CONDITIONS FOR THE MODULARITY OF AN ORTHOMODULAR LATTICE

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1. Introduction. An *orthomodular lattice* is a lattice L with 0 and 1 which is equipped with an *orthocomplementation* $' : L \rightarrow L$ and which satisfies the *orthomodular identity* $e \leqq f \Rightarrow f = e \vee (f \wedge e')$. Recall that an orthocomplementation $' : L \rightarrow L$ maps each element $e \in L$ onto a complement e' of e in L in such a way that $e'' = e$ and $e \leqq f \Rightarrow f' \leqq e'$ for $e, f \in L$. The "logic" of (non-relativistic) quantum mechanics, i.e., the lattice of closed subspaces of a separable infinite dimensional Hilbert space [5, p. 49], as well as the "logic" of classical mechanics, i.e., the Boolean algebra of all Borel subsets of phase space modulo Borel subsets of measure zero [5, p. 48], are both instances of orthomodular lattices.

L. H. Loomis has shown in [4] that orthomodular lattices provide a natural environment for the abstract study of the dimension theory of operator algebras. I. Kaplansky [3] has obtained an elegant theorem to the effect that if an orthomodular lattice is complete and modular, then it is a continuous geometry.

An *involution semigroup* is a semigroup S equipped with an *involution* *, i.e., an antiautomorphism $^* : S \rightarrow S$ of period 2. An element $e \in S$ is called a *projection* in case $e = e^* = e^2$. In this paper, we use the term *Baer *-semigroup* to refer to an involution semigroup S (with a two-sided zero element 0) which is equipped with a mapping $' : S \rightarrow S$ such that

(i) x' is a projection for $x \in S$ and

(ii) for $x \in S$, $\{y \mid y \in s \text{ and } xy = 0\} = x'S$. A projection $e \in S$ is said to be *closed* in case $e = e''$, and the collection of all closed projections in S is denoted by $P' = P'(S)$. The notion of a Baer *-semigroup was introduced in [2, § 2] in a slightly more general form.

In [2] it is shown that there is an intimate connection between orthomodular lattices and Baer *-semigroups, namely: *If S is a Baer *-semigroup, then $P'(S)$ is an orthomodular lattice with $e \rightarrow e'$ as orthocomplementation and with partial order defined by $e \leqq f \Longleftrightarrow ef = e$ for e, $f \in P'(S)$. The element $0' = 1$ acts as a unit in the semigroup S. Conversely, every orthomodular lattice L is isomorphic to a lattice $P'(S)$ for some Bear *-semigroup S.*

In the sequel, the symbol L always denotes an orthomodular lattice and the symbol S always denotes a Baer *-semigroup. When S and L are so related that there is an orthocomplementation preserving iso-

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morphism from $P'(S)$ onto L, we follow [2, § 3] by saying that S is a *coordinate Baer *-semigroup for L*. We assume the basic facts on orthomodular lattices and Baer *-semigroups as given in [4, pp. 3–6] and [2], respectively.

In view of the afore-mentioned result of Kaplansky in [3], and in view of the important role which questions of modularity seem to play in investigations of the "logic" of quantum mechanics [1], it is natural to seek conditions which guarantee that L is modular. The purpose of this paper is to find conditions on coordinatizing Baer *-semigroups S for L which are equivalent to the modularity of L. One such condition will be given in terms of the notion of *range-closed* elements $x \in S$.

Say that $x \in S$ is *range-closed* in case whenever $g \in P'(S)$ with $g \leqq x''$ and $(gx^*)'' = (x^*)''$, then $g = x''$. S itself is said to be *range-closed* in case every element $x \in S$ is range-closed.

As an illustration of the notion of a range-closed element, consider the case in which S is the multiplicative semigroup of all bounded operators on a Hilbert space H. Let $^* : S \rightarrow S$ be taken, as usual, to mean the passage from an operator T to its adjoint T^*. Let the operators in S be thought of as operating on the right on the vectors of H; and observe that for $A, B \in S$, $AB = 0$ if and only if $B = EB$, where E is the projection onto the orthogonal complement of the range of A. Thus, S becomes a Baer *-semigroup if we define $' : S \rightarrow S$ by $A' =$ the projection onto the orthogonal complement of the range of A, for every $A \in S$. If E is any projection in S, then $(1-E)' = E$, hence, $P'(S)$ is the lattice of all projections in S. Consequently, $P'(S)$ is isomorphic to the lattice of all closed linear subspaces of H.

If $T \in S$ and if $E \in S$ is the projection onto the closed linear subspace M of H, then $(ET^*)'$ is the projection onto the closed linear subspace $(M^{\perp})T^{-1}$; in particular, $(T^*)'$ is the projection onto the null space of T. Let N be the range of T, let E be a projection in S with $E \leqq T''$ $=$ the projection onto the closure of N, and let M be the range of E. Suppose that N is closed and that $(ET^*)'' = (T^*)''$, so that $(M^{\perp})T^{-1} =$ the null space of T. It follows that $M^{\perp} \cap N = 0$, i.e., that $E = T''$.

On the other hand, if N is not a closed linear subspace of H, then $N \neq N^{\perp\perp}$, so there exists a vector x which belongs to $N^{\perp\perp}$ but not to N. Let E_1 be the projection onto the orthogonal complement of the one-dimensional subspace of H spanned by x, and let $E = E_1 \wedge T''$. Then, $(ET^*)'' = (T^*)''$, but $E' \wedge T'' = E_1' \neq 0$; hence $E < T''$.

The above argument shows that an operator $T \in S$ is range-closed if and only if the range of T is a closed linear subspace of H. Consequently, S is range-closed if and only if H is finite dimensional. Since the lattice of closed linear subspaces of a Hilbert space H is modular if and only if H is finite dimensional, we are led by the above remarks to conjecture that *an orthomodular lattice L is modular if and only if* *it can be coordinatized by a range-closed Baer *-semigroup.* This conjecture is verified in the sequel.

2. Hemimorphisms of L. In [2, § 3] we defined a *hemimorphism* ϕ of L to be a mapping $\phi : L \to L$ such that $0\phi = 0$ and $(e \vee f)\phi = e\phi \vee f\phi$ for $e, f \in L$. We also denoted the semigroup (under function composition) of all monotone maps $\phi : L \to L$ by $M(L)$, and decreed that two monotone maps $\phi, \psi \in M(L)$ were to be called *mutually adjoint* in case $(e\phi)'\psi \leqq e'$ and $(e\psi)'\phi \leqq e'$ for all $e \in L$. If $\phi \in M(L)$ has an adjoint in $M(L)$, then this adjoint is unique and is denoted by ϕ^*. $S(L)$ denotes the subset of $M(L)$ consisting of all those monotone maps $\phi \in M(L)$ which possess adjoints $\phi^* \in M(L)$.

We proved in [2, § 3] that *$S(L)$ is a Baer *-semigroup (under function composition), and every $\phi \in S(L)$ is a hemimorphism of L. Moreover, if for $e \in L$ we define a mapping $\phi_e : L \to L$ by $f\phi_e = (f \vee e') \wedge e$ for every $f \in L$, then $\phi_e \in S(L)$ and $\phi_e = \phi_e{}^* = \phi_e^2 = (\phi_e)''$. The mapping $e \to \phi_e$ is an orthocomplement preserving isomorphism of L onto $P'(S(L))$, so $S(L)$ coordinatizes L.*

In [2, § 4], we exhibited a natural *-preserving semigroup homomorphism $\phi : S \to S(P'(S))$ defined by $x\phi = \phi_x \in S(P'(S))$ for $x \in S$, where $e\phi_x = (ex)''$ for all $e \in P'(S)$. In case $x = f \in P'(S)$, there is no notational conflict here; indeed, $(ef)'' = (e \vee f') \wedge f$ for all $e \in P'(S)$.

LEMMA 1. *Let $\phi \in S(L)$, $e \in L$. Then, $1\phi^* = (e \wedge 1\phi^*) \vee e'\phi\phi^*$.*

Proof. Put $g = e' \vee (1\phi^*)'$, $h = g \wedge (g\phi\phi^*)' \wedge 1\phi^*$. Since $g\phi\phi^* \leqq 1\phi^*$, we have $(1\phi^*)' \leqq (g\phi\phi^*)'$. Combining the latter inequality with $(1\phi^*)' \leqq g$, we get $(1\phi^*)' \leqq g \wedge (f\phi\phi^*)'$; hence, by the orthomodular identity, $g \wedge (g\phi\phi^*)' = (1\phi^*)' \vee h$. Now, $g\phi\phi^* = e'\phi\phi^* \vee (1\phi^*)'\phi\phi^* = e'\phi\phi^*$ since $(1\phi^*)'\phi = 0$. Consequently, $g \wedge (e'\phi\phi^*)' = (1\phi^*)' \vee h$, and the lemma will be proved as soon as we show that $h = 0$. But, $h\phi \leqq g\phi \wedge (g\phi\phi^*)'\phi \leqq g\phi \wedge (g\phi)' = 0$, so $h\phi = 0$. Thus, $1\phi^* = (h\phi)'\phi^* \leqq h'$, so $h \leqq (1\phi^*)'$. Since also $h \leqq 1\phi^*$, it follows that $h = 0$, proving the lemma.

THEOREM 2. *For $\phi \in S(L)$, the following conditions are equivalent:*
(i) *ϕ is range-closed.*
(ii) *$(f\phi^*)'\phi = f' \wedge 1\phi$ for $f \in L$.*
(iii) *For $e, f \in L$, $f\phi^* = e\phi^* \Rightarrow f \vee (1\phi)' = e \vee (1\phi)'$.*

Proof. To prove (i) $\Rightarrow$ (ii), note that $(f\phi^*)'\phi \leqq f' \wedge 1\phi$ and put $h' = f' \wedge 1\phi \wedge [(f\phi^*)'\phi]'$. It will suffice to prove $h' = 0$. Now, $h\phi^* = f\phi^* \vee (1\phi)'\phi^* \vee (f\phi^*)'\phi\phi^* = f\phi^* \vee (f\phi^*)'\phi\phi^* = 1\phi^*$ by Lemma 1. Since $(1\phi)'$

$\leqq h$, $h = (1\phi)' \vee (h \wedge 1\phi)$, so $h\phi^* = (h \wedge 1\phi)\phi^*$. Thus, we have $(h \wedge 1\phi)\phi^* = 1\phi^*$. The hypothesis that ϕ is range-closed now yields $h \wedge 1\phi = 1\phi$, so $1\phi \leqq h$. Consequently, $1 \leqq h$ and $h' = 0$.

To prove (ii) $\Rightarrow$ (iii), note that according to (ii), $f\phi^* = e\phi^* \Rightarrow f' \wedge 1\phi = (f\phi^*)'\phi = (e\phi^*)'\phi = e' \vee 1\phi$. Consequently, $f\phi^* = e\phi^* \Rightarrow f \vee (1\phi)' = e \vee (1\phi)'$.

To prove (iii) $\Rightarrow$ (i), suppose $g \leqq 1\phi$ and $g\phi^* = 1\phi^*$. Then, by (iii), $g \vee (1\phi)' = 1 \vee (1\phi)' = 1$, so $1\phi = g \vee (g' \wedge 1\phi) = g \vee 0 = g$.

THEOREM 3. *Let $\phi \in S(L)$ be range-closed and let $f \in L$, $e = (1\phi)'$, $f_1 = e'\phi_f$. Then, the necessary and sufficient condition that $\phi\phi_f$ fails to be range-closed is the existence of an element $g \in L$ such that $g < f_1$ and $g \vee e = f \vee e$.*

Proof. By definition, $\phi\phi_f$ fails to be range-closed in $S(L)$ if and only if there exists $g < 1\phi\phi_f = f_1$ such that $g\phi_f\phi^* = 1\phi_f\phi^* = f\phi^*$. Since $g < f_1 \leqq f$, we have $g\phi_f = g$; hence, $\phi\phi_f$ fails to be range-closed if and only if there exists $g < f_1$ with $g\phi^* = f\phi^*$. Because ϕ is range-closed, the condition $g\phi^* = f\phi^*$ is equivalent to $g \vee e = f \vee e$ by part (iii) of Theorem 2.

The hemimorphism $\phi \in S(L)$ will be called *totally range-closed* in case $\phi_e\phi$ is range-closed for every $e \in L$. (In the special case in which L is the lattice of closed subspaces of a Hilbert space H, every bounded operator T on H with the property that it maps closed subspaces of H onto closed subspaces of H induces a totally range-closed hemimorphism ϕ_T on L.)

LEMMA 4. *$\phi \in S(L)$ is totally range-closed if and only if $[(g\phi^*)' \wedge e]\phi = g' \wedge e\phi$ for all $g, e \in L$.*

Proof. Let $e \in L$. Then, by part (ii) of Theorem 2, $\phi_e\phi$ is range-closed if and only if $(g\phi^*\phi_e)'\phi_e\phi = g' \wedge e\phi$ for every $g \in L$. It is easy to verify that ϕ_e is range-closed, so, again by part (ii) of Theorem 2, $(g\phi^*\phi_e)'\phi_e = (g\phi^*)' \wedge e$. Hence, $\phi_e\phi$ is range-closed if and only if $[(g\phi^*)' \wedge e]\phi = g' \wedge e\phi$ for every $g \in L$.

Denote by $S_{TRC}(L)$ the subset of $S(L)$ consisting of those hemimorphims $\phi \in S(L)$ such that both ϕ and ϕ^* are totally range-closed. Suppose that ϕ and ψ are totally range-closed hemimorphisms in $S(L)$. Then, for $g, e \in L$, $[(g\psi^*\phi^*)' \wedge e]\phi\psi = [(g\psi^*)' \wedge e\phi]\psi = g' \wedge e\phi\psi$; hence, by Lemma 4, $\phi\psi$ is totally range-closed. It follows that $S_{TRC}(L)$ is a $*$-subsemigroup of $S(L)$.

3. *-Regular Baer *-semigroups. Borrowing some terminology from [3, p. 525], we say that $f \in P'(S)$ is a *right projection* for $a \in S$ in case

$Sf = Sa$, and we say that S is **-regular* in case every element $a \in S$ has a right projection. It is plain that $a \in S$ has a right projection $f \in P'(S)$ if and only if $f = a''$ and $f = ba$ for some $b \in S$.

Now, suppose for a moment that L is complete and modular and that L contains four or more independent perspective elements. By the afore-mentioned theorem of Kaplanksy [3], L is a continuous geometry, and by the well-known coordinatization theorem for continuous geometries, L can be coordinatized by a *-regular ring R. If S represents the multiplicative semigroup of R, then S is a *-regular Baer *-semigroup coordinatizing L.

Thus, we are led to a second conjecture: *An orthomodular lattice L is modular if and only if it can be coordinatized by a *-regular Baer *-semigroup.* This conjecture will also be verified in the sequel.

Slight modifications of the proof of [3, Lemma 4, p. 525] give the following lemma:

LEMMA 5. *Let $a \in S$ have a right projection f and let a^* have a right projection e. Then, there is a uniquely determined element $a^{-1} \in S$ such that $a^{-1}a = f$ and $a^{-1}e = a^{-1}$. Moreover, $aa^{-1} = e$ and $fa^{-1} = a^{-1}$.*

We will follow Kaplansky in [3, p. 525] by calling the element a^{-1} of Lemma 5 the *relative inverse* of a in S. Evidently, $(a^{-1})^{-1} = a$ and $(a^*)^{-1} = (a^{-1})^*$.

THEOREM 6. *Let $\phi \in S_{TRC}(L)$. Then ϕ and ϕ^* both have right projections in $S(L)$ and ϕ^{-1}, the relative inverse of ϕ in $S(L)$, is given by the prescription $g\phi^{-1} = [(g' \wedge 1\phi)\phi^*]' \wedge 1\phi^*$ for $g \in L$.*

Proof. Let $e = 1\phi^*$, $f = 1\phi$, and let $\phi^{-1} : L \to L$ be the mapping given by the prescription of the theorem. For $g \in L$, $g\phi^{-1}\phi = \{[(g' \wedge f)\phi^*]' \wedge e\}\phi = [(g' \wedge f)\phi^*]'\phi$. Since $\phi \in S_{TRC}(L)$, it is range-closed, so by part (ii) of Theorem 2, $[(g' \wedge f)\phi^*]'\phi = (g' \wedge f)' \wedge f = g\phi_f$. This proves that $\phi^{-1}\phi = \phi_f = \phi''$. Since, for $g \in L$, $g\phi^{-1} \leqq e$, we have $g\phi^{-1}\phi_e = g\phi^{-1}$; hence, $\phi^{-1}(\phi^*)'' = \phi^{-1}\phi_e = \phi^{-1}$. It only remains to prove that $\phi^{-1} \in S(L)$.

Define $(\phi^{-1})^* : L \to L$ by $h(\phi^{-1})^* = [h' \wedge e)\phi]' \wedge f$ for $h \in L$. It is plain that ϕ^{-1} and $(\phi^{-1})^*$ are monotone maps on L. For $g \in L$, $(g\phi^{-1})'(\phi^{-1})^* = \{[((g' \wedge f)\phi^*)' \wedge e]\phi\}' \wedge f = [(g' \wedge f)' \wedge e\phi]' \wedge f = [(g' \wedge f)' \wedge f]' \wedge f = (g' \wedge f)\phi_f = g' \wedge f$. Similarly, for $h \in L$, $[h(\phi^{-1})^*]'\phi^{-1} = h' \wedge e$; hence, ϕ^{-1} and $(\phi^{-1})^*$ are mutually adjoint and $\phi^{-1} \in S(L)$.

THEOREM 7. *Let L be modular. Then, $\phi \in S_{TRC}(L) \Rightarrow \phi^{-1} \in S_{TRC}(L)$.*

Proof. Let $g, h \in L$ and let $e = 1\phi^*$, $f = 1\phi$, $k = [(h' \wedge e)\phi]'$. Since L is modular, $((k \wedge f) \vee g') \wedge f = (k \wedge f) \vee (g' \vee f)$. Thus, by Theorem

6, $[[h(\phi^{-1})^*]' \wedge g]\phi^{-1} = [[((k \wedge f) \vee g') \wedge f]\phi^*]' \wedge e = [[(k \wedge f) \vee (g' \wedge f)]\phi^*]' \wedge e = [(k \wedge f)\phi^* \vee (g' \wedge f)\phi^*]' \wedge e$. Since ϕ^* is totally range-closed, $(k \wedge f)\phi^* = (h' \wedge e)' \wedge f\phi^* = (h' \wedge e)' \wedge e$. Consequently, $[[h(\phi^{-1})^*]' \wedge g]\phi^{-1} = [(h' \wedge e) \vee e'] \wedge [(g' \wedge f)\phi^*]' \wedge e = (h' \wedge e)\phi_e \wedge g\phi^{-1} = h' \wedge e \wedge g\phi^{-1} = h' \wedge g\phi^{-1}$, so ϕ^{-1} is totally range-closed. A dual argument shows that $(\phi^{-1})^*$ is also totally range-closed, completing the proof.

LEMMA 8. *If L is not modular, there exist elements $e, f, g \in L$ such that $g < e'\phi_f$ and $g \vee e = f \vee e$.*

Proof. If L is not modular, there exist elements $a, b, c \in L$ such that $b < c, b \vee a = c \vee a$ and $b \wedge a = c \wedge a$. Let $h = (b \wedge a)' = (c \wedge a)'$, $e = a\phi_h$, $f = c\phi_h$ and $g = b\phi_h$. Since $h' \leqq a, b, c$, we have $e = a \wedge h$, $f = c \wedge h$ and $g = b \wedge h$. Furthermore, since $b \vee a = c \vee a$, $g \vee e = b\phi_h \vee a\phi_h = (b \vee a)\phi_h = (c \vee a)\phi_h = c\phi_h \vee a\phi_h = f \vee e$. Also, $g = b \wedge h \leqq c \wedge h = f$. If $g = f$, then $b\phi_h = c\phi_h$, so by part (iii) of Theorem 2 and the fact that ϕ_h is range-closed we deduce $b \vee h' = c \vee h'$, i.e., $b = c$, contradicting $b < c$. Thus, we have $g < f$. Finally, $e \wedge f = a \wedge c \wedge h = 0$, so $f = (e \wedge f)' \wedge f = e'\phi_f$, completing the proof.

THEOREM 9. *L is modular if and only if $\phi_f \in S_{TRC}(L)$ for every $f \in L$.*

Proof. Suppose that L is modular and that $f, g, h \in L$. Then, since $f' \leqq (g\phi_f)'$, $[(g\phi_f)' \wedge h] \vee f' = (g\phi_f)' \wedge (h \vee f')$. Consequently, $[(g\phi_f)' \wedge h]\phi_f = \{[(g\phi_f)' \wedge h] \vee f'\} \wedge f = (g\phi_f)' \wedge (h \vee f') \wedge f = (g\phi_f)' \wedge f \wedge h\phi_f = (g' \wedge f)\phi_f \wedge h\phi_f = g' \wedge f \wedge h\phi_f = g' \wedge h\phi_f$, proving that $\phi_f \in S_{TRC}(L)$.

Conversely, suppose that $\phi_f \in S_{TRC}(L)$ for every $f \in L$. If L were not modular, there would exist, according to Lemma 8, elements $e, f, g \in L$ such that $g < e'\phi_f$ and $g \vee e = f \vee e$. By Theorem 3, this would imply that $\phi_{e'}\phi_f$ fails to be range-closed, contradicting $\phi_f \in S_{TRC}(L)$.

4. Conditions for the Modularity of L. In this section we prove our main result, namely:

THEOREM 10. *Let L be an orthomodular lattice. Then, the following conditions are mutually equivalent:*

(i) *L is modular.*

(ii) *L can be coordinatized by a *-regular Baer *-semigroup.*

(iii) *L can be coordinatized by a range-closed Baer *-semigroup.*

Proof. If L is modular, then Theorems 7 and 9 imply that $S_{TRC}(L)$ is a *-regular Baer *-semigroup coordinatizing L; hence, (i) $\Rightarrow$ (ii).

In order to prove that (ii) $\Rightarrow$ (iii), we will have to recall that if a and y are elements of a Baer *-semigroup S, then $(ay)'' = (a''y)''$. This was shown in the course of the proof of [2, Theorem 8, p. 654]. Now, suppose that S is a *-regular Baer *-semigoup coordinatizing L. For $b \in S$ and $g \in P'(S)$; $g \leqq b''$ and $(gb^*)'' = (b^*)'' \Rightarrow b'' = b^*(b^*)^{-1} = [b^*(b^*)^{-1}]'' = [(b^*)''(b^*)^{-1}]'' = [(gb^*)''(b^*)^{-1}]'' = [gb^*(b^*)^{-1}]'' = (gb'')'' = g'' = g$. Thus, any element $b \in S$ is range-closed. This proves (ii) $\Rightarrow$ (iii).

Finally, let S be a range-closed Baer *-semigroup, and let $e, f \in L = P'(S)$. Then, $\phi_e\phi_f = \phi_{ef}$ by [2, Theorem 8, p. 654]. Since S is range-closed, ef is range-closed in S, so $\phi_{ef} = \phi_e\phi_f$ is range-closed in S(L). It follows that $\phi_f \in S_{TRC}(L)$ for every $f \in L$; hence, that L is modular by Theorem 9. Consequently, (iii) $\Rightarrow$ (i).

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