C in E^3 is tame if for each neighborhood N of a point p of C there is a simple neighborhood N' of p such that $N' \subset N$ and* $\operatorname{Bd} N'\cdot C = 0$.

4. Local tameness.

DEFINITION. A Cantor set C is locally tame at a point p of C if there is a neighborhood N of p such that $p\cdot N$ is a tame Cantor set. The following result follows from Corollary 3.2.

THEOREM 4.1. *A Cantor set in E^3 is tame if it is locally tame at each of its points.*

The proof of Theorem 3.1 reveals even more—namely, C is tame if it is locally tame except possibly on a discrete set of points. Theorem 6.1 of a following section shows that a Cantor set is tame if it is locally tame except possibly at the points of a tame Cantor set.

THEOREM 4.2. *Suppose C is a Cantor set in E^3, $p \in C$, N is a neighborhood of p, and C is locally tame at each point of $N\cdot(C - p)$. Then C is locally tame at p.*

THEOREM 4.3. *Each wild Cantor set in E^3 contains a Cantor set that is not locally tame at any point.*

Proof. Let C be a wild Cantor set and C' be the set of all points of C at which C is not locally tame. It follows from Theorem 4.2 that C' contains no isolated point and is therefore a Cantor set. We show that C' is not locally tame at any point.

If C' were locally tame at a point p_0, there would be a neighborhood N' of p_0 such that $N'\cdot C'$ is a tame Cantor set. Let N be a second neighborhood of p_0 such that $N \cdot C$ is a Cantor set and $N \cdot C' = N' \cdot C'$. We show that $N \cdot C$ is tame, thereby contradicting the assumption that $p_0 \in C'$.

We show that $N \cdot C$ is tame by showing that if p, q are two points of $N \cdot C$, then there is an ε 2-sphere in $E^3 - N \cdot C$ separating p from q. We suppose the distance from p to q is more than ε and consider only the more difficult case where $p \in N \cdot C'$.

Let C_0 be a polyhedral cube of diameter less than $\varepsilon/3$ and such that $p \in C_0$, $\operatorname{Bd} C_0 \cdot (N \cdot C') = 0$.

Let $(C_1, C_2, \cdots, C_n)$ be a finite collection of polyhedral $\varepsilon/3$ cubes covering $C \cdot \operatorname{Bd} C_0$ such that $C \cdot (\operatorname{Bd} C_1 + \operatorname{Bd} C_2 + \cdots + \operatorname{Bd} C_n) = 0$ and let X be the part of $C \cdot C_0$ not contained in $\operatorname{Int} C_1 + \operatorname{Int} C_2 + \cdots + \operatorname{Int} C_n$. By using Theorem 2.3 we suppose with no loss of generality that the C_i's are mutually exclusive. It follows from Theorem 2.2 that there is a 2-sphere K in $\operatorname{Ext} C_1 + \operatorname{Ext} C_2 + \cdots + \operatorname{Ext} C_n$ such that $X \subset \operatorname{Int} K$ and K is so close to $\operatorname{Bd} C_0 + \operatorname{Bd} C_1 + \cdots + \operatorname{Bd} C_n$ that it misses C and is of diameter less than ε.

Question. Is there a universally wild Cantor set C in E^3 in the sense that for each Cantor set C' in E^3 there is a homeomorphism h of E^3 onto itself taking C' into C?

5. The fundamental group of the complement. The complement of each tame Cantor set C in E^3 is 1-*ULC*—that is, for each positive number ε there is a positive number δ such that each closed curve of diameter less than δ in $E^3 - C$ can be shrunk to a point on subset of $E^3 - C$ of diameter less than ε. We find that this property of the complement characterized a tame Cantor set. Kirkor has given [10] an example of a wild Cantor set whose complement is simply connected.

THEOREM 5.1. *A Cantor set C in E^3 is tame if and only if $E^3 - C$ is* 1-*ULC*.

Proof. Let p be a point of C and K a small polyhedral 2-sphere whose interior contains p. Let $D_1, D_2, \cdots, D_n$ be a finite number of mutually exclusive disks in K whose interiors cover $K \cdot C$. It follows from that fact that $E^3 - C$ is 1-*ULC* that these D_i's can be replaced by singular disks E_i that miss C and from Dehn's lemma as proved by Papakyriakopoulos in [13] that these singular disks E_i can be replaced by disks F_i so that $(K - \Sigma D_i) + \Sigma F_i$ is a small polyhedral 2-sphere which misses C and has p on its interior. Then Theorem 5.1 follows from Theorem 3.1.

6. The sum of Cantor sets. In this section we prove some theorems that will be used in the next sections about the sums of Cantor sets.

THEOREM 6.1. *Any Cantor set which is the sum of a countable number of tame Cantor sets is tame.*

Proof. Suppose the Cantor set C is the sum of the tame Cantor sets $C_1, C_2, \cdots$. If C is wild, it follows from Theorem 4.3 that C contains a wild Cantor set C' that is not locally tame at any point. The Baire category theorem assures us that such a C' would contain a Cantor set C'' which is open in C' and which lies in one C_i. The C'' is not locally tame at any point since C' is not. This contradicts the fact that C'' is a tame Cantor set since it lies in a tame C_i.

Another proof of Theorem 6.1 also follows from Theorem 1.1 and Theorem 7.2 of the next section.

Examples of Cantor sets that cast dense shadows are known [9] but we include the example described in Theorem 6.2 for completeness.

THEOREM 6.2. *If R is a rectangular solid in E^3, there is a tame Cantor set C in R such that each straight line intersecting two opposite faces of R also intersects C.*

Proof. Let F, F' be two opposite faces of R. First, we construct a tame Cantor set C' in R such that each line intersecting each of F, F' also intersects C'.

1. Let $R = M_1$.

2a. Let P be the plane halfway between F and F'. For convenience we suppose that P is horizontal, F lies above P, and F' lies below. Let X_1, X_2, X_3 be three collections of rectangles in $P \cdot R$ such that the elements of X_i $(i = 1, 2, 3)$ are mutually exclusive and of diameter less than 1/2 while any line intersecting each of F, F', Int R also intersects the interior of an element of $X_1 + X_2 + X_3$.

2b. Push the elements of X_1 up and the elements of X_2 down but so slightly that if X_2', X_2' denotes the collection of adjusted rectangles, any line that intersects each of F, F', Int R also intersects the interior of a rectangle of $X_1' + X_2' + X_3$.

2c. Expand each rectangle in $X_2' + X_2' + X_3$ slightly so as to form a collection of mutually exclusive horizontally based rectangular solids in R of mesh less than 1/2 such that any line that intersects each of F_1, F_2 intersects both the upper and lower face of some one of the solids. Denote the sum of these solids by M_2.

3. Let M_3 be the sum of a collection of horizontally based rectangular solids in M_2 of mesh less than 1/3 such that any line intersecting each of F, F' intersects both the top and bottom face of one of these solids. Continuing in this fashion we obtain $M_4, M_5, \cdots$. The intersection of $M_1, M_2, \cdots$ is the Cantor set C'. It follows from Theorem 1.1 that C' is tame.

For each pair of opposite faces of R there is a tame Cantor set in R such that any line intersecting each of these opposite faces also intersects the Cantor set. Since R has only three pairs of opposite faces and it follows from Theorem 6.1 that the sum of three tame Cantor sets is a tame Cantor set, there is a Cantor set C in R such that any line intersecting two opposite faces of R also intersects C.

THEOREM 6.3. *There is a countable collection of tame Cantor sets in E^3 such that each straight line interval in E^3 intersects one of these Cantor sets.*

Proof. Let $R_1, R_2, \cdots$ be the set of all rectangular solids in E^3 with rational vertices and C_i be a tame Cantor set in R_i such that each line intersecting two opposite faces of R_i also intersects C_i. Then ΣC_i intersects each straight line interval in E^3.

7. Avoiding Cantor sets.

In [6] it is shown that a 2-sphere S in E^3 is tame provided that for each positive number ε there are 2-spheres S', S'' in different complementary domains of S and homeomorphisms h', h'' of S onto S', S'' respectively that move no point by as much as ε. As a first step in this proof, a homeomorphism h of E^3 onto itself is used such that $h(S)$ contains no vertical interval. Theorem 7.3, which is proved by means of tame Cantor sets, assures us that there is such a homeomorphism h. An alternate proof of this result is found in Theorem 10.1 of [6].

The following result is used as a lemma in getting Theorem 7.3. We use ρ to denote the distance function and if f, g are two maps of a metric space X onto a metric space Y, $\rho(f, g)$ is the least upper bound of $\rho(f(x), g(x))$ for all points x of X.

THEOREM 7.1. *Suppose $h_1, h_2, \cdots$ is a sequence of homeomorphisms of E^3 onto itself such that*

1. $\rho(h_{i+1}, h_i) < \varepsilon/2^i$ *and* h_1 *is the identity,*
2. $\rho(h_{i+1}, h_i) < \rho(h_i(x_1), h_i(x_2))/2^i$ *if* $\rho(x_1, x_2) > 1/i$.

Since this paper was written, a proof of Theorem 7.1 by T. Homma has appeared in the Yokohama Mathematics Journal, vol. 7, pp. 191-195.

Then the limit of $h_1, h_2, \cdots$ *is a homeomorphism of* E^3 *onto itself that moves no point by as much as* ε.

Proof. That the limit of $h_1, h_2, \cdots$ is continuous and moves no point by as much as ε follows from Condition 1 of the hypothesis. That no two points go into the same point under this limit follows from Condition 2. The limit is onto because each continuous map of E^3 onto itself that moves no point by more than ε is an onto map. It is a homeomorphism because it is a homeomorphism on each closed bounded set.

THEOREM 7.2. *If* X *is a closed 2 dimensional set in* E^3, ε *is a positive number, and* $C_1, C_2, C_3, \cdots$ *is a sequence of tame Cantor sets, there is a homeomorphism* h *of* E^3 *onto itself that moves no point by as much as* ε *and* $h(X)\cdot\Sigma C_i = 0$.

Proof. The homeomorphism h is the inverse of the limit of a sequence of homeomorphism as considered in Theorem 7.1. The identity homeomorphism is denoted by h_1.

Let $y_1, y_2, \cdots, y_n$ be finite collection of mutually exclusive simple neighborhoods covering C_1 of mesh less than $\varepsilon/2$. Let h_2 be a homeomorphism of E^3 onto itself that is fixed outside Σy_i and which takes each $C_1\cdot y_i$ into $y_i - X$.

We now describe h_3. Suppose δ is a positive number such that $\delta < \rho(X, h_2(C_1))$, $\delta < \varepsilon/4$, and $\delta < \rho(h_2(x_1), h_2(x_2))/4$ if $p(x_1, x_2) > 1/2$. Let $z_1, z_2, \cdots, z_m$ be a finite collection of mutually exclusive simple neighborhoods covering $h_2(C_2)$ of mesh less than δ. Let g be a homeomorphism of E^3 onto itself that is fixed outside Σz_i, g is fixed on each z_i missing X, and $gh_2(C_2)$ misses X. Then $h_3 = gh_2$ is fixed on $h_2(C_1)$ and $h_3(C_1 + C_2)$ misses X.

Similarly we obtain $h_4, h_5, \cdots$ such that the sequence $h_1, h_2, \cdots$ satisfies the hypothesis of Theorem 7.1, $h_{i+1} = h_i$ on $C_1 + C_2 + \cdots + C_{i-1}$, and $h_{i+1}(C_1 + C_2 + \cdots + C_i)$ misses X.

It follows from Theorem 7.1 that the limit h_∞ of $h_1, h_2, \cdots$ is a homeomorphism of E^3 onto itself such that h_∞ moves no point by as much as ε and $X\cdot h(\Sigma C_i) = 0$. Then $h = h_\infty^{-1}$ satisfies the conclusion of Theorem 7.2.

EXAMPLE. It is necessary in the hypothesis of Theorem 7.2 to assume that the Cantor sets are tame. The wild Cantor set C described by Antoine [1] has a complement which is not simply connected. There is a disk D in E^3 such that $C\cdot \text{Bd}\, D = 0$ but $\text{Bd}\, D$ can not be shrunk to a point in $E^3 - C$. If $e = \rho(C, \text{Bd}\, D)$, there is no homeomorphism h of E^3 onto itself that moves no point by as much as ε and such that $h(D)\cdot C = 0$.

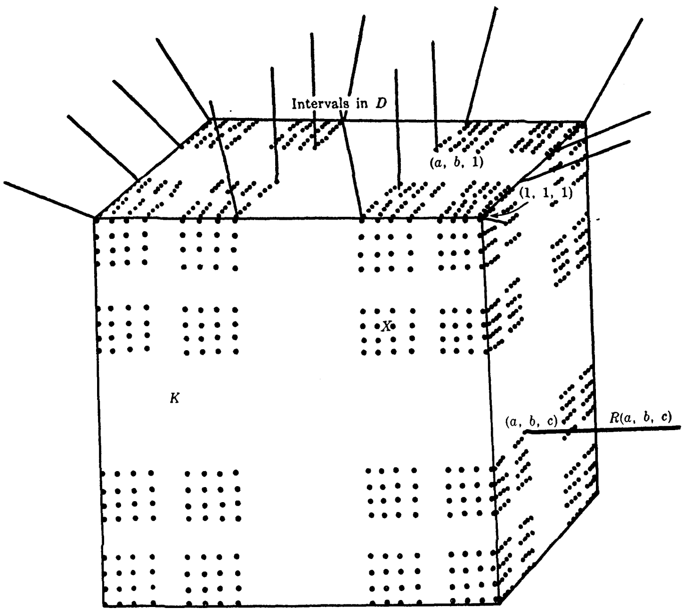
THEOREM 7.3. *If X is a closed 2 dimensional set in E^3 and ε is a positive number, there is a homeomorphism h of E^3 onto itself such that h moves no point by as much as ε and $h(X)$ contains no straight line interval.*

Proof. The result follows from Theorems 6.3 and 7.2.

EXAMPLE. Theorem 7.3 was used in [6] to adjust a 2-sphere S in E^3 so that the adjusted S will contain no vertical interval. One might hope to get such an adjustment by tilting S with a rigid motion. However, we give an example of a tame disk D such that for each direction there is a unit interval in D pointing in that direction.

Let X be the Cantor set obtained by starting with the interval $[-1, 1]$ and removing middle thirds. Let f be the map of $[-1, 1]$ onto $[-1, 1]$ such that $f(-1) = -1$, $f(1) = 1$, $f[-1/3, 1/3] = 0$, $f[-7/9, -5/9] = -1/2$, $f[5/9, 7/9] = 1/2$, $f[-25/27, 23/27] = -3/4, \cdots$. Then f is a function that is continuous and has a derivative equal to 0 except on X.

Let K be the cube with edges parallel to the axes and opposite vertices at $(-1, -1, -1)$, $(1, 1, 1)$. Let X' be the set of all points (x, y, z) on Bd K such that each coordinate of (x, y, z) is a point of X. See Figure 1.



From each point (x, y, z) of Bd K let $R(x, y, z)$ be the ray from the point (x, y, z) in the direction $f(x) : f(y) : f(z)$. We shall show in the next two paragraphs that no two of these rays intersect. For each direction there is one of these rays from a point of X' pointing in that direction. There is an arc A on Bd K that contains X'. A disk D containing a unit interval from each of the rays $R(x, y, z)$ intersecting A contains intervals pointing in all directions.

The ray $R(a_i, b_i, 1)$ has the parametric equation

$$x = a_i + f(a_i)t\ ,\quad y = b_i + f(b_i)t\ ,\quad z = 1 + t \qquad (t \geqq 0)\ .$$

If $a_1 \neq a_2$, $a_1 + f(a_1)t \neq a_2 + f(a_2)t$. Since the same thing is true for b's, $R(a_1, b_1, 1)$ does not intersect $R(a_2, b_2, 1)$ if $(a_1, b_1, 1) \neq (a_2, b_2, 1)$. Hence, no two of the rays starting on the same face of K intersect.

If $c_3 \neq 1$, $R(a_1, b, 1)$ does not intersect $R(a_3, b_3, c_3)$ because if (x, y, z) is a point of $R(a_1, b_1, 1)$

$$z \geqq \max(|x|, |y|)$$

but this inequality does not hold for points of $R(a_3, b_3, c_3)$. Hence, no two of the rays emanating from different faces of K intersect.

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