MOMENT INEQUALITIES OF PÓLYA
FREQUENCY FUNCTIONS

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1. Introduction and summary. The theory of totally positive kernels and Pólya frequency functions has been applied fruitfully in several domains of mathematics and statistics. In this paper we derive moment inequalities governing Pólya frequency functions of various orders. In particular, Pólya frequency functions on the positive axis are characterized in terms of inequalities on the moments.

We begin by introducing the necessary definitions and notation, and we also review some of the fundamental background. A function $f(x, y)$ of two real variables, x ranging over X and y ranging over Y, is said to be totally positive of order k (TP_k) if for all $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$ ($x_i \in X$, $y_j \in Y$), and all $1 \leq m \leq k$,

$$(1) \qquad f\begin{pmatrix} x_1, x_2, \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix} = \begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) & \cdots & f(x_1, y_m) \\ f(x_2, y_1) & f(x_2, y_2) & \cdots & f(x_2, y_m) \\ \vdots & \vdots & & \vdots \\ f(x_m, y_1) & f(x_m, y_2) & \cdots & f(x_m, y_m) \end{vmatrix} \geqq 0$$

Typically, X is an interval of the real line, or a countable set of discrete values on the real line such as the set of all integers or the set of nonnegative integers; similarly for Y. When X or Y is a set of integers, we use the term "sequence" rather than "function".

We record for later reference the following consequence of (1) proved in [3], p. 284.

If $f(x, y)$ is TP_k where X and Y represent open intervals on the line and all the indicated derivatives exist, then

$$(2) \qquad \begin{vmatrix} f(x, y) & \dfrac{\partial}{\partial y} f(x, y) & \cdots & \dfrac{\partial^{n-1}}{\partial y^{n-1}} f(x, y) \\ \dfrac{\partial}{\partial x} f(x, y) & \dfrac{\partial^2}{\partial x \partial y} f(x, y) & \cdots & \dfrac{\partial^n}{\partial x \partial y^{n-1}} f(x, y) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^{n-1}}{\partial x^{n-1}} f(x, y) & \dfrac{\partial^n}{\partial x^{n-1} \partial y} f(x, y) & \cdots & \dfrac{\partial^{2n-2}}{\partial x^{n-1} \partial y^{n-1}} f(x, y) \end{vmatrix} \geqq 0$$

for all $n \leqq k$, $x \in X$ and $y \in Y$.

A related concept to total positivity is that of sign reverse regularity.

A function $f(x, y)$ is *sign reverse regular of order* k (SRR_k), if for every $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$, and $1 \leqq m \leqq k$,

$$(-1)^{[m(m-1)/2]} f \begin{pmatrix} x_1, & x_2, & \cdots, & x_m \\ y_1, & y_2, & \cdots, & y_m \end{pmatrix} \geqq 0 \ .$$

In this case (2) holds with the factor $(-1)^{[m(m-1)/2]}$ multiplying the determinant.

If a TP_k function $f(x, y)$ is a probability density in one of the variables, say x, with respect to a σ-finite measure $\mu(x)$, for each fixed value of y, and if $f(x, y)$ is expressible as a function $f_0(x - y)$ of the difference of x and y, then f_0 is said to be a *Pólya frequency function* (density) *of order* k (PF_k). The argument traverses the real line. If the argument is confined to the integers we shall speak of a *Pólya frequency sequence of order* k $(PF_k$ sequence).

When the subscript ∞ is attached to any of the above definitions, then the property in question will be understood to hold for all positive integers.

Many of the structural properties derived for TP_k functions are based upon the following identity (see [6], p. 48, prob. 68).

LEMMA 1. *If* $r(x, w) = \int p(x, t)q(t, w)d\sigma(t)$, *then*

$$r \begin{pmatrix} x_1, & x_2, & \cdots, & x_k \\ w_1, & w_2, & \cdots, & w_k \end{pmatrix} = \iint \cdots \int_{t_1 < t_2 < \cdots < t_k} p \begin{pmatrix} x_1, & x_2, & \cdots, & x_k \\ t_1, & t_2, & \cdots, & t_k \end{pmatrix} q \begin{pmatrix} t_1, & t_2, & \cdots, & t_k \\ w_1, & w_2, & \cdots, & w_k \end{pmatrix}$$
$$\times \ d\sigma(t_1)d\sigma(t_2)\cdots d\sigma(t_k) \ .$$

An important feature of totally positive functions is their variation diminishing property. Let $V(g)$ denote the number of variations of sign of a function $g(x)$ as x traverses the real line from left to right.

If $h(x)$ is given by the absolutely convergent integral

$$(3) \qquad\qquad h(x) = \int f(x, w)g(w)d\sigma(w)$$

where $f(x, w)$ is TP_k, $\sigma(w)$ is a nonnegative σ-finite measure and $V(g) \leqq k - 1$, then

$$(4) \qquad\qquad V(h) \leqq V(g) \ .$$

Moreover, if actually $V(h) = V(g)$ then h and g have the same arrangement of signs, i.e., the first sign of each traversing the line from left to right, either both positive or both negative. When (2) prevails with strict inequality a stronger version of (4) is valid. In fact, suppose $h(x)$ is k times continuously differentiable and that interchange of integral and derivative is permissible in (3). Let $Z(h)$ denote the number of zeros

of $h(x)$, counting multiplicities.   Under the conditions of (3) we have

(5) $$Z(h) \leq V(g) .$$

An important special case is $f(x, y) = e^{xy}$ which is readily shown to be TP_∞ satisfying (2), with strict inequality for all n, x and y.

The same holds for the function $f(x, y) = \exp[\varphi(x)\psi(y)]$ where φ and ψ represent strictly monotone infinitely differentiable increasing functions.   In particular, $f(x, y) = x^y$ $(x > 0, y > 0)$ is TP_∞ satisfying (2) with strict inequalities.

Throughout the paper, we shall use μ_s to represent the sth moment of the density $f(t)$; i.e.

$$\mu_s = \int \xi^s f(\xi) d\xi \qquad\qquad \text{for } s > -1 .$$

The expression $r_s = \mu_s/\Gamma(s + 1)$ will be referred to as the normalized moment.

In § 2 we show that if f is PF_k density of a nonnegative random variable with "normalized" moments $r_s = \mu_s/\Gamma(s + 1)$, $s > -1$, then r_{s+t} is SRR_k for $s > -1/2$, $t > -1/2$.   We also prove that if f is a density of a nonnegative random variable with normalized moments, r_s, then $\{r_{s-t}\}_{s,t=0,1,2,\cdots}$ is TP_∞ is equivalent to f a PF_∞ function.   Similar results are obtained for Pólya frequency sequences p_j, $j = 0, 1, 2, \cdots$, where we use binomial moments $B_k = \Sigma p_j \binom{j}{k}$ in place of normalized moments.

In § 3 we develop several results describing the rate of decrease of Pólya frequency functions.   For example, it is proved that for $f = 0$ when $x < 0$ and otherwise f a PF_2 and $\not\equiv (1/\mu_1)e^{-t/\mu_1}$, there exists t_0 such that for $t > t_0$

$$0 \leq f(t) < \frac{1}{\mu_1}e^{-t/\mu_1} .$$

Also inequalities on $f(0)$ and on $f'(0)$ are given.   The key device in these analyses is to compare the PF_2 density at hand with an appropriately selected exponential density, exploiting the general variation diminishing properties of Pólya frequency functions.

2. Total positivity of moments. In this section we shall obtain some interesting relationships between the property of total positivity for the density f of a nonnegative random variable and a corresponding property for its normalized moments.

THEOREM 1. *Let f be PF_k of a nonnegative random variable with normalized moments r_s, $s > -1$.   Then r_{s+t} is strictly SRR_k in $s > -1/2$, $t > -1/2$.*

Proof. Write

$$r_{s+t} = \int_0^\infty \frac{\xi^{s+t}}{\Gamma(s+t+1)} f(\xi) d\xi$$

$$= \int_0^\infty \left(\int_0^\xi \frac{x^{s-1/2}}{\Gamma(s+\frac{1}{2})} \frac{(\xi-x)^{t-1/2}}{\Gamma(t+\frac{1}{2})} dx \right) f(\xi) d\xi .$$

The change of variable $y = \xi - x$, yields

$$r_{s+t} = \int_0^\infty \frac{x^{s-1/2}}{\Gamma(s+\frac{1}{2})} \int_0^\infty \frac{y^{t-1/2}}{\Gamma(t+\frac{1}{2})} f(y+x) dx dy$$

$$= \int_0^\infty \frac{x^{s-1/2}}{\Gamma(s+\frac{1}{2})} L(x,t) dx , \quad L(x,t) = \int_0^\infty \frac{y^{t-1/2}}{\Gamma(t+\frac{1}{2})} f(y+x) dy .$$

Since f is PF_k, we know that $f(y+x)$ is SRR_k for $y > 0$ and $x > 0$; also $[y^{t-1/2}/\Gamma(t+1/2)]$ is strictly TP_∞ in $y \geq 0$ and $t > -1/2$ as noted in § 1. Hence by Lemma 1, $L(x,t)$ is strictly SRR_k for $t > -1/2$ and $x > 0$. Since $[x^{s-1/2}/\Gamma(s+1/2)]$ is strictly TP_∞ in $x \geq 0$ and $s > -1/2$, then by another application of Lemma 1, we conclude that r_{s+t} is strictly SRR_k in $s > -1/2$, $t > -1/2$.

A similar result holds for frequency functions on the nonnegative integers using the binomial moments:

THEOREM 2. *Let* $\{p_j\}_{j=0,1,2,\cdots}$ *be a* PF_k *sequence, with binomial moments*

$$B_m = \Sigma p_j \binom{j}{m} .$$

Then B_{m+n} *is strictly* SRR_k *in* $m = 0, 1, 2, \cdots$ *and* $n = 0, 1, 2, \cdots$.

Proof. The proof follows the lines of that of Theorem 1. Write

$$B_{m+n} = \sum_j p_j \binom{j}{m+n} = \sum_j \sum_h p_j \binom{h-1}{n-1} \binom{j-h}{m}$$

where the last equality follows by virtue of the identity [2], p. 71,

$$\binom{z+1}{n+m+1} = \sum_x \binom{x}{n} \binom{z-x}{m} .$$

Employing the change of variable $i = j - h$, we may write

$$B_{m+n} = \sum_i \sum_h p_{i+h} \binom{h-1}{n-1} \binom{i}{m} .$$

Since p_{i+h} is SRR_k, $\binom{h-1}{n-1}$ is TP_∞ (see [4]) and $\binom{i}{m}$ is TP_∞, the conclusion follows.

It is of interest to investigate the converse statement of Theorems

1 and 2. The following theorem asserts that total positivity of order infinity of the normalized moments associated with a density function f of a nonnegative random variable implies that f is PF_∞. The implications when we merely know that r_{s-t} is TP_k $(k < \infty)$ remain unresolved.

THEOREM 3. *Let f be a probability density of a nonnegative random variable having normalized moments r_s, $s = 0, 1, 2, \cdots$. Then f a PF_∞ density is equivalent to $\{r_{s-t}\}_{s,t=0,1,\cdots}$ a TP_∞ sequence.*

Proof. Assume that $\{r_{s-t}\}_{s,t=0,1,\cdots}$ is a TP_∞ sequence with generating function $G(z) = \sum_{s=0}^{\infty} r_s z^s$. Then

$$G(z) = \sum_{s=0}^{\infty} r_s z^s = \sum_{s=0}^{\infty} \int_0^\infty \frac{(tz)^s}{s!} f(t) dt = \int_0^\infty e^{tz} f(t) dt = f^*(z) \, ,$$

where $f^*(z)$ represents the Laplace transform of $f(t)$. Interchange of integral and summation sign is justified since $\sum_{s=0}^{\infty} r_s z^s$ converges uniformly inside a complex neighborhood $|z| \le r$ for some $r > 0$, [1], p. 305 and the fact that $f(t)$ is nonnegative. Now from [1], p. 305, we know that $G(z)$ is of the form:

$$(6) \qquad G(z) = e^{\gamma z} \frac{\prod_{\nu=1}^{\infty} (1 + \alpha_\nu z)}{\prod_{\nu=1}^{\infty} (1 - \beta_\nu z)}$$

$$(\gamma \ge 0, \ \alpha_\nu \ge 0, \ \beta_\nu \ge 0, \ \Sigma \alpha_\nu, \ \Sigma \beta_\nu \text{ convergent}).$$

Hence $f^*(z)$ is of the form (6). But if any $\alpha_\nu > 0$, then $f^*(z)$ would vanish for $z = -1/\alpha_\nu$, which is impossible. Thus $f^*(z)$ is of the form

$$(7) \qquad f^*(z) = \frac{e^{\gamma z}}{\prod_{\nu=1}^{\infty} (1 - \beta_\nu z)}$$

$$(\gamma \ge 0, \ \beta_\nu \ge 0, \ \Sigma \beta_\nu \text{ convergent}).$$

By the Schoenberg representation theorem [7], p. 333, f is PF_∞.

Next assume f is a PF_∞ density. Then $f^*(z)$ is of the form (7) for $|z| < r$, $r > 0$. As above, we conclude that $G(z) = f^*(z)$ so that $G(z)$ is of the form (6). Thus $G(z)$ is the generating function of a TP_∞ sequence.

In a similar fashion we may establish the equivalence of PF_∞ for $\{p_j\}_{j=0,1,2,\cdots}$ and TP_∞ for $\{B_{k-l}\}_{k,l=0,1,2,\cdots}$, where $\{p_j\}$ is a probability frequency function on the nonnegative integers with binomial moments $\{B_k\}$. Again the equivalence is confined to order infinity:

THEOREM 4. *Let $\{p_j\}_{j=0,1,2,\cdots}$ have binomial moments $\{B_k\}$. Then $\{p_j\}$ a PF_∞ sequence is equivalent to $\{B_{k-l}\}_{k,l=0,1,2,\cdots}$, a TP_∞ sequence.*

Proof. Let $P(z) = \sum_{j=0}^{\infty} p_j z^j$ and $B(z) = \sum_{k=0}^{\infty} B_k z^k$ be the generating functions of $\{p_j\}$ and $\{B_k\}$ respectively. Then $B(z) = P(1 + z)$.

Suppose $\{p_j\}$ is a PF_∞ sequence. Then $P(z)$ is of the form (6). Thus

$$B(z) = P(1 + z) = e^\gamma \frac{\prod (1 + \alpha_\nu)}{\prod (1 - \beta_\nu)} e^{\gamma z} \frac{\prod \left(1 + \dfrac{\alpha_\nu}{1 + \alpha_\nu} z\right)}{\prod \left(1 - \dfrac{\beta_\nu}{1 - \beta_\nu} z\right)} .$$

First observe that $\prod (1 + \alpha_\nu) < \infty$ since $\Sigma \alpha_\nu < \infty$ and trivially

$$\sum \frac{\alpha_\nu}{1 + \alpha_\nu} \leqq \sum \alpha_\nu < \infty .$$

Moreover we cannot have $\beta_\nu \geqq 1$, since in this event, $P(1/\beta_\nu)$ diverges, contradicting the fact that $0 \leqq P(z) < 1$ for $0 \leqq z \leqq 1$. Thus, $\sup \beta_\nu < 1$ since $\Sigma \beta_\nu < \infty$, so that

$$\sum \frac{\beta_\nu}{1 - \beta_\nu} \leqq \sum \frac{\beta_\nu}{1 - \sup \beta_\nu} < \infty$$

and

$$\prod_{\nu=1}^{\infty} (1 - \beta_\nu) > 0 .$$

We note further that the coefficient

$$e^\gamma \frac{\prod (1 + \alpha_\nu)}{\prod (1 - \beta_\nu)}$$

is actually 1, as may be verified by substituting $z = 0$ in $B(z)$. Thus $B(z)$ is of the form (6) so that $\{B_k\}$ is a TP_∞ sequence.

Now suppose $\{B_k\}$ is a TP_∞ sequence. Then

$$P(z) = B(z - 1) = e^{-\gamma} \frac{\prod\limits_{\alpha_\nu \neq 1} (1 - \alpha_\nu)}{\prod (1 + \beta_\nu)} e^{\gamma z} z^m \frac{\prod\limits_{\alpha_\nu \neq 1} \left(1 + \dfrac{\alpha_\nu}{1 - \alpha_\nu} z\right)}{\prod \left(1 - \dfrac{\beta_\nu}{1 + \beta_\nu} z\right)} .$$

A factor z enters for each $\alpha_\nu = 1$. Clearly $\sum [\beta_\nu/(1 + \beta_\nu)] < \infty$. Also no $\alpha_\nu > 1$ since $P((\alpha_\nu - 1)/\alpha_\nu) = 0$ contradicting the fact that $P(z) > 0$ for $1 > z > 0$. It follows that $\sup_{\alpha_\nu \neq 1} \alpha_\nu < 1$ since $\Sigma \alpha_\nu < \infty$. Thus

$$\sum_{\alpha_\nu \neq 1} \frac{\alpha_\nu}{1 - \alpha_\nu} < \sum_{\alpha_\nu \neq 1} \frac{\alpha_\nu}{1 - \sup \alpha_\nu} < \infty .$$

Hence apart from the constant factor and the factor z^m, $P(z)$ qualifies as the generating function of a TP_∞ sequence. Of course, the constant factor does not affect the total positivity of the sequence, while the

factor z^m simply implies that the first m elements of $\{p_j\}$ are 0. Since $\Sigma p_j = 1$ we conclude that $\{p_j\}$ is a PF_∞ sequence.

3. Extensions and bounds on the behavior of PF_2 functions for large values of the argument. In this section we develop several further applications by exploiting the variation diminishing property characteristic of totally positive functions. We begin with some preliminary discussions.

1. The function

$$(8) \qquad h(u) = \begin{cases} u^r & u > 0 \\ 0 & u \leq 0 \end{cases} \qquad r = \text{integer}$$

for $r > 0$ generates a PF_∞ function. This follows from the fundamental representation theorem which exhibits the form of the Laplace transform of Pólya frequency functions (see Schoenberg [7]).

2. Let $f(t)$ denote a PF_2 density which is continuous throughout except for at most a single discontinuity of the first kind and consider

$$(9) \qquad g(x) = \int f(t + x)h(t)dt = \int f(u)h(u - x)du$$

where $h(t)$ represents the function specified in (8) with $r = 1$.

By Lemma 1, we infer that $g(x + y)$ is SRR_2. Thus, relation (2) applies (but with a negative sign for $n = 2$) and we have the inequality

$$0 \geqq g(x)g''(x) - [g'(x)]^2$$

or

$$(10) \qquad f(x)\int_x^\infty (u - x)f(u)du \leqq \left(\int_x^\infty [f(u)]du\right)^2.$$

Consider now the case that f is continuous on the interval $[0, \infty)$ and vanishes for negative x.

Then for $x = 0$ (10) reduces to the inequality

$$(11) \qquad f(0) \leqq \frac{1}{\mu_1}, \qquad \mu_1 = \int_0^\infty tf(t)dt .$$

We shall establish below that equality can occur in (11) if and only if

$$f(t) \equiv \begin{cases} \dfrac{1}{\mu_1}e^{-t/\mu_1} & \text{for } t \geqq 0 \\ 0 & \text{for } t < 0 . \end{cases}$$

3. Now consider (9) where $h(u)$ is determined according to (8) where $r = 2$. Then the function $g(x + y)$ is SRR_3 (applying Lemma 1) provided f is assumed to be PF_3. The inequality (2) for determinants of size 3×3 on $g(x + y)$ reduces to

$$f'(0) \leqq \frac{2}{2\mu_1^2 - \mu_2} \quad \text{provided} \quad f(0) = 0 \quad \text{where} \quad \mu_i = \int_0^\infty t^i f(t)$$

and $f(t) = 0$ for $t < 0$. We will see below that $2\mu_1^2 > \mu_2$. Other inequalities for higher order derivatives may be deduced by similar considerations.

We now establish

THEOREM 5. *Let $f(t)$ be continuous on $[0, \infty)$, PF_2, and 0 for $t < 0$.*
(i) *If $f(t) \not\equiv 1/\mu_1 \exp(-t/\mu_1)$, then*

(a)

$$(12) \qquad \begin{aligned} \mu_s &< \Gamma(s + 1)\mu_1^s \quad \text{for} \quad -1 < s < 0 \text{ and } s > 1 \\ \mu_s &> \Gamma(s + 1)\mu_1^s \quad \text{for} \quad 0 < s < 1 \,, \end{aligned}$$

(b) $f(0) < 1/\mu_1$,

(c) *there exists t_0 such that for $t > t_0$*

$$(13) \qquad 0 \leqq f(t) < \frac{1}{\mu_1} \exp(-t/\mu_1) \,.$$

(ii) *Equality in (12) for any $s \neq 0, 1$ implies $f(t) = 1/\mu_1 \exp(-t/\mu_1)$.*

REMARK. The inequalities in (12) are partially contained in the assertion of Theorem 1. However the present method of proof involving the concept of variation diminishing transformations also yields results like (13) which does not seem to follow from the theorems of § 2.

Proof. Consider the function

$$k(x) = \frac{1}{\mu_1} \exp(-x/\mu_1) - f(x) \qquad x \geqq 0 \,.$$

By virtue of (11), $k(0) \geqq 0$. Now let

$$g(x) = \exp(x/\mu_1)k(x) = \frac{1}{\mu_1} - \exp(x/\mu_1)f(x) \,.$$

It is easy to verify that $\exp[(x - y)/\mu_1]$ is TP_2. This implies that $e^{x/\mu_1}f(x)$ is either monotone or PF_2; in either case $g(x)$ has at most 2 sign changes (a PF_2 density is unimodal). Equivalently, $V(k) \leqq 2$ on $[0, \infty)$. We note the relevant fact needed later that if $k(0) = 0$ then $k(x)$ cannot exhibit two sign changes. Otherwise $g(x) + \varepsilon$ for $\varepsilon < 0$ and sufficiently small

displays three sign changes which is impossible in view of the above discussion.

Consider the transformation

$$h(x) = \int_0^\infty x^s k(x) dx \qquad\qquad s > -1 \ .$$

The kernel $M(t, s) = t^s$ is TP_∞ for $t > 0, s$ real and satisfies (2) with strict inequality.

Relation (5) applies and we deduce

$$Z(h) \leqq V(k) \ .$$

Since $V(k) \leqq 2$ we have $Z(h) \leqq 2$. But, by design $h(0) = h(1) = 0$ which means that $s = 0$ and 1 are simple zeros (unless $h(x) \equiv 0$ and we exclude this case as trivial). Therefore, $h(s)$ has two sign changes and $V(k) = V(h) = 2$. Taking account of the fact that $k(0) \geqq 0$ and the discussion preceding (5) we deduce that $k(x) > 0$ for x large and $h(s) > 0$ for $s > 1$ and $-1 < s < 0$ while $h(s) < 0$ for $0 < s < 1$. In case $k(0) = 0$, then as noted above $V(k) = 1$. But $Z(h) = 2$ means that $h(s) \equiv 0$ and consequently $1/\mu_1 \exp(-x/\mu_1) \equiv f(x)$, $0 \leqq x < \infty$. This completes the proof of the theorem.

The inequality (13) shows that f decreases to zero at an exponential rate of a specified amount. We can sharpen the bounds indicated there in the following manner. Let us determine constants a and b such that

$$(14) \qquad \int_0^\infty x^{s_i} a e^{-bx} dx = \int_0^\infty x^{s_i} f(x) dx = \mu_{s_i} \qquad\qquad i = 1, 2$$

where s_1, s_2 are prescribed satisfying $s_1 < s_2$. Performing the left hand integrations we obtain

$$\Gamma(s_1 + 1) a b^{-(s_1+1)} = \mu_{s_1} \ , \qquad \Gamma(s_2 + 1) a b^{-(s_2+1)} = \mu_{s_2} \ .$$

Thus,

$$(15) \qquad \frac{1}{b} = \left(\frac{\mu_{s_2}}{\mu_{s_1}}\right)^{1/(s_2-s_1)} \left(\frac{\Gamma(s_1 + 1)}{\Gamma(s_2 + 1)}\right)^{1/(s_2-s_1)}$$

$$a = \left(\frac{\mu_{s_1}}{\Gamma(s_1 + 1)}\right)^{[(s_2+1)/(s_2-s_1)]} \left(\frac{\mu_{s_2}}{\Gamma(s_2 + 1)}\right)^{-[(s_1+1)/(s_2-s_1)]} \ .$$

Similar to (12) we can achieve other moment inequalities under the same assumptions as prevail in Theorem 5. Throughout we exclude the trivial situation when $f(x) = 1/\mu_1 \exp(-x/\mu_1)$.

We illustrate with two examples:

EXAMPLE 1. Put in (15) $s_1 = 0$ and $s_2 = s > 1$; then the inequality (12) implies $[\mu_s/\Gamma(s + 1)]^{-1/s} = a > 1/\mu_1$. Now employing the analogous argument as in Theorem 5 we secure an extension of (12) to the effect

that

$$(16) \qquad \left(\frac{\mu_t}{\Gamma(t+1)}\right)^{1/t} < \left(\frac{\mu_s}{\Gamma(s+1)}\right)^{1/s} \qquad t > s > 0 \, .$$

This result derives further interest by comparing with the classical moment inequality

$$\mu_t^{1/t} > \mu_s^{1/s} \qquad t > s > 0 \, ,$$

valid for all density functions on the positive axis.

EXAMPLE 2. If we set $s_1 = 1$ and $s_2 = s > 1$ in (15) we get

$$(17) \qquad a = \left[\frac{\mu_1^{s+1}\{\Gamma(s+1)\}^2}{\mu_s^2}\right]^{1/(s-1)} , \qquad b = \left\{\frac{\mu_1 \Gamma(s+1)}{\mu_s}\right\}^{1/(s-1)} .$$

It follows from (12) that $a > 1/\mu_1$ and $b > 1/\mu_1$. Also by the argument of Theorem 4, we may show that there exists t_0 such that for $t \geqq t_0$

$$0 \leqq f(t) < ae^{-bt}$$

for a and b as given in (15). Thus we achieve a sharper statement concerning the rate of exponential decrease of the PF_2 density. By involving higher moments we may strengthen and refine these assertions employing the same techniques.

We close this section by indicating an application of the **above** method to symmetric PF_2 densities. Inequality (11) becomes

$$(18) \qquad \frac{1}{2\mu_1} \geqq f(0) \quad \text{where} \quad \mu_s = \int_{-\infty}^{\infty} |t|^s f(t)dt \, .$$

The result corresponding to Theorem 5 is as follows:

THEOREM 6. *If f is a symmetric PF_2 density then*

$$\mu_s \leqq \Gamma(s+1)\mu_1^s \qquad\qquad for \qquad s > 1 \, .$$

Moreover

$$(19) \qquad 0 \leqq f(t) \leqq \frac{1}{2\mu_1} \exp\left(|t|/\mu_1\right) \qquad\qquad |t| \geqq |t_0|$$

and strict inequality holds in (19) unless $f(t) = 1/2\mu \exp\left(|t|/\mu_1\right)$.

The proof of Theorem 6 entails slight modifications of that of Theorem 5, and thus will be omitted. Results similar to those of Examples 1 and 2 may be obtained for symmetric PF_2 densities.

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