A GENERALIZATION OF THE STONE-WEIERSTRASS THEOREM

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1. Introduction. Consider a compact Hausdorff space X and the set $C(X)$ of all continuous complex-valued functions on X. Consider also a subset $\mathfrak{A}$ of $C(X)$ which is an algebra, which is closed in the uniform topology of $C(X)$, which contains the constant functions, and which contains sufficiently many functions to distinguish points of X. Such an algebra $\mathfrak{A}$ is called *self-adjoint* if the complex conjugate of each function in $\mathfrak{A}$ is in $\mathfrak{A}$. The classical Stone-Weierstrass Theorem states that if $\mathfrak{A}$ is self-adjoint then $\mathfrak{A} = C(X)$. If $\mathfrak{A}$ has the property that the only functions in $\mathfrak{A}$ which are real at every point of X are the constant functions then $\mathfrak{A}$ is called *anti-symmetric*. Clearly anti-symmetry and self-adjointness are opposite properties, in the sense that if $\mathfrak{A}$ has both properties then X must consist of a single point.

Hoffman and Singer [2] have studied these two properties and given several interesting examples. The present paper was inspired by their work but it more directly relates to a previous paper of Šilov [3]. The purpose of the present paper is to prove the following decomposition theorem for a general algebra $\mathfrak{A}$ of the type defined above.

THEOREM. *There exists a partition P of X into disjoint closed sets such that*

(i) *for each S in P the restriction $\mathfrak{A}_S$ of $\mathfrak{A}$ to S is anti-symmetric,*

(ii) *if a function f in $C(X)$ has, for each S in P, a restriction to S which belongs to $\mathfrak{A}_S$, then f is in $\mathfrak{A}$,*

(iii) *for each S in P, each closed subset T of $X - S$, and each $\varepsilon > 0$ there exists g in $\mathfrak{A}$ with $\|g\| \leqq 1$, with $|g(x) - 1| < \varepsilon$ for x in S, and with $|g(x)| < \varepsilon$ for x in T.*

Property (ii) of this theorem is the essential new fact of this paper. The construction given below which leads to the partition P is due to Šilov [3], who in essence proved (i) and (iii). Šilov proved a weaker property than (ii). Our proofs are different from those of Šilov, although the construction is the same.

The fact that the Stone-Weierstrass theorem is a special case of the theorem to be proved here is clear. If $\mathfrak{A}$ is self-adjoint then each $\mathfrak{A}_S$ is self-adjoint. Since $\mathfrak{A}_S$ is also anti-symmetric, each set S in P consists of a single point. Therefore $\mathfrak{A}_S = C(S)$. By the theorem to

Received June 15, 1960.

be proved, it follows that each function in $C(X)$ is in $\mathfrak{A}$. Thus $\mathfrak{A} = C(X)$, which is the conclusion of the Stone-Weierstrass Theorem.

2. **Proof of the theorem.** The key step in the proof will be the following lemma.

LEMMA. *Let Y be a compact Hausdorff space and $\mathfrak{B}$ be a subalgebra of $C(Y)$ which contains the constant functions. Let $\mathfrak{R}$ be all real functions in $\mathfrak{B}$. Define $y_1 \equiv y_2$, for y_1 and y_2 in Y, to mean that $f(y_1) = f(y_2)$ for all f in $\mathfrak{R}$. Let Q be the set of all equivalence classes for this equivalence relation, so that Q is a partition of Y into disjoint closed sets. Let μ be a finite complex-valued Baire measure on Y and f a function in $C(Y)$ such that*

(a) $\|\mu\| \leqq 1$,

(b) $\int g d\mu = 0$ *for all g in $\mathfrak{B}$*,

(c) $\int f d\mu \neq 0$.

Then there exists S_0 in Q and a finite complex-valued Baire measure ν on S_0 such that

(a_1) $\|\nu\| \leqq 1$,

(b_1) $\int g d\nu = 0$ *for all g in $\mathfrak{B}$*,

(c_1) $\left|\int f d\nu\right| \geqq \left|\int f d\mu\right|$.

Proof. It is clearly no loss of generality to assume that $\mathfrak{B}$ is closed in $C(Y)$. Let $\gamma = \{g_i\}$ be a finite set of functions in $\mathfrak{R}$ such that

$$g_i \geqq 0 \text{ for all } i,$$

$$\Sigma g_i = 1 .$$

Let Γ denote the class of all such γ. Define a partial ordering on Γ by writing

$$\{g_i\} \leqq \{g'_j\}$$

if there exists a mapping φ of the set of indices j onto the set of indices i such that

$$g_i = \sum_{\varphi(j)=i} g'_j$$

for all i. To see that Γ is a directed set relative to this partial ordering, let $\{g_i\}$ and $\{g'_j\}$ be any two elements of Γ. Then the set $\{g_i g'_j\}$ is clearly a common successor of $\{g_i\}$ and $\{g'_j\}$.

Consider $\gamma = \{g_i\}$ in Γ. For each index i let μ_i be the measure defined by

$$\mu_i(H) = \int_H g_i d\mu$$

for each Baire subset H of S. Clearly

$$\Sigma \| \mu_i \| = \| \mu \| \leqq 1$$

and

$$\int f d\mu = \sum \int f d\mu_i \ .$$

Thus for at least one value of i with $\| \mu_i \| \neq 0$ we have

$$\left| \int f d\mu_i \right| \Big/ \| \mu_i \| \geqq \left| \int f d\mu \right| \ .$$

Choose such a value of i and write

$$\mu_\gamma = \frac{\mu_i}{\| \mu_i \|} \ .$$

It follows that

$$(\alpha) \qquad \| \mu_\gamma \| = 1$$

and

$$(\beta) \qquad \left| \int f d\mu_\gamma \right| \geqq \left| \int f d\mu \right| \ .$$

By the compactness in the weak* topology of the unit sphere of the set of Baire measures on X, it follows that the net $\{\mu_\gamma\}$ has a cluster point ν in the weak* topology. Let x_0 be any point in the support of the measure ν, and let S_0 be that member of the partition P which contains x_0. Let x_1 be any point in $X - S_0$. Thus there exists h_0 in $\mathfrak{R}$ with $h_0(x_0) \neq h_0(x_1)$. Let

$$h_1 = c_1 h_0 + c_2 \ .$$

If the real constants c_1 and c_2 are chosen properly then

$$h_1(x_1) = 1 \ , \quad h_1(x_0) = 0 \ .$$

It follows that there exists a neighborhood U_0 of x_0 and a neighborhood U_1 of x_1 such that

$$h_1(x) < \tfrac{1}{4} \ , \quad x \in U_0$$

and

$$h_1(x) > \frac{3}{4}\,, \quad x \in U_1\,.$$

Let λ be continuous real-valued function on the range of h_1 with $0 \leqq \lambda \leqq 1$, $\lambda(t) = 0$ for $t \leqq \frac{1}{4}$, $\lambda(t) = 1$ for $t \geqq \frac{1}{4}$. By the Weierstrass approximation theorem, $\lambda(t)$ is a uniform limit of polynomials in t. Therefore the function

$$h_2 = \lambda \circ h_1$$

is in $\mathfrak{R}$. Clearly $0 \leqq h_2 \leqq 1$, $h_2(x) = 0$ for x in U_0, and $h_2(x) = 1$ for x in U_1.

Define $g_1 = h_2$ and $g_2 = 1 - h_2$, so that $\{g_i\} \in \Gamma$. If $\gamma = \{g'_j\}$ is an element of Γ which follows $\{g_i\}$, then each g'_j vanishes on either U_0 or U_1. Therefore the support of μ_γ is either a subset of $X - U_0$ or of $X - U_1$. Thus the support of ν is either a subset of $X - U_0$ or $X - U_1$. By the choice of x_0, it follows that the support of ν cannot be a subset of $X - U_0$ and is therefore a subset of $X - U_1$. Therefore x_1 is not in the support of ν. Since x_1 was any point in $X - S_0$, it follows that the support of ν is a subset of S_0. Thus ν is a Baire measure on S_0. It is clear from the definition of ν and from (α) and (β) that (a_1) and (c_1) are valid. It only remains to prove (b_1). Now for each g in $\mathfrak{B}$ and each γ in Γ we have

$$\int g d\mu_\gamma = \int g g_i d\mu = 0\,,$$

by (b) and the fact that $gg_i \in \mathfrak{B}$. Passing to the limit gives (b_1). This completes the proof of the lemma.

Let Ω be the class of ordinal numbers whose cardinal numbers are less than or equal to 2^β, where β is the cardinal number of X. For each σ in Ω we define by transfinite induction a partition P_σ of Ω into disjoint closed sets. This is to be done in such a way that P_σ is a refinement of P_τ for $\sigma > \tau$. The definition is started by defining $P_1 = \{X\}$, so that the first partition P_1 consists of the set X alone. Assume that P_τ has been defined for all ordinals $\tau < \sigma$. If σ has a predecessor σ_0, let S be any element of P_{σ_0}, and let $\mathfrak{F}$ be the set of all functions in $\mathfrak{A}$ which are real on S. Partition S by defining $x_1 \equiv x_2$ for x_1 and x_2 in S to mean that $f(x_1) = f(x_2)$ for all f in $\mathfrak{F}$. Clearly S is partitioned into disjoint closed sets by this equivalence relation. The totality of all sets into which the elements S of P_{σ_0} are partitioned in this way is defined to be the class P_σ.

If σ has no predecessor, define $x_1 \equiv x_2$, for x_1 and x_2 in X, to mean that x_1 and x_2 belong to the same element of P_τ for all $\tau < \sigma$. The equivalence classes of this equivalence relation clearly form a partition P_σ of X into disjoint closed sets. Thus the classes P_σ are defined for all σ in Ω, and it is clear that P_σ is a refinement of P_τ whenever $\sigma > \tau$.

Assume that $P_{\sigma+1}$ is a proper refinement of P_σ for all σ in Ω, i.e., that $P_{\sigma+1} \neq P_\sigma$. Then $P_{\sigma+1}$ contains a set not in any P_τ for $\tau < \sigma + 1$. Therefore the cardinal number of subsets of X is at least equal to the cardinal number of the set Ω. This contradicts the choice of Ω. Therefore there exists an ordinal in ρ such that $P_{\rho+1} = P_\rho$. We shall show that the partition $P = P_\rho$ satisfies all requirements of the theorem.

The fact (i) that $\mathfrak{A}_S$ is anti-symmetric for each S in $P = P_\rho$ is a consequence of the fact $P_\rho = P_{\rho+1}$.

We next prove (ii). Consider to this end f in $C(X)$ such that the restriction of f to S belongs to $\mathfrak{A}_S$ for all S in P. Assume that f is not in $\mathfrak{A}$. By the Hahn-Banach theorem, there exists a bounded linear functional on $C(X)$ which vanishes on $\mathfrak{A}$ and does not vanish at f. By the Riesz representation theorem, this functional can be realized as a measure μ on X. Thus

$$\int g d\mu = 0 , \qquad g \in \mathfrak{A}$$

$$\int f d\mu \neq 0 .$$

we may clearly assume that $\|\mu\| \leqq 1$. We now construct, by transfinite induction, a set S_σ in P_σ for each σ in Ω and a finite complex-valued Baire measure μ_σ on S_σ with

$$(1) \quad S_\sigma \subset S_\tau \quad \text{for} \quad \tau < \sigma ,$$

$$(2) \quad \left|\int f d\mu_\sigma\right| \geqq \left|\int f d\mu\right| ,$$

$$(3) \quad \|\mu_\sigma\| \leqq 1 ,$$

$$(4) \quad \int g d\mu_\sigma = 0 , \quad \text{all } g \text{ in } \mathfrak{A}.$$

Clearly the measure $\mu_1 = \mu$ does the trick for $\sigma = 1$. Assume therefore that the sets S_τ and the measures μ_τ have been constructed for all $\tau < \sigma$. If σ is a limit ordinal, let

$$S_\sigma = \bigcap_{\tau<\sigma} S_\tau$$

and let μ_σ be any cluster point in the weak* topology of the net $\{\mu_\tau\}_{\tau<\sigma}$. Clearly this set S_σ and the measure μ_σ do the trick.

If σ is not a limit ordinal, then there exists τ with $\sigma = \tau + 1$. Let Y be S_τ and let $\mathfrak{B}$ be the restriction to Y of the algebra $\mathfrak{A}$. By the above lemma, applied to the measure μ_τ, there exists a finite complex-valued Baire measure μ_σ on some $S_\sigma \in P_\sigma$, with $S_\tau \subset Y = S_\sigma$, such that

$$(3) \quad \|\mu_\sigma\| \leqq 1 ,$$

$$(4) \quad \int g d\mu_\sigma = 0 , \qquad \text{all } g \text{ in } \mathfrak{B}$$

$$(2) \quad \left|\int f d\mu_\sigma\right| \geqq \left|\int f d\mu_\tau\right| \geqq \left|f d\mu\right| .$$

Thus the set S_σ and the measure μ_σ do the trick.

This completes the construction of the sets S_σ and the measures μ_σ. For $\sigma = \rho$ we have

$$\left|\int f d\mu_\rho\right| \geqq \left|\int f d\mu\right| \neq 0$$

and $\int g d\mu_\rho = 0$ for all g in $\mathfrak{A}$. Therefore the restriction of f to S_ρ is not in $\mathfrak{A}_{S_\rho}$. This contradicts the assumption of (ii), thereby proving that f is in $\mathfrak{A}$. This proves (ii).

It remains to prove (iii). To do this we prove by induction on σ that for each σ in Ω, each S in P_σ, each closed subset T of $X - S$, and each $\varepsilon > 0$ there exists g in $\mathfrak{A}$ with $\|g\| \leqq 1$, $|g(x) - 1| < \varepsilon$ for all x in S, and $|g(x)| < \varepsilon$ for all x in T. Once this is done (iii) is obviously obtained by letting σ equal ρ. Since the existence of g is clear if $\sigma = 1$, consider $\sigma > 1$ and assume that all smaller values of σ have been disposed of. If σ is a limit ordinal, there exists $\tau < \sigma$ and R in P_τ with $S \subset R$ and $T \subset X - R$. By the induction hypothesis, there exists g in $\mathfrak{A}$ with $\|g\| \leqq 1$, $|g(x) - 1| < \varepsilon$ for all x in R, and $|g(x)| < \varepsilon$ for all x in T. Since $S \subset R$ this function g has the required properties.

It remains to consider the case of an ordinal σ which has a predecessor τ, so that $\sigma = \tau + 1$. Let R be that element of P_τ for which $S \subset R$. Let $\mathfrak{D}$ be the closure in $C(R)$ of $\mathfrak{A}_R$. Let x_1 be any point in S and x_0 any point in $T \cap R$. By the definition of the partition P_σ there exists h_1 in $\mathfrak{A}$ such that the restriction of h_1 to $\mathfrak{R}$ is real and such that $h_1(x_0) \neq h_1(x_1)$. Let λ be a function on $h_1(R)$ with $0 \leqq \lambda \leqq 1$, $\lambda(h_1(x_0)) = 0$, $\lambda(h_1(x_1)) = 1$. Define the function h_2 in $C(R)$ by

$$h_2 = \lambda \circ h_1 .$$

Since λ is a uniform limit of polynomials, $h_2 \in \mathfrak{D}$. Clearly $\|h_2\| \leqq 1$, $h_2(x_0) = 0$, and $h_2(x_1) = 1$. Thus $h_2(x) = 1$ for all x in S. By the compactness of $T \cap R$, the product of a certain finite number of such functions h_2 will be a function h_3 in $\mathfrak{D}$ with $\|h_3\| \leqq 1$, $h_3(x) = 1$ for x in S, $|h_3(x)| < \delta$ for x in $T \cap R$, where δ is an arbitrarily small positive number. By the definition of $\mathfrak{D}$, there exists h_4 in $\mathfrak{A}$ with

$$|h_3(x) - h_4(x)| < \delta$$

for all x in R. Define

$$h_5 = (1 + 2\delta)^{-1} h_4 ,$$

so that $h_5 \in \mathfrak{A}$. Also

$$|h_5(x)| \leqq (1 + 2\delta)^{-1}(|h_3(x)| + \delta) < 1$$

for all x in R. Thus there exists an open set $U_0 \subset X$ with $R \subset U_0$ such that $|h_5(x)| < 1$ for all x in U_0. If δ is sufficiently small it is clear that

$$|h_5(x) - 1| < \frac{\varepsilon}{2}$$

for all x in S and

$$|h_5(x)| < \varepsilon$$

for all x in $T \cap R$. Thus there exists an open set U_1 in X with $T \cap R \subset U_1$ such that $|h_5(x)| < \varepsilon$ for all x in U_1. By the induction hypothesis at stage τ there exists h_6 in $\mathfrak{A}$ with $\|h_6\| \leqq 1$, $|h_6(x) - 1| < \varepsilon/2$ for x in R,

$$(*) \qquad |h_6(x)| < \min\{|h_5(x)|^{-1} : x \in X - U_0\}$$

for all x in $X - U_0$, and

$$(\substack{*\\*}) \qquad |h_6(x)| < \min\{\varepsilon|h_5(x)|^{-1} : x \in T - U_1\}\ ,$$

for all x in $T - U_1$.

Define $g = h_5 h_6$, so that $g \in \mathfrak{A}$. For x in U_0 we have $|h_5(x)| \leqq 1$, so that $|g(x)| \leqq 1$ since $\|h_6\| \leqq 1$. For x in $X - U_0$ we have $|g(x)| \leqq 1$ because of $(*)$. Thus $\|g\| \leqq 1$. For x in S we have $|h_6(x) - 1| < \varepsilon/2$ and $|h_5(x) - 1| < \varepsilon/2$, so that

$$|g(x) - 1| \leqq |h_6(x)|\,|h_5(x) - 1| + |h_6(x) - 1| < \varepsilon\ .$$

For x in U_1 we have $|h_5(x)| < \varepsilon$, so that $|g(x)| < \varepsilon$ since $\|h_6\| \leqq 1$. For x in $T - U_1$ we have $(\substack{*\\*})$, so that $|g(x)| < \varepsilon$. Thus $|g(x)| < \varepsilon$ for all x in T. Thus g has all of the required properties. This completes the proof of (iii) and thereby the proof of the theorem.

We note in conclusion that property (iii) and some results to be found in [**1**] imply that $\mathfrak{A}_S$ is closed in $C(S)$ for each S in P.

References

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