IMBEDDING COMPACT RIEMANN SURFACES IN 3-SPACE

TILLA KLOTZ

1. Any sufficiently smooth surface in E^3 has a conformal structure imposed upon it by the metric of the surrounding space. If there is a conformal homeomorphism between a Riemann surface and some C^k surface in E^3, then the Riemann surface is said to be C^k imbedded in E^3. We deal below with some aspects of the problem of C^∞ imbedding compact Riemann surfaces in E^3.

Since every compact Riemann surface of genus zero is conformally equivalent to the sphere, the problem becomes non-trivial only when genus $g \geqq 1$. Recently Garsia and Rodemich [4] proved that every compact Riemann surface of genus 1 can be C^∞ imbedded in E^3. We therefore restrict our attention compact Riemann surfaces of genus $g \geqq 2$.

2. Before stating the main result, we recall some definitions. For each fixed genus $g \geqq 2$, choose a fixed compact Riemann surface R_g of genus g. Then a marked Riemann surface of genus g is an equivalence class

$$\mathscr{S} = \langle (R, \alpha) \rangle$$

of pairs, where R is a compact Riemann surface of genus g, and α is a homotopy class of orientation preserving topological mappings of R_g onto R. The equivalence

$$(R, \alpha) \sim (R', \alpha')$$

holds if and only if R and R' are conformally equivalent under a homeomorphism in the homotopy class $\alpha^{-1}\alpha'$. A marked Riemann surface is said to be C^k imbedded in E^3 if the first member of some representative pair is C^k imbedded in E^3.

It is well known (see, for example, [1]) that the set of all marked Riemann surfaces of genus g may be made into a metric space in a natural manner, thereby becoming the Teichmüller space T_g. We define $\Sigma_g \subset T_g$ to be the set of all $\mathscr{S} \in T_g$ which can be C^∞ imbedded in E^3. Note that Σ_g is never empty.

But then, the conjecture that every compact Riemann surface of genus $g \geqq 2$ is C^∞ imbeddable in E^3 is equivalent to the conjecture that Σ_g is both open and closed in T_g.[1] In what follows we deal exclusively

Received June 15, 1960. Part of this research was completed while the author was a National Science Foundation Fellow.

[1] It is in this form that the problem was suggested to the author by Professor Lipman Bers, to whom we express our gratitude.

with the problem of showing Σ_g to be open in T_g. But we succeed in proving, basically, only the following.

THEOREM. $(\Sigma_g - \Sigma_g^0)$ *is open in* T_g.

The set $\Sigma_g^0 \subset \Sigma_g$ is, fortunately, both small and interesting. But its definition is most conveniently stated toward the end of the proof.[2]

3. We recall some facts before proceeding to prove the theorem (see [2]). Let

$$m = \mu(z, \bar{z})\frac{d\bar{z}}{dz}$$

be a Beltrami differential on R, and, thereby, on $\mathscr{S} = \langle(R, \alpha)\rangle$. Consider the new Riemann surface R^m defined when we take as new conformal parameters homeomorphic solutions to the Beltrami equations

$$w_{\bar{z}} = \mu w_z$$

on R. It is usual to call

$$\mathscr{S}^m = \langle(R^m, \alpha)\rangle .$$

Then m is trivial, written $m \equiv 0$, if and only if

$$|\mathscr{S} - \mathscr{S}^{\varepsilon m}| = o(\varepsilon) ,$$

instead of the usual $O(\varepsilon)$.

It is an important fact that $m \equiv 0$ if and only if for every holomorphic quadratic differential $\Omega = fdz^2$ on $\mathscr{S}$,

$$(\Omega, m) = \iint_{\mathscr{S}} f\mu dxdy = 0 . \tag{1}$$

For W, the space of holomorphic quadratic differentials on $\mathscr{S}$, is a complex, linear space of dimension $3g - 3$. It follows that $B_{\mathscr{S}}$, the space of Beltrami differentials modulo trivial Beltrami differentials on $\mathscr{S}$, is a real linear space of dimension $6g - 6$.

Let $\tau = 3g - 3$. Bers has shown that T_g can be made into a C^ω manifold by coordinate mappings

$$t = (t_1, \cdots, t_{2\tau}) \rightarrow \mathscr{S}^{t_1m_1+\cdots+t_{2\tau}m_{2\tau}}$$

for each $\mathscr{S} \in T_g$ and each basis $m_1, \cdots, m_{2\tau}$ in $B_{\mathscr{S}}$. It is a trivial consequence of Bers' work (see § 6 in [2]) that if now

$$m = t_1m_1 + \cdots + t_{2\tau}m_{2\tau} + \cdots$$

where the $m_1, \cdots, m_{2\tau}$ form a basis in $B_{\mathscr{S}}$, then the mapping

[2] See the last paragraphs of sections 4 and 5.

$$t \to \mathscr{S}^m \tag{2}$$

has non-zero Jacobian at $t = 0$ and gives a mapping of some 2τ dimensional ball in $E^{2\tau}$ onto a neighborhood of $\mathscr{S}$ in T_g.

4. Now to the proof of the theorem. Let $\mathscr{S} \in \Sigma_g$ be C^∞ imbedded in E^3 as the surface S. Let δ be a fixed, arbitrarily small patch on S described in terms of local isothermal coordinates x, y on S by $z = |x + iy| < 2$. We seek to describe a family of C^∞ surfaces $S(t_1, \cdots, t_{2\tau})$ in E^3 which coincide with S except on δ, and which yield imbeddings of all marked Riemann surfaces in some neighborhood of $\mathscr{S}$ in T_g. To this end, if the vector ξ describes S in E^3, let $S(t_1, \cdots, t_{2\tau})$ be a described by the vector

$$\xi(t_1, \cdots, t_{2\tau}) = \xi + \sum_{j=1}^{2\tau} t_j \varphi^j + \cdots$$

where all (vector valued) coefficients in the power series are $C^\infty(x, y)$ and vanish identically outside $|z| < 1$, and thereby outside of δ on S.

It follows that the coefficients of the first fundamental form on $S(t_1, \cdots, t_{2\tau})$ are given by

$$g_{11} = \lambda + 2\sum_{j=1}^{2\tau} (\xi_x \cdot \varphi_x^j) t_j + \cdots$$

$$g_{22} = \lambda + 2\sum_{j=1}^{2\tau} (\xi_y \cdot \varphi_y^j) t_j + \cdots$$

$$g_{12} = \sum_{j=1}^{2\tau} (\xi_x \cdot \varphi_y^j + \xi_y \cdot \varphi_x^j) t_j + \cdots .$$

But it is well known that $S(t_1, \cdots, t_{2\tau})$ is conformally equivalent to $S^{m(t_1, \cdots, t_{2\tau})}$ where

$$m(t_1, \cdots, t_{2\tau}) = \mu(t_1, \cdots, t_{2\tau}) \frac{d\bar{z}}{dz} \tag{4}$$

and

$$\mu(t_1, \cdots, t_{2\tau}) = \frac{\frac{1}{2}(g_{11} - g_{22}) + ig_{12}}{\frac{1}{2}(g_{11} + g_{22}) + \sqrt{g_{11}g_{22} - g_{12}^2}} ,$$

i.e. $\mu(t_1, \cdots, t_{2\tau}) \equiv 0$ outside $|z| < 1$. This means that $S(t_1, \cdots, t_{2\tau})$ yields a C^∞ imbedding in E^3 of $\mathscr{S}^{m(t_1, \cdots, t_{2\tau})}$.

Computations show that

$$\left.\frac{\partial \mu(t_1, \cdots, t_{2\tau})}{\partial t_j}\right|_{t=0} d\bar{z}/dz = \frac{2\xi_{\bar{z}} \cdot \varphi_{\bar{z}}^j}{\lambda} d\bar{z}/dz \overset{\text{def}}{=} \mu_j d\bar{z}/dz \overset{\text{def}}{=} m_j . \tag{5}$$

Each m_j is determined therefore by the choice of the vector φ^j. We seek to choose the 2τ vectors φ^j so that the corresponding $m_1, \cdots, m_{2\tau}$

form a basis in $B_{\mathscr{S}}$. Then, by the remark closing § 3, the mapping (2) with m given by (4) is onto a neighborhood of $\mathscr{S}$ in T_g, thereby showing *$\mathscr{S}$ to be an interior point of Σ_g.*

Note that the $\{m_j\}$ form a real basis in $B_{\mathscr{S}}$ if there is a basis $\{\Omega_k\}$ in W for which the following matrix of scalar products is obtained.

$$\begin{array}{c|cccc|cccc|} & m_1 & \cdots & & m_\tau & m_{\tau+1} & \cdots & & m_{2\tau} \\ \hline \Omega_1 & 1 & & & & i & & & \\ \vdots & & \ddots & 0 & & & \ddots & 0 & \\ & 0 & & & & & & & \\ \Omega_\tau & & & & 1 & & & & i \\ \hline \end{array} \tag{6}$$

Using (1), (5), integration by parts, the vanishing of φ^j outside $|z| < 1$, and the analyticity f_k, we obtain

$$(m_j, \Omega_k) = -2\iint_{|z|<1}\left(\frac{\xi_{\bar z}}{\lambda}\right)_{\bar z} f_k \cdot \varphi^j dxdy \ .$$

If, for convenience, we set

$$\psi_k = -2\left(\frac{\xi_{\bar z}}{\lambda}\right)_{\bar z} f_k \tag{7}$$

then

$$(m_j, \Omega_k) = \iint_{|z|<1} \psi_k \cdot \varphi^j dxdy \ . \tag{8}$$

Now, choose an arbitrary basis $\{\Omega_k\}$ in W. We will try to choose the φ_j so as to obtain the matrix (6) of scalar products, altering as necessary *our choice of the basis $\{\Omega_k\}$ in W. Recall that each φ^j must* be real, $C^\infty(x, y)$, and vanish outside of $|z| < 1$.

First take φ^1 such that $(m_1, \Omega_1) \neq 0$. If there is no such φ^1 then, by (8),

$$\iint_{|z|<1} \psi_1 \cdot \varphi dxdy = 0$$

for all appropriate choices of φ, i.e. $\psi_1 \equiv 0$ in $|z| < 1$ so that, by (7),

$$\left(\frac{\xi_{\bar z}}{\lambda}\right)_{\bar z} \equiv 0 \tag{9}$$

in $|z| < 1$. We therefore make an assumption which will be justified in § 5.

ASSUMPTION 1. (9) does not hold in $|z| < 1$.

Now φ^1 may be chosen as specified. If $(m_1, \Omega_k) = \gamma_k$, take as new basis in W Ω_1/γ_1, $\Omega_2 - (\gamma_2\Omega_1/\gamma_1), \cdots, \Omega_\tau - (\gamma_\tau\Omega_1/\gamma_1)$ and call this the basis $\{\Omega_k\}$. Thus we get the first column of (6). But now take φ^2 such that $(m_2, \Omega_2) \neq 0$ and call $(m_2, \Omega_k) = \gamma_k$. If we take as new basis in W $\Omega_1 - (\gamma_1\Omega_2/\gamma_2)$, $\Omega_2/\gamma_2, \cdots, \Omega_\tau - (\gamma_\tau\Omega_2/\gamma_2)$, calling this the basis $\{\Omega_k\}$ we obtain the first two, and, by the corresponding procedure, the first τ columns of (6).

Next, choose a φ for which $(m, \Omega_k) = \alpha_k + i\beta_k$ with $\beta_1 \neq 0$. If this is impossible then

$$\Im\mathrm{m}\iint_{|z|<1} \psi_1 \cdot \varphi dxdy = 0$$

for all appropiate choices of φ, i.e., ψ_1 is real in $|z| < 1$, and, by (7), would be real under any change of local parameter in $|z| < 1$. In order not to have this difficulty here or further on, we make a second assumption. In § 5 this assumption is weakened but it is never fully eliminated.

ASSUMPTION 2. No expression ψ of the form

$$\psi = \left(\frac{\xi_{\bar{z}}}{\lambda}\right)_{\bar{z}} f \tag{10}$$

with $fdz^2 \in W$ is real throughout $|z| < 1$. Note that if the assumption were violated by two expressions of the form (10) then each expression would be a real scalar multiple of the other.

But now, φ may be chosen as specified. We can therefore set $\varphi^{\tau+1} = (\varphi - \alpha_1\varphi^1)/\beta_1$, and obtain the $(\tau + 1)^{\text{st}}$ column of (6). But under assumption 2 there is a φ for which $(m, \Omega_k) = \alpha_k + i\beta_k$ with $\beta_2 \neq 0$. By subtracting a suitable real multiple of φ^2 from $\varphi^{\tau+1}$ (so as to get a new equally acceptable $\varphi^{\tau+1}$) and a suitable real multiple of φ^1 from φ (so as to get a new equally acceptable φ) the following scalar products can be attained.

$$\begin{array}{c|cc|} & \varphi_{\tau+1} & \varphi \\ \hline \Omega_1 & i & i\beta_1 \\ \Omega_2 & i\gamma & i \\ \hline \end{array} \tag{11}$$

If now $\beta_1 \neq 1/\gamma$, take $\varphi^{\tau+2} = (\varphi - \beta_1\varphi^{\tau+1})/(1 - \beta_1\gamma)$, so as to obtain the $(\tau + 2)^{\text{nd}}$ column of (6).

But suppose there were no appropriate φ for which (11) holds with $\beta_1 \neq 1/\gamma$. Then $\psi = (\psi_2 - \gamma\psi_1)$ would be real in $|z| < 1$, since, for every appropriate choice of φ we would have

$$\Im\mathrm{m}\iint_{|z|<1} \psi \cdot \varphi dxdy = 0 .$$

But ψ would have the form (10), and assumption 2 outlaws exactly this situation. The procedure for obtaining the rest of (6) is clear.

We can now complete the proof of the theorem by defining Σ_g^0 to be the set of all $\mathscr{S} \in \Sigma_g$ which are C^∞ imbeddable in E^3 *only* as surfaces which violate assumption 1 or 2 in every coordinate patch. A less artificial definition of Σ_g^0 is given at the end of § 5.

5. The preceding considerations can be clarified by a study of the assmptions 1 and 2. First note that the Gauss equations yield

$$\left(\frac{\xi_{\bar{z}}}{\lambda}\right)_{\bar{z}} = \lambda(L - N + 2iM)\xi^3 = 2\lambda\bar{\phi}\xi^3 \tag{12}$$

where ξ^3 is the unit normal vector to S, L, N and M the coefficients of the second fundamental form, and

$$\phi = \frac{L - N}{2} - iM\,.$$

But then the violation of assumption 1 means that $|z| < 1$ is a spherical piece. In short, assumption 1 is always valid so long as δ is chosen to be, as is always possible, a non-spherical patch on S. As a second alternative, when assumption 1 is violated, $\mathscr{S}$ can be reimbedded in the following manner. Replace (say) $|z| < 1/2$ on δ by a conformally equivalent piece of a surface of revolution in a C^∞ manner. Note that all points with $|z| = 1/2$ are fixed under the conformal correspondence.

We need only worry therefore about assumption 2. By (10) and (12) if the second assumption is violated then there is an $\Omega = fdz^2 \in W$, such that

$$\mathfrak{Im}(\bar{\phi}f) = 0 \tag{13}$$

in $|z| < 1$. Moreover, if $\mathfrak{Im}(\bar{\phi}g) = 0$ in $|z| < 1$ for $\hat{\Omega} = gdz^2 \in W$, then $\hat{\Omega} = \alpha\Omega$ with α real. It is easy to show that if there is a patch on S in which (13) does not hold, then there is a patch δ' on S for which no expression of the form (10) can be real throughout $|z'| < 1$. Simply, chose for δ' a patch in which (13) holds on only part of $|z'| < 1$.

Assumption 2 can always be justified therefore unless (13) holds eveywhere on S. But even then we can reimbed S so as to satisfy assumption 2 in some patch so long as S has a spherical portion. For in this case we can again replace some spherical δ on S by a conformally equivalent piece of a surface of revolution. On the new piece there is an isolated umbilic with index $j = 1$ at (say) $z = 0$.

But (see chapter 6 of [5]) j can be computed by setting

$$j = \frac{-1}{4\pi}\Delta \arg \phi \tag{14}$$

where the change in argument is taken going once about $|z| = \varepsilon$ in the positive sense. If (13) still held on the new piece, we would have

$$j = \frac{-1}{4\pi} \Delta \arg f \tag{15}$$

so that, by $j = 1$, f would have a pole at $z = 0$. From this contradiction it follows that assumption 2 causes no trouble on the reimbedded surface.

We call S a *critical surface* if it is compact, has no spherical patches and if there is an $\Omega \in W$ on S for which (13) holds everywhere. It is now possible to give a slightly more reasonable definition of Σ_g^0.

DEFINITION. Σ_g^0 is the set of all $\mathscr{S} \in \Sigma_g$ which can be C^∞ imbedded in E^3 only as critical surfaces.

6. Before studying critical surfaces, we note that the arguments of § 4 do yield some information even when $\mathscr{S} \in \Sigma_g^0$. For, if assumption 2 is violated, only a slight alteration of procedure shows that the φ^j may be chosen so as to determine the matrix

$$\begin{array}{c|ccc|c|ccc|} & m_1 & \cdots & m_\tau & & m_{\tau+1} & \cdots & m_{2\tau} \\ \hline \Omega_1 & 1 & & & 0 & \cdots & & 0 \\ \vdots & & \ddots & 0 & \vdots & i & & \\ & & & & & & \ddots & 0 \\ & 0 & & & & & & \\ \Omega_\tau & & & 1 & 0 & & & i \\ \hline \end{array}$$

of scalar products.

But then, every $\mathscr{S}' \in T_g$ in some neighborhood of $\mathscr{S}$ is describable in the form

$$\mathscr{S}' = \mathscr{S}^{m(t_1, \cdots, t_{2\tau}) + it_{\tau+1} m_1},$$

since $m_1, \cdots, m_\tau, im_1, m_{\tau+2}, \cdots, m_{2\tau}$ form a basis in $B_{\mathscr{S}}$. This means however that the mapping

$$t \longrightarrow \mathscr{S}^{m(t_1, \cdots, t_{2\tau})}$$

is onto a $6g - 7$ dimensional sub-neighborhood of $\mathscr{S}$ in T_g. We can therefore make the following remark.

REMARK. If $\mathscr{S} \in \Sigma_g^0$, then every $\mathscr{S}'$ in some $6g - 7$ dimensional subneighborhood of $\mathscr{S}$ in T_g is in Σ_g.

7. Our study of critical surfaces has two well defined goals. First we want to determine "how many" critical surfaces there are if any. Next, we ask whether critical surfaces can in general be reimbedded as non-critical surfaces, in which case Σ_g^0 would be empty, and Σ_g open. The discussion which follows is at best a first step in these directions.

To begin with, consider the net of curves formed on S by the curves along which $\Omega > 0$ and $\Omega < 0$ respectively. These curves are usually called the trajectories and orthogonal trajectories respectively of Ω on S. For convenience, we refer to the net they form as the Ω-net on S.

It follows from (13) that the Ω-net is a net of lines of curvature in the neighborhood of any point where $\Omega \neq 0$. Moreover, since Ω has $4g-4$ zeroes (counted with multiplicities), each zero of Ω corresponds to an isolated singularity in this Ω-net of lines of curvature on S. But then (13) and (15) imply that any n-fold zero of Ω is an umbilic point on S with index $j = (-n)/2$ in the Ω-set of lines of curvature.

A critical surface can therefore be described as a compact surface with no spherical portions on which there is an Ω-net of lines of curvature. As a consequence, every critical surface has a net of lines of curvature with $\leqq 4g-4$ singularities, each with negative index. Note that there may be umbilic points even where $\Omega \neq 0$, so that a critical surface need not have a finite number of umbilic points.[3]

We can offer as yet no example of a critical surface of genus $g \geqq 2$. The torus of revolution is an example for $g = 1$ and none can exist for $g = 0$. But it is worth noting that if there were a compact surface of constant mean curvature of genus $g \geqq 1$, which A. D. Alexandrov has shown (see chapter 7 of [5]) to be impossible, it would be critical, with $\Omega = \phi dz^2$. Moreover, the surface obtained by reflecting such a surface in a sphere would be critical and of non-constant mean curvature. In general, a critical surface differs from a surface of constant mean curvature only in that ϕdz^2 must be multiplied by a real valued expression before becoming an element of W.

Finally, note that critical surfaces go into critical surfaces under conformal mappings of E^3 onto itself. Thus the first trivial approach to the reimbedding of critical surfaces as non-critical surfaces fails. It remains to be seen whether on a critical surface one may replace a patch by a conformally equivalent patch so as to get a non-critical surface. Note that all points on the boundary of the patch are required to be fixed under the conformal correspondence.

8. Some closing comments are in order. First, imbeddings of all $\mathscr{S}' \in T_g$ can not be attained by our method of deforming S in one or even several patches. This follows from result of Oikawa [6] on the boundedness in T_g of the set of surfaces obtained in this manner.

But, imbeddings of all nearby $\mathscr{S}' \in T_g$ can probably still be attained

[3] By an umbilic point we mean a point where $\phi = 0$. Please note that in the preliminary abstract of this report, on p. 193 of the April, 1960 issue of A. M. S. *Notices*, the term was used differently, to denote a singularity in the Ω-net.

by this method. For note that our procedure was very crude. We concluded that mappings $t \rightarrow \mathscr{S}^{m(t)}$ were onto a neighborhood of $\mathscr{S}$ in T_g only when we could show that their Jacobians were non-zero at $t = 0$. Needless to say, such mappings may still be onto a neighborhood of $\mathscr{S}$ in T_g even when their Jacobians vanish at $t = 0$.

References

1. L. V. Ahlfors, *On quasi-conformal mappings*, J. Anal. Math., **3** (1953-4), 1-58.
2. Lipman Bers, *Spaces of Riemann surfaces*, Proc. Int. Cong. Math., Edinburgh, (1958), 349-361.
3. A. M. Garsia, *The calculation of conformal parameters for some imbedded Riemann surfaces*, Pacific J. Math., **10** (1960), 121-67.
4. A. M. Garsia, and E. Rodemich, *An imbedding of Riemann surfaces of Genus one*, Pacific J. Math., **11** (1961), 193-204.
5. Heinz Hopf, *Lectures on Differential Geometry in the Large* (mimeographed notes), Stanford University (1956).
6. Kotaro Oikawa, *On the prolongation of an open Riemann surface of finite genus*, Kodai Math. Sem. Reports, **9** (1957), 34-41.